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On stability of periodic solutions of the Navier-Stokes equations in unbounded domains

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Abstract. We consider the stability of periodic solutions of the Navier-Stokes equations in unbounded domains $\Omega\subset \boldsymbol{R}^{n}$ $(n\geq 3)$, which belong to $BC(\boldsymbol{R};L^{m_{1}}\cap L^{m_{2}})$ for some $n/2<m_{1}<n<m_{2}$. We show that if the periodic solution w is small in $L^{\infty}(0,\infty;L^{m_{1}}\cap L^{m_{2}})$ for some $m_{1}<n<m_{2}$ and if the initial disturbance a is small in $L^{n}(\Omega)$, then w is stable.

Key words: Navier-Stokes equations, unbounded domains, stability, periodic solutions.

1. Introduction

Let Ω be an *exterior* domain in $\boldsymbol{R}^{n}$ $(n\geq 4)$, i.e., a domain having a compact complement $\boldsymbol{R}^{n}\backslash\Omega$, the half space $\boldsymbol{R}_{+}^{n}$ $(n\geq 3)$, or the whole space $\boldsymbol{R}^{n}$ $(n\geq 3)$ and assume that the boundary $\partial\Omega$ is of class $C^{2+\mu}(0<\mu<1)$. The motion of the incompressible fluid occupying Ω is governed by the Navier-Stokes equations:

$$(N-S)\quad \begin{cases}\dfrac{\partial w}{\partial t}-\Delta w+w\cdot\nabla w+\nabla\pi=f, & \operatorname{div} w=0\quad x\in\Omega,\ t\in\boldsymbol{R},\\ w=0\quad \text{on}\ \partial\Omega, & w(x,t)\to 0\quad \text{as}\ |x|\to\infty,\end{cases}$$

where $w=w(x,t)=(w^{1}(x,t),\ldots,w^{n}(x,t))$ and $\pi=\pi(x,t)$ denote the unknown velocity vector and the unknown pressure of the fluid, respectively, while $f=f(x,t)=(f^{1}(x,t),\ldots,f^{n}(x,t))$ is the given external force. In [13], Kozono-Nakao constructed periodic strong solutions in unbounded domains for some periodic external force f. Their solutions belong to $BC(\boldsymbol{R};L^{r}\cap L^{\infty})$ for some $n/2<r<n$.

The purpose of the present paper is to show the *stability* of such solutions. If $w(x,0)$ is initially perturbed by a, then the perturbed flow $v(x,t)$ is governed by the following Navier-Stokes equations:

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$$(N-S_1)\quad \begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + v\cdot\nabla v + \nabla q = f, \quad \operatorname{div} v = 0 \quad \text{in } \Omega,\, t>0,\\ v = 0 \quad \text{on } \partial\Omega, t>0, \qquad v(x,t)\to 0 \quad \text{as } |x|\to\infty,\\ v(x,0) = w(x,0) + a(x) \quad \text{for } x\in\Omega. \end{cases}$$

We show that if the periodic solution w is small in $L^\infty(0,\infty; L^{m_1}\cap L^{m_2})$ for some $m_1 < n < m_2$ and if the initial disturbance a is small in $L^n(\Omega)$, then there is a unique *global strong solution* v of $(N-S_1)$ such that the integrals

$$\int_\Omega |v(x,t) - w(x,t)|^r dx \quad \text{for } n\le r<\infty$$

converge to zero with *definite decay rates* as $t\to\infty$.

Let w and v be solutions of $(N-S)$ and $(N-S_1)$, respectively. Then the pair of functions $u\equiv v-w$, $p\equiv q-\pi$ satisfies

$$(N-S')\quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + w\cdot\nabla u + u\cdot\nabla w + u\cdot\nabla u + \nabla p = 0,\\ \qquad\qquad\qquad\qquad \operatorname{div} u = 0 \quad \text{in } \Omega, t>0,\\ u=0 \quad \text{on } \partial\Omega, t>0, \quad u(x,t)\to 0 \quad \text{as } |x|\to\infty, \quad u|_{t=0} = a. \end{cases}$$

Thus our problem on the stability for $(N-S)$ can now be reduced to investigation into existence of global strong solutions to $(N-S')$ and their asymptotic behavior. If $w\equiv 0$, our problem coincides with the initial boundary value problem for the usual nonstationary Navier-Stokes equations. Kato [11] constructed a global strong solution of $(N-S)$ having a decay property by the iteration method. His method needs the global estimate $\sup_{0<t<\infty} t^{1/2}\|\nabla u(t)\|_n < \infty$. On the other hand, the periodic solution w prevents us from getting this estimate. Hence we introduce a notion of *mild* solution as Kozono-Ogawa [15]. We first construct a global mild solution having a decay property. Then we shall show that this mild solution can be identified locally in time with the strong solution. Since the time interval of existence of strong solutions is characterized by the L^{2n}-norm of the initial data, we may conclude that our mild solution is actually a strong one.

In Section 2, we shall state the main results. Section 3 is devoted to preparing some fundamental lemmas. Finally, we shall prove the main results in Sections 4 and 5.

2. Results

Throughout this paper we impose the following assumption on the domain.

Assumption 2.1 (Case 1) *Ω is the half-space $\boldsymbol{R}^n_+$ or the whole space $\boldsymbol{R}^n$, where $n \geq 3$.*

(Case 2) *Ω is an exterior domain in $\boldsymbol{R}^n$ with $C^{2+\mu}(\mu > 0)$-boundary $\partial\Omega$, where $n \geq 4$.*

