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Norm estimates for function starlike or convex of order alpha

Shinji YAMASHITA

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Abstract. For holomorphic functions f with $\mathrm{Re}\{zf'(z)/f(z)\}>\alpha$ and $\mathrm{Re}\{zf''(z)/f'(z)\}>\alpha-1$, $(0\leq\alpha<1)$, respectively, in $\{|z|<1\}$, estimates of $\sup_{|z|<1}(1-|z|^{2})|f''(z)/f'(z)|$ are given. Functions Gelfer-close-to-convex of exponential order (α,β) will also be considered.

Key words: starlike and convex of order α; Gelfer-starlike, Gelfer-convex, and Gelfer-close-to-convex; Schwarz's and Schwarz-Pick's inequalities.

1. Introduction

Sharp upper estimates of the norm

$$\|f\|=\sup_{|z|<1}\left(1-|z|^{2}\right)\left|\frac{f''(z)}{f'(z)}\right|$$

are given for f holomorphic in $D=\{z;|z|<1\}$ under additional conditions.

Throughout the present paper, by f we always mean a function holomorphic in D with the Taylor expansion

$$f(z)=z+a_{2}z^{2}+a_{3}z^{3}+\cdots. \tag{1.1}$$

If f is univalent in D, then $\|f\|\leq 6$ and $\|k\|=6$ for the Koebe function $k(z)=z/(1-z)^{2}$. Conversely if $\|f\|\leq 1$, then f is univalent in D; see [B, p. 36, Korollar 4.1]. A necessary and sufficient condition for $\|f\|<+\infty$ is that there exists a constant ρ, $0<\rho\leq 1$, such that f is univalent in each Appolonius disk,

$$\left\{w;\left|\frac{w-z}{1-\overline{z}w}\right|<\rho\right\},\qquad z\in D;$$

see [Y1, Y2]. The set of all f with finite $\|f\|$ is a nonseparable Banach space with the norm $\|\cdot\|$ under the Hornich operation; see [Y1, Theorem 1].

For a constant α, $0\leq\alpha<1$, the set $S^{*}(\alpha)$ consists of all f such that

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$zf'(z)/f(z)$ is pole-free and

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha$$

in D, whereas, the set $C(\alpha)$ consists of all f such that $zf''(z)/f'(z)$ is pole-free and

$$\mathrm{Re}\frac{zf''(z)}{f'(z)} > \alpha - 1$$

in D. Each function of $S^*(\alpha)$ is called starlike of order α and that of $C(\alpha)$ is called convex of order α. Each $f \in S^*(\alpha)$ is univalent in D, and, in particular, the image $f(D)$ of D is starlike with respect to the origin 0, whereas, each $f \in C(\alpha)$ is univalent in D, and, in particular, $f(D)$ is convex. As typical examples we consider

$$\Phi(z) = \frac{z}{(1-z)^{2(1-\alpha)}}, \quad \text{and,}$$

$$\Psi(z) = \begin{cases} \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1}, & \alpha \neq \dfrac{1}{2}, \\ \log\dfrac{1}{1-z} \quad, & \alpha = \dfrac{1}{2}, \end{cases}$$

for which

$$\frac{z\Phi'(z)}{\Phi(z)} = \frac{z\Psi''(z)}{\Psi'(z)} + 1 = \frac{1+(1-2\alpha)z}{1-z}.$$

Then $\Phi \in S^*(\alpha)$ and $\Psi \in C(\alpha)$. An Alexander-type criterion can easily be proved: $f \in C(\alpha)$ if and only if $h(z) \equiv zf'(z) \in S^*(\alpha)$. Consequently, $h''(0) = 2f''(0)$. In particular, $\Phi(z) = z\Psi'(z)$ in D.

