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Note on C^{∞} functions with the zero property

(Dedicated to the memory of Etsuo Yoshinaga (1946-1995))

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Abstract. Suppose that all of C^{∞} functions $f_{1}, \ldots, f_{k}$ have the zero property. We give a necessary and sufficient condition for their product to have the same property. This is a generalization of Bochnak's result ([1]).

Key words: zero property, theorem of zeros.

1. Introduction

The theorem of zeros for ideals of C^{∞} functions was studied by J. Bochnak and J.J. Risler in the 1970's.

Let M be a connected manifold of class C^{∞} and J an ideal in the ring $C^{\infty}(M)$ of C^{∞} functions on M. We say that J *has the zero property* if all functions in $C^{\infty}(M)$ vanishing on the zeros of J belong to J. Also, we say that $f \in C^{\infty}(M)$ *has the zero property* if the principal ideal (f) has the zero property.

J. Bochnak shows that for an ideal J in $C^{\infty}(M)$ generated by a finite number of real analytic functions, J has the zero property if and only if J is real ([1]). He conjectures that for a finitely generated ideal J in $C^{\infty}(M)$, J has the zero property if and only if J is real and closed with respect to C^{∞} topology ([1]).

J.J. Risler shows that for a finitely generated ideal J in $C^{\infty}(\mathbb{R}^{2})$, J has the zero property if and only if J is real and closed ([3]). Moreover for $f \in C^{\infty}(\mathbb{R}^{3})$, he shows that if (f) is real and closed and the zero set of f satisfies a certain condition then f has the zero property ([3]). It is still an open problem to give a complete characterization of those finitely generated ideals of C^{∞} functions which have the zero property.

We are interested in the characterization of C^{∞} functions with the zero property. In this paper we treat the C^{∞} functions that can be expressed as a product of C^{∞} functions with the zero property. Namely, suppose that

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$f_1, \dots, f_k$ have the zero property and consider the following condition.

The product $f = f_1 \cdots f_k$ has the zero property.

In the case when the functions f_i are real analytic, J. Bochnak proves the following.

Theorem (Bochnak [1]) *Let M be a connected real analytic manifold and k a positive integer. Suppose that real analytic functions $f_i : M \to \mathbb{R}$ have the zero property and that $f_i \not\equiv 0$ $(1 \leqq i \leqq k)$. Set $f = f_1 \cdots f_k$. Then the following two conditions are equivalent.*

(1) *f has the zero property.*

(2) *$\overline{G(f)} = V(f)$, where $V(f)$ denotes the zero set of f and $G(f)$ denotes the set of regular points of f in $V(f)$.*

We get rid of the condition of analyticity. Moreover, we add five conditions which are equivalent to (1). We have the following.

Theorem *Let M be a connected manifold of class C^∞ and k a positive integer. Suppose that $f_i \in C^\infty(M)$ have the zero property and that $f_i \not\equiv 0$ $(1 \leqq i \leqq k)$. Set $f = f_1 \cdots f_k$. Then the following seven conditions are equivalent.*

(1) *f has the zero property.*

(2) *(f) is real, i.e., ${g_1}^2 + \cdots + {g_p}^2 \in (f)$ implies $g_i \in (f)$ for $1 \leqq i \leqq p$.*

(3) *(f) is a radical, i.e., for some $k \in \mathbb{N}$, $g^k \in (f)$ implies $g \in (f)$.*

(4) *$\overline{G(f)} = V(f)$, where $V(f)$ denotes the zero set of f and $G(f)$ denotes the set of regular points of f in $V(f)$.*

(5) *$V(f_i) = \overline{V(f_i) \setminus V(f_j)}$ for $1 \leqq i, j \leqq k$, $i \neq j$.*

(6) *$V(f_i) = \overline{V(f_i) \setminus V(f_{j_1} \cdots f_{j_m})}$ for $1 \leqq m \leqq k-1$, $1 \leqq i, j_1, \dots, j_m \leqq k$, $i \neq j_1, \dots, j_m$.*

(7) *$V(f_i) = \overline{V(f_i) \setminus V(f_1 \cdots f_{i-1})}$ for $1 < i \leqq k$.*

The conditions (2) and (3) are algebraic conditions. The conditions (5), (6) and (7) are purely topological conditions. The condition (7) depends on the numbering of f_i, but the weakest condition among them. In fact, (5) and (6) are always equivalent but (5) and (7) are not equivalent in general without the hypothesis that f_i have the zero property. (Example: $k = 2$,

$f_1 = x^2+y^2$, $f_2 = x$). Namely, the hypothesis that f_i have the zero property is necessary for the equivalence of (5), (6) and (7). The equivalence of (1) and (2) shows that Bochnak's conjecture is affirmative in this situation.