Before stating our results, we introduce some notations and function spaces. Let $C^\infty_{0,\sigma}$ denote the set of all C^∞-real vector functions $\phi = (\phi^1, \ldots, \phi^n)$ with compact support in Ω such that $\operatorname{div}\phi = 0$. L^r_σ is the closure of $C^\infty_{0,\sigma}$ with respect to the L^r-norm $\| \quad \|_r$; $(\cdot,\cdot)$ denotes the L^2-inner product and the duality pairing between L^r and $L^{r'}$, where $1/r + 1/r' = 1$. $\| \quad \|_{r,\infty;T}$ and $\| \quad \|_{r,\infty}$ denote the $L^\infty(0,T;L^r)$ and $L^\infty(0,\infty;L^r)$-norms, respectively. $H^{1,r}_{0,\sigma}$ denotes the closure of $C^\infty_{0,\sigma}$ with respect to the norm

$$\|\phi\|_{H^{1,r}} = \|\phi\|_r + \|\nabla\phi\|_r,$$

where $\nabla\phi = (\partial\phi^i/\partial x_j; i,j = 1,\ldots,n)$. When X is a Banach space, its norm is denoted by $\|\cdot\|_X$. Then $C^m([t_1,t_2);X)$ is the usual Banach space, where $m = 0,1,2,\ldots$ and t_1 and t_2 are real numbers such that $t_1 < t_2$. $BC^m([t_1,t_2);X)$ is the set of all functions $u \in C^m([t_1,t_2);X)$ such that $\sup_{t_1<t<t_2} \|\frac{d^m u(t)}{dt^m}\|_X < \infty$. In this paper, we denote by C various constants. In particular, $C = C(*,\ldots,*)$ denotes the constant depending only on the quantities appearing in the parentheses.

Let us recall the Helmholtz decomposition:

$$L^r = L^r_\sigma \oplus G_r (\text{direct sum}), \quad 1 < r < \infty,$$

where $G_r = \{\nabla p \in L^r; p \in L^r_{loc}(\overline{\Omega})\}$. For the proof, see Fujiwara-Morimoto [6], Miyakawa [18], Simader-Sohr [19] and Borchers-Miyakawa [1]. P_r denotes the projection operator from L^r onto L^r_σ along G_r. The Stokes operator A_r on L^r_σ is then defined by $A_r = -P_r\Delta$ with domain $D(A_r) = \{u \in W^{2,r}(\Omega); u|_{\partial\Omega} = 0\} \cap L^r_\sigma$. It is known that

$$(L^r_\sigma)^* (\text{the dual space of } L^r_\sigma) = L^{r'}_\sigma,$$

$$A^*_r (\text{the adjoint operator of } A_r) = A_{r'},$$

where $1/r + 1/r' = 1$. It is shown by Giga [7], Giga-Sohr [9] and Borchers-

Miyakawa [1] that for every $\frac{\pi}{2} < \omega < \pi$ and every $1 < r < \infty$, the resolvent set $\rho(-A_r)$ of $-A_r$ contains the sector $\Sigma_\omega \equiv \{\lambda \in \boldsymbol{C}; |\arg\lambda| < \omega\}$ and there is a constant $M_{r,\omega}$ depending only on r and ω such that

$$\|(A_r+\lambda)^{-1}\|_{\mathbf{B}(\mathbf{L}^r_\sigma,\mathbf{L}^r_\sigma)} \le M_{r,\omega}|\lambda|^{-1} \tag{2.1}$$

holds for all $\lambda \in \Sigma_\omega$, where $\mathbf{B}(\mathbf{X},\mathbf{Y})$ is the set of bounded operators from X to Y. Therefore $-A_r$ generates a uniformly bounded holomorphic semigroup $\{e^{-tA_r}; t \ge 0\}$ of class C_0 in L^r_σ. Moreover, there holds

$$\|u\|_{W^{2,r}} \le C\|(1+A_r)u\|_r \quad \text{for all} \quad u \in D(A_r) \tag{2.2}$$

with a constant $C = C(r)$.

Since $P_r u = P_q u$ for all $u \in L^r \cap L^q (1 < r, q < \infty)$ and since $A_r u = A_q u$ for all $u \in D(A_r) \cap D(A_q)$, for simplicity, we shall abbreviate $P_r u$, $P_q u$ as Pu for $u \in L^r \cap L^q$ and $A_r u$, $A_q u$ as Au for $u \in D(A_r) \cap D(A_q)$, respectively.

Our definition of strong and mild solutions of $(N-S)$ and $(N-S')$ are as follows:

Definition 1 Let $a \in L^n_\sigma$. A measurable function u on $\Omega \times (0,T)$ is called a strong solution of $(N-S')$ on $(0,T)$ if
(i) $u \in C([0,T); L^n_\sigma) \cap C^1((0,T); L^n_\sigma)$;
(ii) $u(t) \in D(A_n)$ for $t \in (0,T)$ and $A_n u \in C((0,T); L^n_\sigma)$;
(iii) u satisfies

$$\frac{\partial}{\partial t}u + Au + P(u\cdot\nabla u) + P(u\cdot\nabla w) + P(w\cdot\nabla u) = 0$$
$$\text{in} \quad L^n_\sigma \quad \text{on} \quad (0,T) \quad \text{and} \quad u(0) = a.$$

Similarly as above, for an external force $f \in C((0,T); L^n_\sigma)$ we define the strong solution of $(N-S)$ on $(0,T)$, so we do not write its definition here. Next we define a mild solution of $(N-S')$ as Kozono-Ogawa [15].

Definition 2 Let $a \in L^n_\sigma$ and let $w \in L^\infty(0,T; L^m_\sigma)$ for some $m > n$. Suppose that $n < r < \infty$. A measurable function u on $\Omega \times (0,T)$ is called a mild solution of $(N-S')$ in the class $S_r(0,T)$ if
(i) $u \in BC([0,T); L^n_\sigma)$ and $t^{(1-n/r)/2}u(\cdot) \in BC([0,T); L^r_\sigma)$;
(ii) $\lim_{t\to+0} t^{(1-n/r)/2}\|u(t)\|_r = 0$;

(iii) u satisfies

$$\begin{aligned}(u(t), \phi) &= (e^{-tA}a, \phi) + \int_0^t (u(s) \cdot \nabla e^{-(t-s)A}\phi, u(s))ds \\ &\quad + \int_0^t (w(s) \cdot \nabla e^{-(t-s)A}\phi, u(s))ds \\ &\quad + \int_0^t (u(s) \cdot \nabla e^{-(t-s)A}\phi, w(s))ds\end{aligned}$$

for all $\phi \in C_{0,\sigma}^{\infty}$ and all $0 < t < T$.