It is well known that both Φ and Ψ are extremal in the following estimate of a_2. For each $f \in S^*(\alpha)$ we have $|a_2| \leq 2(1-\alpha)$ and the equality $|a_2| = 2(1-\alpha)$ holds if and only if

$$f(z) \equiv \overline{\mu}\Phi(\mu z), \tag{1.2}$$

where μ is a unimodular constant, that is, μ is complex with $|\mu|^2 = \mu\overline{\mu} = 1$. On the other hand, for each $f \in C(\alpha)$ we have $|a_2| \leq 1-\alpha$ and the equality $|a_2| = 1-\alpha$ holds if and only if

$$f(z) \equiv \overline{\mu}\Psi(\mu z) \tag{1.3}$$

for a unimodular constant μ. The Alexander-type criterion shows that the $C(\alpha)$ case follows from the $S^*(\alpha)$ case and *vice versa.* See, for example, [Go, I, p. 138 *et seq.*] for reference of these facts, where $S^*(\alpha) = ST(\alpha)$ and $C(\alpha) = CV(\alpha)$. These familiar estimates of $|a_2|$ for $S^*(\alpha)$ and $C(\alpha)$ will be observed again in the proofs of the following Theorems 1 and 2.

We begin with the $C(\alpha)$ case.

Theorem 1 *The following two propositions hold for* $0 \le \alpha < 1$.

(I) *Suppose that* $f \in C(\alpha)$. *Then,* $\|f\| = 4(1-\alpha)$ *if and only if* f *is of the form* (1.3).

(II) *If* $f \in C(\alpha)$ *is not of the form* (1.3), *then*

$$\|f\| \le 4(1-\alpha)\frac{B+A+1}{B-A+3}, \tag{1.4}$$

which reflects personality of f, *where*

$$0 \le A = \frac{|a_2|}{1-\alpha} < 1, \quad \textit{and} \tag{1.5}$$

$$0 \le B = \frac{|(3-3\alpha)a_3 + (2\alpha-3)a_2^2|}{(1-\alpha)(1-\alpha-|a_2|)} \le 1 + A < 2, \tag{1.6}$$

so that

$$\frac{1}{3} \le \frac{B+A+1}{B-A+3} \le \frac{1+A}{2} < 1.$$

The $S^*(\alpha)$ case is not an immediate consequence of Theorem 1.

Theorem 2 *The following two propositions hold for* $0 \le \alpha < 1$.

(III) *Suppose that* $f \in S^*(\alpha)$. *Then,*

$$\|f\| = 4(1-\alpha) + 2 = 6 - 4\alpha$$

if and only if f *is of the form* (1.2).

(IV) *If* $f \in S^*(\alpha)$ *is not of the form* (1.2), *then*

$$\|f\| \le 4(1-\alpha)\frac{B'+A'+1}{B'-A'+3} + 2, \tag{1.7}$$

which reflects personality of f, *where*

$$0 \le A' = \frac{|a_2|}{2(1-\alpha)} < 1, \quad \textit{and} \tag{1.8}$$

$$0 \leq B' = \frac{|(4-4\alpha)a_3 + (2\alpha-3)a_2^2|}{2(1-\alpha)(2(1-\alpha)-|a_2|)} \leq 1 + A' < 2, \tag{1.9}$$

so that

$$\frac{1}{3} \leq \frac{B' + A' + 1}{B' - A' + 3} \leq \frac{1 + A'}{2} < 1.$$

Theorems 1 and 2 claim roughly that $\|f\| \leq 4(1-\alpha)$ for $f \in C(\alpha)$ and $\|f\| \leq 6-4\alpha$ for $f \in S^*(\alpha)$, respectively. These norm inequalities themselves are actually obtained under far general settings which will be clarified in Theorem 3 in Section 3 in terms of Gelfer functions. See Remark (ii) in Section 3.