2. Proof of Theorem

Proposition 1 *Let M be a manifold of class C^∞ and V be open in M. If $g \in C^\infty(M)$ has the zero property then $g|_V \in C^\infty(V)$ also has the zero property. Conversely, if g has the zero property locally, it has the zero property globally.*

Proof. Suppose that $\psi \in C^\infty(V)$ vanishes on $V(g|_V)$. It is known that there exists an $\eta \in C^\infty(M)$ such that $\eta\psi \in C^\infty(M)$ and $\eta(x) \neq 0$ for $x \in V$, $\eta(x) = 0$ for $x \notin V$. Then $\eta\psi$ vanishes on $V(g)$. Since g has the zero property, there exists a $Q \in C^\infty(M)$ such that $\eta\psi = gQ$. Hence $\psi = (g|_V)(Q/\eta)$ on V. The converse immediately follows from partition of unity. □

This means that the zero property is a local property. Hence it is sufficient to prove our theorem in the case of $M = \mathbb{R}^n$. First, we remember the following three propositions.

Proposition 2 *If $g \in C^\infty(\mathbb{R}^n)$ has the zero property and $g \not\equiv 0$ then $\operatorname{Int} V(g) = \emptyset$.*

Proof. Suppose that $\operatorname{Int} V(g) \neq \emptyset$. If $\overline{\operatorname{Int} V(g)} = \operatorname{Int} V(g)$ then $\operatorname{Int} V(g) = \mathbb{R}^n$, since $\mathbb{R}^n$ is connected. Then $V(g) = \mathbb{R}^n$, which contradicts $g \not\equiv 0$. Hence there exists a point $p \in \overline{\operatorname{Int} V(g)} \setminus \operatorname{Int} V(g)$. On the other hand, it is known that if $\phi \in C^\infty(\mathbb{R}^n)$ is flat on $V(\psi)$, where $\psi : \mathbb{R}^n \to \mathbb{R}$ is real analytic, then $\phi/\psi \in C^\infty(\mathbb{R}^n)$ ([2, Chapter IV]). Now, g is flat at p. Hence $g/||x-p||^2 \in C^\infty(\mathbb{R}^n)$. Obviously, $g/||x-p||^2$ vanishes on $V(g)$. Since g has the zero property, there exists $Q \in C^\infty(\mathbb{R}^n)$ such that $g/||x-p||^2 = gQ$, then $Q = 1/||x-p||^2$ off $V(g)$. For any open neighborhood $U(p)$ of p in $\mathbb{R}^n$, we have $U(p) \not\subset V(g)$. In fact, if $U(p) \subset V(g)$, then it follows $U(p) \subset \operatorname{Int} V(g)$. This contradicts the fact that $p \in \overline{\operatorname{Int} V(g)} \setminus \operatorname{Int} V(g)$. Hence there exists a sequence of points $\{p_i\}$ which converges to p such that $p_i \notin V(g)$ for all i. Then $Q(p_i) = 1/||p_i - p||^2 \to \infty$ $(i \to \infty)$. This contradicts that Q is continuous at p and proves that $\operatorname{Int} V(g) = \emptyset$. □

Proposition 3 *If $g \in C^\infty(\mathbb{R}^n)$ has the zero property and $g \not\equiv 0$ then g*

is not a zerodivisor.

Proof. If g is a zerodivisor, then there exists an $h \in C^\infty(\mathbb{R}^n)$ such that $h \not\equiv 0$ and $gh \equiv 0$. Hence $V(g) \cup V(h) = \mathbb{R}^n$. Therefore $V(g) \supset V(g) \setminus V(h) = \mathbb{R}^n \setminus V(h) \neq \emptyset$. Hence $V(g)$ has an interier point. This contradicts that Proposition 2. □

Proposition 4 *If $g \in C^\infty(\mathbb{R}^n)$ has the zero property and $g \not\equiv 0$, then $\overline{G(g)} = V(g)$.*

Proof. See [1], Proposition 1. □

(1) $\Longrightarrow$ (2). This follows immediately from the definitions of the zero property and a real ideal.

(2) $\Longrightarrow$ (3). This is trivial.

(3) $\Longrightarrow$ (4). Suppose that $\psi \in C^\infty(\mathbb{R}^n)$ vanishes on $V(f)$. Since f_i have the zero property, it follows $\psi \in (f_i)$. Hence $\psi^k \in (f)$. Since (f) is a radical, it follows $\psi \in (f)$. Therefore f has the zero property. From Proposition 3, it follows $f = f_1 \cdots f_k \not\equiv 0$. Hence from Proposition 4, we have $\overline{G(f)} = V(f)$.