As Kozono-Ogawa [15], we can show that if u is a mild solution in the class $S_r(0,T)$, then the integrals on the right-hand side of (iii) in Definition 2 is well-defined and that (iii) holds for all $\phi \in L_{\sigma}^{n'}$ $(1/n' = 1 - 1/n)$.

Remark 2.1. By the similar argument given by Brezis [4] and Kato [12], we see that the condition (ii) follows from (i) and (iii), so (ii) is not necessary. The proof of this fact, however, is not brief. Hence we impose the condition (ii) for simplicity.

Our results are stated as follows.

Theorem 2.1 *Let $a \in L_{\sigma}^n$ and let $w \in L^{\infty}(0,T; L_{\sigma}^{m_1} \cap L_{\sigma}^{m_2})$ for some m_1, m_2 with $2n/(2n-3) \le m_1 < n < m_2$. There are positive numbers $\lambda_1(m_1, m_2, n)$, $\lambda_2(n)$ such that if*

$$\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty} < 1/\lambda_1, \tag{2.3}$$

$$\|a\|_n < \lambda_2(1 - \lambda_1(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}))^2, \tag{2.4}$$

then there is a unique mild solution u of $(N-S')$ in the class $S_{2n}(0,\infty)$ with the decay property

$$\|u(t)\|_l \le Ct^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{l})} \quad \textit{for} \;\; n \le l \le 2n, \tag{2.5}$$

$$\lim_{t\to\infty} \|u(t)\|_n = 0. \tag{2.6}$$

Theorem 2.2 *Let (2.3) and (2.4) hold. For every $2n < r < \infty$, there are positive numbers $\eta_1(m_1, m_2, n, r)$, $\eta_2(n, r)$ such that if*

$$\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty} < 1/\eta_1, \tag{2.7}$$

$$\|a\|_n < \eta_2(1 - \eta_1(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}))^2, \tag{2.8}$$

then the mild solution u given in Theorem 2.1 has the additional decay property

$$\|u(t)\|_l \leq Ct^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{l})} \quad \text{for} \ \ 2n \leq l \leq r. \tag{2.9}$$

Theorem 2.3 *In addition to the hypotheses of Theorem 2.1, assume moreover that w is a strong solution of $(N-S)$ on $(0,\infty)$ for some external force $f \in C((0,\infty);L^n_\sigma)$. Then the mild solution given in Theorem 2.1 is a strong solution of $(N-S')$ on $(0,\infty)$.*

Remark 2.2. When $\Omega = \boldsymbol{R}^n, \boldsymbol{R}^n_+$ with $n \geq 3$ and when Ω is an exterior domain in $\boldsymbol{R}^n$ with $n \geq 4$, for small periodic force f, Kozono-Nakao [13] constructed the strong periodic solution w with (2.3); their solution w belongs to $BC(\boldsymbol{R};L^r)$ for $2 < r < n$ with $\nabla w \in BC(\boldsymbol{R};L^q)$ for $n/2 < q < n$. If f is sufficiently small, then $\|w\|_{L^\infty(0,\infty;L^r)} + \|\nabla w\|_{L^\infty(0,\infty;L^q)}$ is also sufficiently small. By the Sobolev inequality, $w \in BC(\boldsymbol{R};L^p)$ for all $p \in [r, nq/(n-q)]$. Since $nq/(n-q) > n$, this implies (2.3). Maremonti [16], [17] also showed the existence of the periodic solutions in the three-dimensional whole space $\boldsymbol{R}^3$ and the half space $\boldsymbol{R}^3_+$. It seems to be an open question whether there exists a periodic solution in *three-dimensional exterior* domain.

3. Preliminaries

Let us first recall the following $L^q - L^r$-estimate for the semigroup $\{e^{-tA}\}_{t\geq 0}$.

Lemma 3.1 (Kato [11], Ukai [21], Giga-Sohr [9], Iwashita [10], Borchers-Miyakawa [1], [2])

$$\|e^{-tA}a\|_r \leq M_{q,r}t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|a\|_q, \quad 1 < q \leq r < \infty, \tag{3.1}$$

$$\|\nabla e^{-tA}a\|_r \leq M'_{q,r}t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|a\|_q, \quad 1 < q \leq r < \infty \ \ \textit{in} \text{ (Case 1)}, \tag{3.2}$$

$$\|\nabla e^{-tA}a\|_r \leq M'_{q,r}t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|a\|_q, \quad 1 < q \leq r \leq n \ \ \textit{in} \text{ (Case 2)} \tag{3.3}$$

for all $a \in L^q_\sigma$ and all $t > 0$, where $M_{q,r}$, $M'_{q,r}$ are constants depending only on q, r.

Concerning $r = \infty$, we have

Lemma 3.2 (Chen[5], Borchers-Miyakawa [1], [3])

$$\|e^{-tA}a\|_\infty \le M_{q,\infty} t^{-\frac{n}{2q}} \|a\|_q, \quad 1 < q \le 2n, \tag{3.4}$$

for all $a \in L^q_\sigma$ *and all* $t > 0$, *with the constant* $M_{q,\infty}$ *depending only on* q.

By Lemma 3.1, we have the following lemmas.

Lemma 3.3 *Let* $0 < T \le \infty$. (i) *Suppose that* u *is a measurable function with* $t^{\frac{1-\alpha}{2}} u(\cdot) \in L^\infty(0,T; L^{n/\alpha}_\sigma)$ *for some* $0 < \alpha < 1$ *and that* $w \in L^\infty(0,T; L^{m_1}_\sigma \cap L^{m_2}_\sigma)$ *for some* m_1, m_2 *with* $\frac{n}{n-\alpha-1} \le m_1 < n < m_2$. *Then there holds*

$$\begin{aligned} &\left|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u(s))ds\right| + \left|\int_0^t (u(s)\cdot\nabla e^{-(t-s)A}\phi, w(s))ds\right| \\ &\quad \le C(\alpha, m_1, m_2, n)(\|w\|_{m_1,\infty;T} + \|w\|_{m_2,\infty;T}) \\ &\qquad \cdot \left(\sup_{0<s<t} s^{\frac{1-\alpha}{2}} \|u(s)\|_{n/\alpha}\right) t^{\frac{\alpha-1}{2}} \|\phi\|_{\frac{n}{n-\alpha}} \end{aligned} \tag{3.5}$$

for all $0 < t < T$.