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2. Proof of Theorem 1

The function

$$F(z) \equiv F_\alpha(z) = \frac{1 + (1-2\alpha)z}{1-z} \tag{2.1}$$

is univalent in D satisfying the identities

$$F'(0) = 2(1-\alpha), \quad F''(0) = 4(1-\alpha), \quad \text{and}$$
$$F(D) = \left\{z; \mathrm{Re}z > \alpha\right\}.$$

For $f \in C(\alpha)$ we set

$$g(z) = \frac{zf''(z)}{f'(z)} + 1, \quad z \in D.$$

Then the composed function

$$\phi \equiv F^{-1} \circ g : D \to D,$$

first g and then the inverse of F, is holomorphic with $\phi(0) = 0$ and $g = F \circ \phi$ in D; in short, g is subordinate to F. Since

$$g'(0) = 2a_2 \qquad \text{and} \qquad g''(0) = 12a_3 - 8a_2^2,$$

it follows that

$$\phi'(0) = \frac{a_2}{1-\alpha} \quad \text{and}$$
$$\phi''(0) = \frac{2}{(1-\alpha)^2}\left((3-3\alpha)a_3 + (2\alpha-3)a_2^2\right). \tag{2.2}$$

In particular, the Schwarz lemma for ϕ shows that

$$A = \frac{|a_2|}{1-\alpha} = |\phi'(0)| \le 1$$

and further $A = 1$ if and only if

$$\phi(z) \equiv \mu z \tag{2.3}$$

for a unimodular constant μ, or f is of the form (1.3). On the other hand, it follows from $g = F \circ \phi$ that

$$\frac{f''(z)}{f'(z)} = \frac{2(1-\alpha)\phi(z)}{z(1-\phi(z))} \tag{2.4}$$

in D.

For the proof of (II), we remark that ϕ is not of the form (2.3). It then follows from [Y5, p. 313, (6.8**a)] that

$$|\phi(z)| \le |z|Q(|z|), \quad z \in D, \tag{2.5}$$

where

$$Q(x) = \frac{x^2 + Bx + A}{Ax^2 + Bx + 1}, \qquad 0 \le x \le 1.$$

Here,

$$B = \frac{|\phi''(0)|}{2(1-|\phi'(0)|)}$$

which, together with (2.2), yields the expression of B in terms of a_2 and a_3. With the aid of the Schwarz-Pick inequality at 0 applied to $\chi(z) = \phi(z)/z$, where $|\chi| < 1$, we furthermore observe that

$$\frac{B}{1+|\phi'(0)|} = \frac{|\chi'(0)|}{1-|\chi(0)|^2} \le 1.$$

Hence

$$B \le 1 + A = 1 + \frac{|a_2|}{1-\alpha} < 2$$

by $|\phi'(0)| = A < 1$. Combining (2.4) and (2.5) one now has

$$\left(1 - |z|^2\right)\left|\frac{f''(z)}{f'(z)}\right| \le 2(1-\alpha)\frac{(1-|z|^2)\,Q(|z|)}{1-|z|Q(|z|)} = 2(1-\alpha)G(|z|), \tag{2.6}$$

where

$$G(x) = \frac{(x+1)(x^2+Bx+A)}{x^2+(B-A+1)x+1}, \qquad 0 \le x \le 1.$$

To prove that

$$G(x) \le G(1) = \frac{2(B+A+1)}{B-A+3}, \qquad 0 \le x \le 1, \tag{2.7}$$

we let $H(x)$ be the numerator of the derivative $G'(x)$. Then,

$$H(0) = (1-A)B + A^2 \ge 0, \qquad H'(0) = 2(B-A+1) > 0,$$

$$H''(0) = 2\left(B^2 + (1-A)B + 2(2-A)\right) > 0,$$

and, furthermore,

$$H'''(x) = 12(2x+B-A+1) > 0 \quad \text{for} \quad 0 \le x \le 1.$$

Hence $H(x) \ge 0$ or $G(x)$ is nondecreasing in $0 \le x \le 1$, which shows (2.7). Combining (2.6) with (2.7) one finally has (1.4).

Since (II) has been proved, we have only to prove that

$$\|f\| = 4(1-\alpha) \tag{2.8}$$

for f of the form (1.3). Since

$$z\frac{f''(z)}{f'(z)} + 1 = \mu z\frac{\Psi''(\mu z)}{\Psi'(\mu z)} + 1 = F(\mu z),$$

it follows that

$$\left(1 - |z|^2\right)\left|\frac{f''(z)}{f'(z)}\right| = 2(1-\alpha)\frac{1-|z|^2}{|1-\mu z|} \le 4(1-\alpha).$$

Since $(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| = 2(1-\alpha)(1+x)$ for $z = \overline{\mu}x$, $0 < x < 1$, tends to $4(1-\alpha)$ as $x \to 1-0$ we finally have (2.8).