(5) $\Longrightarrow$ (6). We consider the non trivial case when $V(f_i) \neq \emptyset$. Then

$$V(f_i) \setminus V(f_{j_1} \cdots f_{j_m}) = \bigcap_{p=1}^{m} \{V(f_i) \setminus V(f_{j_p})\}.$$

Since $V(f_i) \setminus V(f_{j_p})$ are open dense in $V(f_i)$, so is $V(f_i) \setminus V(f_{j_1} \cdots f_{j_m})$.

(6) $\Longrightarrow$ (7). This is trivial.

(7) $\Longrightarrow$ (1). We proceed by induction on k. In the case of $k = 1$, it is trivial. Suppose that it holds in the case of $k-1$ and that $V(f_i) = \overline{V(f_i) \setminus V(f_1 \cdots f_{i-1})}$ $(1 < i \leqq k)$ and $V(f) \subset V(\psi)$. Clearly, $V(f_i) = \overline{V(f_i) \setminus V(f_1 \cdots f_{i-1})}$ $(1 < i \leqq k-1)$ and $V(f_1 \cdots f_{k-1}) \subset V(\psi)$. From the induction hypothesis, we can write $\psi = f_1 \cdots f_{k-1} Q_{k-1}$ for some $Q_{k-1} \in C^\infty(\mathbb{R}^n)$. It follows that $V(f_k) = \overline{V(f_k) \setminus V(f_1 \cdots f_{k-1})} \subset \overline{V(\psi) \setminus V(f_1 \cdots f_{k-1})} \subset \overline{V(Q_{k-1})} = V(Q_{k-1})$. Since f_k has the zero property, there exists a $Q_k \in C^\infty(\mathbb{R}^n)$ such that $Q_{k-1} = f_k Q_k$. Therefore $\psi = f_1 \cdots f_k Q_k$.

(4) $\Longrightarrow$ (5). We proceed by induction on k. In the case of $k = 1$, it is trivial. Let us assume that Theorem is proved in the case of $k-1$.

Suppose that there exist i and j with $i \neq j$ such that $V(f_i) \supsetneqq \overline{V(f_i) \setminus V(f_j)}$. If we put $g = f_1 \cdots f_{i-1} f_{i+1} \cdots f_k$, then it follows that $V(f_i) \supsetneqq \overline{V(f_i) \setminus V(g)}$. Set $W = V(f_i) \setminus \overline{V(f_i) \setminus V(g)}$. Then W is nonempty and open in $V(f_i)$. Hence there exists an open set U in $\mathbb{R}^n$ such that $W = U \cap V(f_i)$. Clearly, $W \subset V(g)$. Since f_i has the zero property, we can write $g = f_i Q$ on U from Proposition 1. Therefore $f = f_i g = {f_i}^2 Q$ on U. Hence $W \cap G(f) = \emptyset$. It is easily seen that

$$G(f) \cap U = [\{G(f_i) \setminus V(g)\} \cup \{G(g) \setminus V(f_i)\}] \cap U \subset W \cup G(g).$$

Since $W \cap G(f) = \emptyset$ it follows $G(f) \cap U \subset G(g) \cap U$. Therefore from the hypothesis $\overline{G(f)} = V(f)$ it follows

$$V(g) \cap U \subset V(f) \cap U = \overline{G(f)} \cap U \subset \overline{G(g)} \cap U.$$

Clearly, $\overline{G(g)} \cap U \subset V(g) \cap U$. Hence $\overline{G(g)} \cap U = V(g) \cap U$. Since we now suppose that Theorem holds in the case of $k-1$, this equality shows that g has the zero property in U.

Now, suppose that $W \subset \overline{V(g) \setminus W}$. Then it follows that $V(g) \cap U = V(Q) \cap U$. Since g has the zero property in U, we can write $Q = gQ'$ on U. Hence $g = f_i Q = f_i g Q'$. Therefore $f_i Q' = 1$ on $U \setminus V(g)$. From Proposition 2, $U \setminus V(g)$ is open dense in U. Therefore $f_i Q' = 1$ on U. This contradicts that $f_i = 0$ on W and proves that $W \not\subset \overline{V(g) \setminus W}$. Therefore there exists a point $p \in W$ such that $V(g) \setminus W$ is not adherent to p. Namely, there exists an open neighborhood $V \subset U$ of p such that $V(f) \cap V = W \cap V \subset V \setminus G(f)$. Then $V(f) \cap V \neq \emptyset$ and $V(f) \cap G(f) \cap V = \emptyset$. This contradicts the assumption that $\overline{G(f)} = V(f)$. Thus we have completed.

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