(ii) *In* (Case 1), *let* $0 < \beta < n$, *and let* $1 < m_1' < n < m_2$.

In (Case 2), *let* $0 < \beta < n-2$, *and let* $\frac{n}{n-\beta-1} \le m_1' < n < m_2$. *Suppose that* $u \in L^\infty(0,T; L^{n/\beta}_\sigma)$ *and that* $w \in L^\infty(0,T; L^{m_1'}_\sigma \cap L^{m_2}_\sigma)$. *Then there holds*

$$\begin{aligned} &\left|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u(s))ds\right| + \left|\int_0^t (u(s)\cdot\nabla e^{-(t-s)A}\phi, w(s))ds\right| \\ &\quad \le C(\beta, m_1', m_2, n)(\|w\|_{m_1',\infty;T} + \|w\|_{m_2,\infty;T})\|u\|_{\frac{n}{\beta},\infty;T}\|\phi\|_{\frac{n}{n-\beta}} \end{aligned} \tag{3.6}$$

for all $0 < t < T$.

Proof. We here prove only (i); statement (ii) is proved similarly. Let $1/\delta_i = 1 - \alpha/n - 1/m_i$ $(i = 1, 2)$. Since $\frac{n}{n-\alpha-1} \le m_1 < n < m_2$ implies $\delta_i \le n$, we have by Lemma 3.1 and the Hölder inequality that for $t > 2$

$$\begin{aligned} &\left|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u(s))ds\right| \\ &\quad \le \int_0^{t-1} |(w(s)\cdot\nabla e^{-(t-s)A}\phi, u(s))|ds \end{aligned}$$

$$
\begin{aligned}
&+\int_{t-1}^{t}|(w(s)\cdot\nabla e^{-(t-s)A}\phi,u(s))|ds\\
&\le\Big\{M'_{\frac{n}{n-\alpha},\delta_1}\Big(\sup_{0<s<t}s^{\frac{1-\alpha}{2}}\|u(s)\|_{\frac{n}{\alpha}}\Big)\|w\|_{m_1,\infty;T}\\
&\qquad\cdot\int_0^{t-1}(t-s)^{-\frac{n}{2m_1}-\frac12}s^{\frac{\alpha-1}{2}}ds\\
&+M'_{\frac{n}{n-\alpha},\delta_2}\Big(\sup_{0<s<t}s^{\frac{1-\alpha}{2}}\|u(s)\|_{\frac{n}{\alpha}}\Big)\|w\|_{m_2,\infty;T}\\
&\qquad\cdot\int_{t-1}^{t}(t-s)^{-\frac{n}{2m_2}-\frac12}s^{\frac{\alpha-1}{2}}ds\Big\}\|\phi\|_{\frac{n}{n-\alpha}}. \qquad (3.7)
\end{aligned}
$$

A direct calculation shows

$$
\begin{aligned}
&\int_0^{\frac{t}{2}}(t-s)^{-\frac{n}{2m_1}-\frac12}s^{\frac{\alpha-1}{2}}ds+\int_{\frac{t}{2}}^{t-1}(t-s)^{-\frac{n}{2m_1}-\frac12}s^{\frac{\alpha-1}{2}}ds\\
&\le\left[\frac{2^{\frac{n}{2m_1}-\frac{\alpha}{2}+1}}{\alpha+1}t^{\frac12(1-\frac{n}{m_1})}+\frac{2^{\frac{3-\alpha}{2}}m_1}{n-m_1}\left\{1-\left(\frac{t}{2}\right)^{\frac12(1-\frac{n}{m_1})}\right\}\right]t^{\frac{\alpha-1}{2}}\\
&\le 2^{\frac{3-\alpha}{2}}\left(\frac{1}{\alpha+1}+\frac{m_1}{n-m_1}\right)t^{\frac{\alpha-1}{2}}\quad\text{for }\ t>2, \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{t-1}^{t}(t-s)^{-\frac{n}{2m_2}-\frac12}s^{\frac{\alpha-1}{2}}ds &\le (t-1)^{\frac{\alpha-1}{2}}\frac{2m_2}{m_2-n}\\
&\le 2^{\frac{3-\alpha}{2}}\frac{m_2}{m_2-n}t^{\frac{\alpha-1}{2}}\quad (t>2), \qquad (3.9)
\end{aligned}
$$

since $\frac{1}{t-1}<\frac{2}{t}$ for $t>2$. By (3.7), (3.8) and (3.9) we have

$$
\begin{aligned}
&\left|\int_0^t(w(s)\cdot\nabla e^{-(t-s)A}\phi,u(s))ds\right|\\
&\le C_0(\alpha,m_1,m_2,n)\left(\|w\|_{m_1,\infty;T}+\|w\|_{m_2,\infty;T}\right)\\
&\qquad\cdot\left(\sup_{0<s<t}s^{\frac{1-\alpha}{2}}\|u(s)\|_{\frac{n}{\alpha}}\right)t^{\frac{\alpha-1}{2}}\|\phi\|_{\frac{n}{n-\alpha}} \qquad (3.10)
\end{aligned}
$$

for $t>2$, where

$$
C_0(\alpha,m_1,m_2,n)=\Big\{M'_{\frac{n}{n-\alpha},\delta_1}\Big(\frac{1}{\alpha+1}+\frac{m_1}{n-m_1}\Big)+M'_{\frac{n}{n-\alpha},\delta_2}\frac{m_2}{m_2-n}\Big\}2^{\frac{3-\alpha}{2}}.
$$