Correction: There is a misprint in the line 3 of [Y5, p. 313]; the quotient

$$\frac{|f''(0)|}{2(1-|f'(0)|)}$$

in min $[\,\cdot\,,\,\cdot\,]$ there should be

$$\frac{|f''(0)|}{2(1-|f'(0)|^2)}.$$

3. Gelfer function

A function g holomorphic in D is called a Gelfer (or Gel'fer) function if $g(0) = 1$ and $g(z) + g(w) \neq 0$ for all $z, w \in D$, possibly, $z = w$. Let $\mathcal{G}$ be the set of all Gelfer functions. Thus, if $g(0) = 1$, then $g \in \mathcal{G}$ if and only if the image $g(D) \subset \mathbf{C}$ of D by g in the complex plane $\mathbf{C}$ and the set

$$-g(D) = \Big\{ -w; w \in g(D) \Big\}$$

are mutually disjoint: $g(D) \cap (-g(D)) = \emptyset$. For example, F_α of (2.1), $0 \le \alpha < 1$, is in $\mathcal{G}$; in particular, $\lambda \equiv F_0 \in \mathcal{G}$ plays important roles in the study of $\mathcal{G}$. Note that $F_\alpha = (1-\alpha)\lambda + \alpha$. See [Ge] and [Go, II, p. 73 *et seq.*] for reference of Gelfer functions.

Among many properties of Gelfer functions we shall make use of the following (3.1) and (3.2) for $g \in \mathcal{G}$. The first is the estimate

$$\left|\frac{g'(z)}{g(z)}\right| \le \frac{\lambda'(|z|)}{\lambda(|z|)} = \frac{2}{1-|z|^2}, \qquad z \in D; \tag{3.1}$$

see [Y3, p. 247, (G6)]. Actually, for each Bieberbach-Eilenberg function h [Go, II, p. 73] one has

$$|h'(z)| \le \frac{|1-h(z)^2|}{1-|z|^2}$$

for all $z \in D$; see [Go, II, p. 82, Exercise 49] and [Ge, p. 35, Theorem 2]. Since $h = (g-1)/(g+1)$ is a Bieberbach-Eilenberg function, one immediately has (3.1). Since each $g \in \mathcal{G}$ is zero-free, the function g^α $(\alpha \ge 0)$ which assumes 1 at 0 is single-valued and holomorphic in D. With the aid of (3.1) one can prove that

$$|g(z)^{\alpha} - 1| \leq \lambda(|z|)^{\alpha} - 1 \tag{3.2}$$

for $g \in \mathcal{G}$, $\alpha \geq 0$, and $z \in D$; see [Y3, p. 255, Lemma 5.1]. For real α, $-\infty < \alpha < +\infty$, and for $\beta \geq 0$ we let $C_G(\alpha, \beta)$ be the set of all f such that there exists a function $g \in \mathcal{G}$ depending on f with

$$\frac{zf''(z)}{f'(z)} + 1 = (1 - \alpha)g(z)^{\beta} + \alpha$$

in D. For real α and for $\beta \geq 0$ we let $S_G^*(\alpha, \beta)$ be the set of all f such that there exists a function $g \in \mathcal{G}$ depending on f with

$$\frac{zf'(z)}{f(z)} = (1 - \alpha)g(z)^{\beta} + \alpha$$

in D. An Alexander-type criterion is valid: $f \in C_G(\alpha, \beta)$ if and only if $zf'(z) \in S_G^*(\alpha, \beta)$. Furthermore,

$$C_G(1, \beta) = C_G(\alpha, 0) = S_G^*(1, \beta) = S_G^*(\alpha, 0) = \{z\}.$$

An exercise is to prove that, for $0 \leq \alpha < 1$,

$$S^*(\alpha) \subset S_G^*(\alpha, 1) \quad \text{and} \quad C(\alpha) \subset C_G(\alpha, 1).$$

For three real parameters, α, β, and γ with $\beta \geq 0$ and $\gamma \geq 0$ we let $K_G(\alpha, \beta, \gamma)$ be the set of all f such that there exist $h \in C_G(\alpha, \beta)$ and $g \in \mathcal{G}$ both depending on f and satisfying

$$\frac{f'}{h'} = g^{\gamma} \tag{3.3}$$

in D. It is obvious that $C_G(\alpha, \beta) \subset K_G(\alpha, \beta, 0)$. Hence $C(\alpha) \subset K_G(\alpha, 1, 0)$. One can further prove that

$$S^*(\alpha) \subset K_G(\alpha, 1, 1) \qquad (0 \leq \alpha < 1). \tag{3.4}$$

For $f \in S^*(\alpha)$ one can find a holomorphic $\phi : D \to D$ with $\phi(0) = 0$ such that $zf'(z)/f(z) = F_{\alpha}(\phi(z))$ in D. On the other hand, we have $h \in C(\alpha) \subset C_G(\alpha, 1)$ satisfying $f(z) = zh'(z)$ in D. Since $F_{\alpha} \circ \phi = f'/h'$ is Gelfer we now observe that $f \in K_G(\alpha, 1, 1)$. It is easy to prove that $S_G^*(0, 1) \subset K_G(0, 1, 1)$. However, it is open to prove whether or not $S_G^*(\alpha, 1) \subset K_G(\alpha, 1, 1)$ for $0 < \alpha < 1$; see Remark (i) at the end of the present Section.

For $0 \leq \alpha \leq 1$, let $\nu(\alpha) = 0$ for $0 \leq \alpha < 1$ and $\nu(1) = 4$. Then for $0 \leq \alpha \leq 1$, the function

$$\Lambda(x) \equiv \Lambda_\alpha(x) = \begin{cases} 2\alpha & , \quad x = 0, \\ \dfrac{1-x^2}{x}\left[\left(\dfrac{1+x}{1-x}\right)^\alpha - 1\right], & 0 < x < 1, \\ \nu(\alpha) & , \quad x = 1, \end{cases}$$

is continuous for $0 \leq x \leq 1$, so that

$$\max_{0 \leq x \leq 1} \Lambda(x) = M(\alpha) \geq 0$$

exists; $M(0) = 0$, $M(1) = 4$, and $M(\alpha) > 0$ for $0 < \alpha < 1$. Further property of $M(\alpha)$ will be given in Section 5.

Theorem 3 *Let* $-\infty < \alpha < +\infty$, $0 \leq \beta \leq 1$, *and* $\gamma \geq 0$. *Then for* $f \in K_G(\alpha, \beta, \gamma)$ *we have*

$$\|f\| \leq |1-\alpha| M(\beta) + 2\gamma. \tag{3.5}$$

There exists an $f \in K_G(\alpha, \beta, \gamma)$ *for which the equality holds in* (3.5).

Proof. For f satisfying (3.3) one has

$$\frac{f''}{f'} = \frac{h''}{h'} + \gamma \frac{g'}{g}. \tag{3.6}$$

On the other hand, there exists $g_o \in \mathcal{G}$ such that

$$\frac{zh''(z)}{h'(z)} + 1 = (1-\alpha) g_o(z)^\beta + \alpha$$

in D. Recalling (3.2) for the present g_o, α being replaced with β, we now have

$$\left(1 - |z|^2\right) \left|\frac{h''(z)}{h'(z)}\right| \leq |1-\alpha| \Lambda_\beta(|z|). \tag{3.7}$$

Recalling (3.1) for the present g and observing (3.1), (3.6), and (3.7) one now has (3.5).