For $0 < t \le 2$, we have by Lemma 3.1 that

$$\begin{aligned}
&\left|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u(s))ds\right| \\
&\le M'_{\frac{n}{n-\alpha},\delta_2}\|w\|_{m_2,\infty;T}\left(\sup_{0<s<t} s^{\frac{1-\alpha}{2}}\|u(s)\|_{\frac{n}{\alpha}}\right) \\
&\quad\cdot B(\tfrac{\alpha+1}{2},\tfrac{1}{2}(1-\tfrac{n}{m_2}))t^{\frac{\alpha}{2}-\frac{n}{2m_2}}\|\phi\|_{\frac{n}{n-\alpha}} \\
&\le 2^{\frac{1}{2}(1-\frac{n}{m_2})}M'_{\frac{n}{n-\alpha},\delta_2}\|w\|_{m_2,\infty;T}\left(\sup_{0<s<t} s^{\frac{1-\alpha}{2}}\|u(s)\|_{\frac{n}{\alpha}}\right) \\
&\quad\cdot B(\tfrac{\alpha+1}{2},\tfrac{1}{2}(1-\tfrac{n}{m_2}))t^{\frac{\alpha-1}{2}}\|\phi\|_{\frac{n}{n-\alpha}}.
\end{aligned} \tag{3.11}$$

From (3.10) and (3.11) we obtain for all $0 < t < T$

$$\begin{aligned}
&\left|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u(s))ds\right| \\
&\quad\le C(\alpha, m_1, m_2, n)\left(\|w\|_{m_1,\infty;T} + \|w\|_{m_2,\infty;T}\right) \\
&\qquad\cdot\left(\sup_{0<s<t} s^{\frac{1-\alpha}{2}}\|u(s)\|_{\frac{n}{\alpha}}\right)t^{\frac{\alpha-1}{2}}\|\phi\|_{\frac{n}{n-\alpha}},
\end{aligned} \tag{3.12}$$

where
$C(\alpha, m_1, m_2, n) = C_0(\alpha, m_1, m_2, n) + 2^{\frac{1}{2}(1-\frac{n}{m_2})}M'_{\frac{n}{n-\alpha},\delta_2}B(\frac{\alpha+1}{2},\frac{1}{2}(1-\frac{n}{m_2}))$.
The second term on the left hand side of (3.5) can be handled in the same way as above and we get the conclusion. □

Lemma 3.4 *Let $0 < T \le \infty$ and let v and w be measurable functions with $w \in L^\infty(0,T; L^{n/\gamma}_\sigma)$ and $t^{\frac{1-\alpha}{2}}v(\cdot) \in L^\infty(0,T; L^{n/\alpha}_\sigma)$ for some $0 < \gamma$, $\alpha < 1$. Then for $\delta \in [\alpha, \alpha+\gamma]$ and $0 < \beta < \frac{1}{2} + \frac{\delta}{2} - \frac{\alpha}{2} - \frac{\gamma}{2} (> 0)$,*

$$\begin{aligned}
F_{w,v}(t,h) \equiv{}& \left|\int_0^{t+h}(w(s)\cdot\nabla e^{-(t+h-s)A}\phi, v(s))ds\right. \\
&\left.\quad - \int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, v(s))ds\right| \\
\le{}& C\left(\sup_{0<s<T}\|w(s)\|_{n/\gamma}\right)\left(\sup_{0<s<T} s^{\frac{1-\alpha}{2}}\|v(s)\|_{n/\alpha}\right) \\
&\times\left(h^\beta t^{\frac{\delta}{2}-\frac{\gamma}{2}-\beta} + h^{\frac{1}{2}+\frac{\delta}{2}-\frac{\alpha}{2}-\frac{\gamma}{2}}t^{\frac{-1+\alpha}{2}}\right)\|\phi\|_{\frac{n}{n-\delta}} \\
F_{v,w}(t,h) \equiv{}& \left|\int_0^{t+h}(v(s)\cdot\nabla e^{-(t+h-s)A}\phi, w(s))ds\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \int_0^t (v(s)\cdot\nabla e^{-(t-s)A}\phi, w(s))ds\Big| \\
&\le C\Big(\sup_{0<s<T}\|w(s)\|_{n/\gamma}\Big)\Big(\sup_{0<s<T} s^{\frac{1-\alpha}{2}}\|v(s)\|_{n/\alpha}\Big) \\
&\qquad \times(h^{\beta}t^{\frac{\delta}{2}-\frac{\gamma}{2}-\beta}+h^{\frac{1}{2}+\frac{\delta}{2}-\frac{\alpha}{2}-\frac{\gamma}{2}}t^{\frac{-1+\alpha}{2}})\|\phi\|_{\frac{n}{n-\delta}},
\end{aligned}
$$

for all $h>0$ *and* $0<t<t+h<T$, *where* C *is independent of* w, v, ϕ *and* T. *For* $\delta\in[\alpha,2\alpha]$ *and* $0<\beta<\frac{1}{2}-\alpha+\frac{\delta}{2}(>0)$,

$$
\begin{aligned}
F_{v,v}(t,h) &\equiv \Big|\int_0^{t+h}(v(s)\cdot\nabla e^{-(t+h-s)A}\phi, v(s))ds \\
&\qquad - \int_0^t (v(s)\cdot\nabla e^{-(t-s)A}\phi, v(s))ds\Big| \\
&\le C\Big(\sup_{0<s<T} s^{\frac{1-\alpha}{2}}\|v(s)\|_{n/\alpha}\Big)^2 \\
&\qquad \times(h^{\beta}t^{\frac{\delta}{2}-\frac{1}{2}-\beta}+h^{\frac{1}{2}-\alpha+\frac{\delta}{2}}t^{-1+\alpha})\|\phi\|_{\frac{n}{n-\delta}},
\end{aligned}
$$

for all $h>0$ *and* $0<t<t+h<T$.