For the equality, suppose first that $\alpha \leq 1$. Let $h \in C_G(\alpha, \beta)$ satisfy

$$\frac{zh''(z)}{h'(z)} + 1 = (1-\alpha)\lambda(z)^\beta + \alpha \tag{3.8}$$

in D, and let $f \in K_G(\alpha,\beta,\gamma)$ satisfy the identity $f'/h' = \lambda^\gamma$ in D. Then

$$\left(1-x^2\right)\frac{f''(x)}{f'(x)} = (1-\alpha)\Lambda_\beta(x) + 2\gamma \qquad (0 \le x < 1),$$

so that $\|f\| = (1-\alpha)M(\beta) + 2\gamma$. In the case $\alpha > 1$ we recall that $1/\lambda \in \mathcal{G}$. Let $h \in C_G(\alpha,\beta)$ satisfy (3.8) and let $f \in K_G(\alpha,\beta,\gamma)$, this time, satisfy the identity $f'/h' = \lambda^{-\gamma}$ in D. Then

$$\left(1-x^2\right)\left|\frac{f''(x)}{f'(x)}\right| = (\alpha-1)\Lambda_\beta(x) + 2\gamma \qquad (0 \le x < 1),$$

so that $\|f\| = (\alpha-1)M(\beta) + 2\gamma$. □

Remark (i) One might suspect that $(1-\alpha)g^\beta + \alpha \in \mathcal{G}$ for real α, for $\beta \ge 0$, and for $g \in \mathcal{G}$. This is not always true. First, for each fixed $\beta > 0$ we observe that $h \equiv (1-\alpha)\lambda^\beta + \alpha \notin \mathcal{G}$ for all $\alpha > 1$. Actually, there exists $z_o \in D$ such that

$$\lambda(z_o) = \left(\frac{\alpha+1}{\alpha-1}\right)^{1/\beta}.$$

Hence $h(z_o) + h(0) = 0$, so that $h \notin \mathcal{G}$. Next, for each fixed $\alpha \ne 1$, we have $h \equiv (1-\alpha)\lambda^\beta + \alpha \notin \mathcal{G}$ for all $\beta > 1$. Actually, in case $\alpha < 0$ or $\alpha > 1$, the set $h(D)$ contains 0. Hence $h \notin \mathcal{G}$. In case $0 \le \alpha < 1$ we set $\beta' = \min(\beta, \frac{3}{2})$. The set $h(D)$ then contains two points,

$$\pm\left(\epsilon - \alpha\tan\frac{\pi\beta'}{2}\right)i \qquad (\epsilon > 0),$$

so that $h \notin \mathcal{G}$. It is plausible that $(1-\alpha)g + \alpha \in \mathcal{G}$ if $g \in \mathcal{G}$ and $0 < \alpha < 1$, but we have no answer for its validity.

Remark (ii) It follows from Theorem 3 that $\|f\| \le 4(1-\alpha)$ for $f \in K_G(\alpha,1,0)$ and $\|f\| \le 6-4\alpha$ for $f \in K_G(\alpha,1,1)$, assuming $\alpha \le 1$ in both cases. Hence it follows from the inclusion formula $C(\alpha) \subset K_G(\alpha,1,0)$ that $\|f\| \le 4(1-\alpha)$ for $f \in C(\alpha)$, $0 \le \alpha < 1$. Furthermore, it follows from (3.4) that $\|f\| \le 6-4\alpha$ for $f \in S^*(\alpha)$, $0 \le \alpha < 1$.

4. Proof of Theorem 2

For the proof of Theorem 2 we need much more analysis.

Proof of (IV). There exists $h \in C(\alpha)$ such that $f(z) = zh'(z)$ in D.

Since f is not of the form (1.2), h is not of the form (1.3). There exists a holomorphic $\phi : D \to D$ with $\phi(0) = 0$ such that

$$g(z) \equiv F_\alpha \circ \phi(z) = \frac{zf'(z)}{f(z)} = \frac{f'(z)}{h'(z)}$$

in D. Hence, in view of

$$\frac{f''}{f'} = \frac{h''}{h'} + \frac{g'}{g}$$

and (3.1), one now has

$$\|f\| \leq \|h\| + 2. \tag{4.1}$$

We can now apply (II) of Theorem 1 to

$$h(z) = z + \frac{a_2}{2}z^2 + \frac{a_3}{3}z^3 + \cdots.$$

Then, A and B for h are A' and B' for f, respectively. Consequently, (1.4) for h, together with (4.1), shows (1.7).

We complete the proof of Theorem 2 by showing that $\|f\| = 4(1-\alpha)+2$ for f of (1.2). Since

$$\frac{f''(z)}{f'(z)} = \frac{2(1-\alpha)\mu}{1-\mu z} + \frac{g'(z)}{g(z)},$$

where $g(z) \equiv F_\alpha(\mu z)$ is in $\mathcal{G}$, it follows that $\|f\| \leq 4(1-\alpha)+2$. Furthermore, letting $x \to 1$, $0 < x < 1$, in

$$(1-x^2)\left|\frac{f''(\overline{\mu}x)}{f'(\overline{\mu}x)}\right| = 2(1-\alpha)(1+x)\left(1 + \frac{1}{1+(1-2\alpha)x}\right),$$

we have $\|f\| = 4(1-\alpha) + 2$.

5. Gelfer - close - to - convex function

Elements of $S^*_G(\alpha) \equiv S^*_G(0,\alpha)$, $C_G(\alpha) \equiv C_G(0,\alpha)$, and $K_G(\alpha,\beta) \equiv K_G(0,\alpha,\beta)$ for $\alpha \geq 0$ and $\beta \geq 0$, are called Gelfer-starlike of exponential order α, Gelfer-convex of exponential order α, and Gelfer-close-to-convex of exponential order (α,β), respectively. These sets are introduced and investigated in [Y3] and [Y4]. In particular,

$$S^*(0) \subset S^*_G(1), \qquad C(0) \subset C_G(1),$$

$$C_G(\alpha) = K_G(\alpha, 0), \quad \text{and} \quad S_G^*(\alpha) \subset K_G(\alpha, \alpha).$$

If $zf'(z)/f(z)$ is zero- and pole-free and

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi\alpha}{2} \qquad (\alpha > 0) \tag{5.1}$$

in D, then $f \in S_G^*(\alpha)$, whereas, if $zf''(z)/f'(z) + 1$ is zero- and pole-free and

$$\left|\arg \left(\frac{zf''(z)}{f'(z)} + 1\right)\right| < \frac{\pi\alpha}{2} \qquad (\alpha > 0) \tag{5.2}$$

in D, then $f \in C_G(\alpha)$.

If $f \in S_G^*(\alpha)$ for $0 \leq \alpha \leq 1$, then $f \in K_G(\alpha, \alpha)$, so that Theorem 3 shows the estimate

$$\|f\| \leq M(\alpha) + 2\alpha. \tag{5.3}$$

The extremal function is obvious. In particular, if f satisfies (5.1) in D for $0 < \alpha \leq 1$, then (5.3) holds because $f \in S_G^*(\alpha)$. T. Sugawa [S, Theorem 1.1] independently obtained (5.3) for the specified f satisfying (5.1) in D. Although his description on $M(\alpha)$ has some overlaps with ours, we here include some detailed properties of $M(\alpha)$ for the sake of the readers' convenience, for example,

$$2\alpha < M(\alpha) < 2\alpha(\alpha + 1) \ \big(< 4\alpha \big) \tag{5.4}$$

for $0 < \alpha < 1$, the priority of which belongs to Sugawa [S].

It might be difficult to express $M(\alpha)$ explicitly in terms of α for $0 < \alpha < 1$. However, we can prove that

$$M(\alpha) = \frac{4\alpha p}{(1-\alpha)p^2 + 1 + \alpha}, \tag{5.5}$$

where $p = p(\alpha)$ is the unique real root of the equation:

$$(\alpha - 1)y^{\alpha+2} - (\alpha + 1)y^{\alpha} + y^2 + 1 = 0 \quad \text{for} \quad y > 1.$$

Sugawa [S] independently obtained (5.5) and the priority is due to him. For the proof of (5.5) we set

$$\Xi(y) = \begin{cases} 2\alpha, & y = 1, \\ \dfrac{4y\,(y^{\alpha} - 1)}{y^2 - 1}, & 1 < y < +\infty. \end{cases}$$

Then

$$\Lambda(x) = \Xi(y) \qquad \text{for} \quad y = \frac{1+x}{1-x}, \quad 0 \le x < 1.$$

For $1 \le y < +\infty$, we set

$$T(y) = (\alpha - 1)y^{\alpha+2} - (\alpha+1)y^{\alpha} + y^2 + 1.$$

Then the numerator of $\Xi'(y)/4$ is $T(y)$ for $1 < y < +\infty$.