Proof. Concerning the inequality for $F_{w,v}(t,h)$, we have

$$
\begin{aligned}
F_{w,v}(t,h) &\le \Big|\int_t^{t+h}(w(s)\cdot\nabla e^{-(t+h-s)A}\phi, v(s))ds\Big| \\
&\qquad + \Big|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}(e^{-hA}-1)\phi, v(s))ds\Big| \\
&\equiv I_1+I_2.
\end{aligned}
$$

Since $0>-\frac{1}{2}+\frac{\delta}{2}-\frac{\alpha}{2}-\frac{\gamma}{2}>-1$ and $\frac{n}{n-\gamma-\alpha}\ge\frac{n}{n-\delta}$, from Lemma 3.1 we obtain

$$
\begin{aligned}
I_1 &\le \int_t^{t+h}\|w(s)\|_{n/\gamma}\|v(s)\|_{n/\alpha}\|\nabla e^{-(t+h-s)A}\phi\|_{\frac{n}{n-\gamma-\alpha}}ds \\
&\le C\Big(\sup_{0<s<T}\|w(s)\|_{n/\gamma}\Big)\Big(\sup_{0<s<T} s^{\frac{1-\alpha}{2}}\|v(s)\|_{n/\alpha}\Big) \\
&\qquad \times h^{\frac{1}{2}+\frac{\delta}{2}-\frac{\alpha}{2}-\frac{\gamma}{2}}t^{\frac{-1+\alpha}{2}}\|\phi\|_{\frac{n}{n-\delta}}.
\end{aligned}
$$

Since $0>-\frac{1}{2}+\frac{\delta}{2}-\frac{\alpha}{2}-\frac{\gamma}{2}-\beta>-1$ and $\|(e^{-hA}-1)\psi\|_p\le Ch^{\beta}\|A^{\beta}\psi\|_p$

$(1 < p < \infty)$, we obtain

$$
\begin{aligned}
I_2 &\le \int_0^t \|w(s)\|_{n/\gamma}\|v(s)\|_{n/\alpha}\|\nabla e^{-(t-s)A}(e^{-hA}-1)\phi\|_{\frac{n}{n-\gamma-\alpha}} ds \\
&\le C\int_0^t \|w(s)\|_{n/\gamma}\|v(s)\|_{n/\alpha}\Big(\frac{t-s}{2}\Big)^{\frac{\delta}{2}-\frac{\alpha}{2}-\frac{\gamma}{2}-\frac{1}{2}} \\
&\qquad \times \|(e^{-hA}-1)e^{-\frac{t-s}{2}A}\phi\|_{\frac{n}{n-\delta}} ds \\
&\le C\Big(\sup_{0<s<T}\|w(s)\|_{n/\gamma}\Big)\Big(\sup_{0<s<T} s^{\frac{1-\alpha}{2}}\|v(s)\|_{n/\alpha}\Big) \\
&\qquad \times h^\beta \int_0^t s^{\frac{-1+\alpha}{2}}(t-s)^{\frac{\delta}{2}-\frac{\alpha}{2}-\frac{\gamma}{2}-\frac{1}{2}-\beta} ds \|\phi\|_{\frac{n}{n-\delta}} \\
&\le C\Big(\sup_{0<s<T}\|w(s)\|_{n/\gamma}\Big)\Big(\sup_{0<s<T} s^{\frac{1-\alpha}{2}}\|v(s)\|_{n/\alpha}\Big) h^\beta t^{\frac{\delta}{2}-\frac{\gamma}{2}-\beta}\|\phi\|_{\frac{n}{n-\delta}}.
\end{aligned}
$$

Hence we get the inequality for $F_{w,v}$. The assertion on $F_{v,w}$ and $F_{v,v}$ can be handled similarly, so we may omit its proof. □

Concerning the mild solution, we have

Lemma 3.5 *Let $h \in (0,T)$ and let u be a mild solution of $(N-S')$ in the class $S_r(0,T)$, $(n < r < \infty)$. Then $u(\cdot + h)$ is also a mild solution of $(N-S')$ in the class $S_r(0,T-h)$ with initial data $u(h)$.*

Proof. Since u is a mild solution, there holds

$$
\begin{aligned}
(u(h),\phi) &= (e^{-hA}a,\phi) + \int_0^h (u(s)\cdot\nabla e^{-(h-s)A}\phi, u(s))ds \\
&\quad + \int_0^h (w(s)\cdot\nabla e^{-(h-s)A}\phi, u(s))ds \\
&\quad + \int_0^h (u(s)\cdot\nabla e^{-(h-s)A}\phi, w(s))ds.
\end{aligned}
$$

for all $\phi \in L^{n'}_\sigma$. Substituting $e^{-tA}\phi$ into ϕ, we have

$$
\begin{aligned}
(e^{-tA}u(h),\phi) &= (e^{-(t+h)A}a,\phi) + \int_0^h (u(s)\cdot\nabla e^{-(t+h-s)A}\phi, u(s))ds \\
&\quad + \int_0^h (w(s)\cdot\nabla e^{-(t+h-s)A}\phi, u(s))ds \\
&\quad + \int_0^h (u(s)\cdot\nabla e^{-(t+h-s)A}\phi, w(s))ds.
\end{aligned}
$$

Hence we see that

$$\begin{aligned}(u(t+h),\phi)-(e^{-tA}u(h),\phi)\\ &=\int_0^t(u(s+h)\cdot\nabla e^{-(t-s)A}\phi,u(s+h))ds\\ &+\int_0^t(w(s+h)\cdot\nabla e^{-(t-s)A}\phi,u(s+h))ds\\ &+\int_0^t(u(s+h)\cdot\nabla e^{-(t-s)A}\phi,w(s+h))ds\end{aligned}$$

for all $\phi\in C_{0,\sigma}^\infty$. This completes the proof of Lemma 3.5. □

Concerning the uniqueness of mild solutions, we have

Lemma 3.6 (Uniqueness) *Let $a\in L_\sigma^n$ and let $w\in L^\infty(0,T;L_\sigma^m)$ for some $m>n$. Suppose that $n<r<\infty$. Then the mild solution of $(N-S')$ is unique within the class $S_r(0,T)$.*

Proof. Following [15] we give the proof. Let u and v be mild solutions of $(N-S')$ in $S_r(0,T)$ with the same initial data a. Set

$$\begin{aligned}D(t)&\equiv\sup_{0<s\le t}\|u(s)-v(s)\|_n\\ K(t)&\equiv\sup_{0<s\le t}s^{(1-\beta)/2}\|u(s)\|_{n/\beta}+\sup_{0<s\le t}s^{(1-\beta)/2}\|v(s)\|_{n/\beta},\end{aligned}$$

where $\beta=n/r$. Similarly to the proof of Lemma 3.3, we have by (iii) in Definition 2 and Lemma 3.1 that

$$|(u(t)-v(t),\phi)|\le\left\{C_*K(t)+B_*t^{\frac12(1-\frac nm)}\right\}D(t)\|\phi\|_{\frac{n}{n-1}},$$

for all $\phi\in C_{0,\sigma}^\infty$ and all $0<t<T$, where $C_*=M'_{\frac{n}{n-1},\frac{n}{n-1-\beta}}B(\frac{1-\beta}{2},\frac{1+\beta}{2})$ and $B_*=\frac{4m}{m-n}M'_{\frac{n}{n-1},\delta}\|w\|_{m,\infty;T}$, $(1/\delta=1-1/m-1/n)$. By duality we have

$$D(t)\le(C_*K(t)+B_*t^{\frac12(1-\frac nm)})D(t),\quad 0<t<T.$$

Since $\lim_{t\to+0}K(t)=0$, we can choose small positive number t_0 such that $D(t_0)\le\frac12D(t_0)$, which implies

$$u(t)\equiv v(t)\quad\text{for}\quad 0\le t\le t_0.$$

Next we show that $u(t) \equiv v(t)$ for $t_0 \le t < T$, by Lemma 3.5. Let

$$\begin{aligned} D^h(t) &\equiv \sup_{0\le s\le t} \|u(s+h) - v(s+h)\|_n, \\ K^h(t) &\equiv \sup_{0\le s\le t} s^{(1-\beta)/2}\|u(s+h)\|_{n/\beta} + \sup_{0\le s\le t} s^{(1-\beta)/2}\|v(s+h)\|_{n/\beta}, \\ K_* &\equiv \sup_{0\le s\le T} s^{(1-\beta)/2}\|u(s)\|_{n/\beta} + \sup_{0\le s\le T} s^{(1-\beta)/2}\|v(s)\|_{n/\beta}, \end{aligned}$$

for $0 < t < t+h < T$. We easily show

$$K^h(t) \le K_* h^{\frac{-1+\beta}{2}} t^{\frac{1-\beta}{2}} \le K_* t_0^{\frac{-1+\beta}{2}} t^{\frac{1-\beta}{2}}$$

for all $h \ge t_0$ and all $0 < t < T - h$.

Suppose that $u(t_1) \equiv v(t_1)$ for some $t_1 \ge t_0$. Then, by Lemma 3.5 we see that $u(\cdot + t_1)$ and $v(\cdot + t_1)$ is mild solutions in the class $S_r(0, T - t_1)$ with same initial data $u(t_1)$. By the above argument we have

$$D^{t_1}(t) \le (C_* K^{t_1}(t) + B_* t^{\frac{1}{2}(1-\frac{n}{m})}) D^{t_1}(t), \quad 0 < t < T - t_1.$$

Letting $\xi \equiv \min\{1/(4C_* t_0^{\frac{-1+\beta}{2}} K_*)^{\frac{2}{\beta-1}}, 1/(4B_*)^{\frac{2m}{m-n}}\}$, we obtain $D^{t_1}(\xi) \le \frac{1}{2} D^{t_1}(\xi)$ which implies

$$u(t) \equiv v(t) \quad \text{for} \quad t_1 \le t \le t_1 + \xi.$$

Since ξ can be chosen independent of t_1, we can repeat the same argument as above for $t \ge t_1 + \xi$ and we have $u(t) \equiv v(t)$ for all $t \in [0, T)$. This proves Lemma 3.6. □

4. Proof of Theorems 2.1 and 2.2

Proof of Theorem 2.1. Let us construct the mild solution according to the following scheme:

$$u_0(t) = e^{-tA} a, \tag{4.1}$$

$$\begin{aligned} (u_{j+1}(t), \phi) = {} & (e^{-tA} a, \phi) + \int_0^t (u_j(s) \cdot \nabla e^{-(t-s)A} \phi, u_j(s)) ds \\ & + \int_0^t (w(s) \cdot \nabla e^{-(t-s)A} \phi, u_j(s)) ds \\ & + \int_0^t (u_j(s) \cdot \nabla e^{-(t-s)A} \phi, w(s)) ds, \quad j = 0, 1 \ldots . \end{aligned} \tag{4.2}$$

for all $\phi \in C_{0,\sigma}^{\infty}$ and all $0 < t < \infty$. Indeed, we can see that there is a function u_{j+1} satisfying (4.2) with $t^{1/4}u_{j+1}(\cdot) \in L^{\infty}(0,\infty); L_{\sigma}^{2n})$ if $t^{1/4}u_j(\cdot) \in L^{\infty}(0,\infty); L_{\sigma}^{2n})$. To see this, we assume that

$$\sup_{0<t<\infty} t^{\frac{1-\alpha}{2}} \|u_j(t)\|_{\frac{n}{\alpha}} \leq K_{\alpha,j} < \infty \quad \text{for some } \ 0 < \alpha \leq 1/2. \qquad (4.3)$$

From Lemma 3.1, we obtain

$$\left| \int_0^t (u_j(s) \cdot \nabla e^{-(t-s)A}\phi, u_j(s))ds \right| \\ \leq M'_{\frac{n}{n-\alpha},\frac{n}{n-2\alpha}} (K_{\alpha,j})^2 B(\alpha, \tfrac{1-\alpha}{2}) t^{\frac{\alpha-1}{2}} \|\phi\|_{\frac{n}{n-\alpha}} \qquad (4.4)$$

for all $\phi \in C_{0,\sigma}^{\infty}$ and all $0 < t < \infty$. By Lemma 3.3 (i), we have

$$\left| \int_0^t (w(s) \cdot \nabla e^{-(t-s)A}\phi, u_j(s))ds \right| + \left| \int_0^t (u_j(s) \cdot \nabla e^{-(t-s)A}\phi, w(s))ds \right| \\ \leq C(\alpha, m_1, m_2, n)(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}) \\ \cdot \left(\sup_{0<s<t} s^{\frac{1-\alpha}{2}} \|u_j(s)\|_{n/\alpha} \right) t^{\frac{\alpha-1}{2}} \|\phi\|_{\frac{n}{n-\alpha}} \qquad (4.5)$$

for all $0 < t < \infty$. Obviously we have

$$|(e^{-tA}a, \phi)| \leq \|e^{-tA}a\|_{\frac{n}{\alpha}} \|\phi\|_{\frac{n}{n-\alpha}} \leq M_{n,\frac{n}{\alpha}} t^{\frac{\alpha-1}{2}} \|a\|_n \|\phi\|_{\frac{n}{n-\alpha}} \qquad (4.6)$$