Since $T'''(y) < 0$ for $1 \le y < +\infty$, $T''(1) = 2\alpha^2$, and $T''(y) \to -\infty$ as $y \to +\infty$, there is only one $y_1 > 1$ such that $T''(y_1) = 0$. Since $T'(1) = 0$ and $T'(y) \to -\infty$ as $y \to +\infty$, there is only one $y_2 > 1$ such that $T'(y_2) = 0$. Finally, since $T(1) = 0$ and $T(y) \to -\infty$ as $y \to +\infty$, there is only one $p > 1$ such that $T(p) = 0$. Note that $1 < y_1 < y_2 < p$.

Consequently, Ξ attains its maximum for $1 \le y < +\infty$ at the point $p > 1$. By eliminating p^{α} in $M(\alpha) = \Xi(p)$ with the aid of $T(p) = 0$, one now has (5.5).

For the proof of $M(\alpha) < 2\alpha(\alpha+1)$ in (5.4) for $0 < \alpha < 1$ we observe the original form $\Xi(p) = M(\alpha)$. Set

$$V(y) = y^{\alpha+1} - ky^2 - y + k$$

for $1 \le y < +\infty$, where $k = \frac{1}{2}\alpha(\alpha+1)$ for the present α, $0 < \alpha < 1$. Since

$$V''(y) = \alpha(\alpha+1)y^{\alpha-1} - 2k \le V''(1) = 0,$$

and since $V'(1) = -\alpha^2 < 0$, it follows that $V'(y) < 0$. Hence V decreases from $V(1) = 0$ to $-\infty$ as y increases from 1 to $+\infty$. Therefore $V(y) < 0$ for $1 < y < +\infty$. In particular, $V(p) < 0$, and this shows that $M(\alpha) < 2\alpha(\alpha+1)$.

There is another set $C(\alpha,\beta)$ of functions described below. For α, β with $0 \le \alpha < 1$ and $0 \le \beta < 1$, we let $C(\alpha,\beta)$ be the set of all f such that there exist a real constant γ and a function $h \in C(\beta)$ both depending on f such that

$$\mathrm{Re}\frac{e^{i\gamma}f'}{h'} > \alpha$$

in D. We actually have

$$C(\alpha,\beta) = \bigcup_{\delta,\, real} CC_{\delta}(\alpha,\beta)$$

in the notation of [Go, II, p. 89]. Each $f \in C(\alpha,\beta)$ is called close-to-convex of order (α,β) and, in particular, each member of $K \equiv C(0,0)$ ($K = CC$ in [Go, II, p. 2]) is simply called close-to-convex. Set $H = e^{i\gamma} f'/h'$ and $\phi = F_\alpha^{-1} \circ H$. Then $f'/h' = e^{-i\gamma} F_\alpha \circ \phi$ is in $\mathcal{G}$ because $f'(0)/h'(0) = 1$. Since $h \in C(\beta) \subset C_G(\beta,1)$, it follows that $C(\alpha,\beta) \subset K_G(\beta,1,1)$. Note that the inclusion formula $S^*(\alpha) \subset C(\alpha,\alpha)$ can be proved with the aid of the Alexander-type criterion for $S^*(\alpha)$ and $C(\alpha)$, $0 \le \alpha < 1$. We again have (3.4).

It is now an exercise to prove that $\|f\| \le 4(1-\beta)+2$ for $f \in C(\alpha,\beta)$; the equality is attained by f satisfying the equation

$$f'(z) = \frac{1+(1-2\alpha)z}{(1-z)^{3-2\beta}}$$

in D.

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Department of Mathematics
Tokyo Metropolitan University
Minami-Osawa, Hachioji
Tokyo 192-0397, Japan
E-mail: yamashin@comp.metro-u.ac.jp