Hence it follows from (4.4), (4.5), (4.6) and duality that under the assumption (4.3), there is a unique function $u_{j+1}(t) \in L_{\sigma}^{n/\alpha}$ satisfying (4.2) for all $t > 0$ with

$$\sup_{0<t<\infty} t^{\frac{\alpha-1}{2}} \|u_{j+1}(t)\|_{\frac{n}{\alpha}} \\ \leq M_{n,\frac{n}{\alpha}} \|a\|_n + C_1(\alpha, n)(K_{\alpha,j})^2 \\ + C_2(\alpha, m_1, m_2, n)(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}) K_{\alpha,j}. \qquad (4.7)$$

Now we have

$$\sup_{0<t<\infty} t^{\frac{1-\alpha}{2}} \|u_0(t)\|_{\frac{n}{\alpha}} = \sup_{0<t<\infty} t^{\frac{1-\alpha}{2}} \|e^{-tA}a\|_{\frac{n}{\alpha}} \leq M_{n,\frac{n}{\alpha}} \|a\|_n,$$

which show (4.3) is true for $j = 0$ with $K_{\alpha,0} = M_{n,\frac{n}{\alpha}} \|a\|_n$. Therefore by induction we see that for all $j = 0, 1 \ldots,$ there is a unique function u_{j+1} satisfying (4.2) and (4.3) with j replaced by $j + 1$ and that

$$K_{\alpha,j+1} = K_{\alpha,0} + C_1(K_{\alpha,j})^2 + C_2(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}) K_{\alpha,j}. \qquad (4.8)$$

Moreover, we can see that $u_j \in C(0,\infty; L^{n/\alpha}_\sigma)$. Indeed we have

$$\begin{aligned}&(u_{j+1}(t+h) - u_{j+1}(t), \phi)\\ &\quad = ((e^{-hA}-1)e^{-tA}a, \phi) + F_{u_j,u_j}(t,h) + F_{w,u_j}(t,h) + F_{u_j,w}(t,h)\end{aligned}$$

for all $\phi \in C^\infty_{0,\sigma}$ and all $0 < t < t+h$, where $F_{u,v}(t,h)$ is defined in Lemma 3.4. From Lemma 3.1 we obtain

$$\begin{aligned}&|((e^{-hA}-1)e^{-tA}a, \phi)|\\ &\quad \le C(\alpha,\beta,n) h^\beta t^{-\beta-\frac{1}{2}+\frac{\alpha}{2}} \|a\|_n \|\phi\|_{\frac{n}{n-\alpha}}, \quad (0<\beta<1).\end{aligned}$$

Hence from this estimate, Lemma 3.4 and duality it follows that $u_j \in C(0,\infty; L^{n/\alpha}_\sigma)$. If we assume for some $0 < \alpha \le 1/2$ that

$$C_2(\alpha, m_1, m_2, n)(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}) < 1; \tag{4.9}$$

$$4M_{n,\frac{n}{\alpha}} \|a\|_n C_1(\alpha,n) < (1 - C_2(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}))^2, \tag{4.10}$$

then the sequence $\{K_{\alpha,j}\}_{j=0}^\infty$ is bounded with

$$K_{\alpha,j} < \frac{1 - C_2|||w||| - \sqrt{(1-C_2|||w|||)^2 - 4K_{\alpha,0}C_1(\alpha,n)}}{2C_1(\alpha,n)} \equiv k_\alpha \tag{4.11}$$

for all $j = 0, 1, \ldots$, where $|||w||| \equiv \|w\|_{m_1,\infty} + \|w\|_{m_2,\infty}$. Note that $k_\alpha \le \frac{2K_{\alpha,0}}{1-C_2|||w|||}$. From now on we assume (4.9) and (4.10) for some $0 < \alpha \le 1/2$. Set $v_j \equiv u_j - u_{j-1} (u_{-1} \equiv 0)$. By Lemma 3.3 (i) we see that

$$\begin{aligned}&|(v_{j+1}(t), \phi)|\\ &\quad = \Big| \int_0^t (v_j(s)\cdot\nabla e^{-(t-s)A}\phi, u_j(s))ds\\ &\qquad + \int_0^t (u_{j-1}(s)\cdot\nabla e^{-(t-s)A}\phi, v_j(s))ds\\ &\qquad + \int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, v_j(s))ds\\ &\qquad + \int_0^t (v_j(s)\cdot\nabla e^{-(t-s)A}\phi, w(s))ds \Big|\\ &\quad \le (2C_1k_\alpha + C_2|||w|||)\left(\sup_{0<s<\infty} s^{\frac{1-\alpha}{2}} \|v_j(s)\|_{\frac{n}{\alpha}}\right) t^{\frac{\alpha-1}{2}} \|\phi\|_{\frac{n}{n-\alpha}}.\end{aligned} \tag{4.12}$$

Letting $C_{\alpha,3} \equiv 2C_1(\alpha,n)k_\alpha + C_2(\|w\|_{m_1,\infty} + \|w\|_{m_2,\infty})$, from duality we obtain

$$\sup_{0<s<\infty} s^{\frac{1-\alpha}{2}}\|v_{j+1}(s)\|_{\frac{n}{\alpha}} \leq C_{\alpha,3}\left(\sup_{0<s<\infty} s^{\frac{1-\alpha}{2}}\|v_j(s)\|_{\frac{n}{\alpha}}\right), \quad j=0,1,\ldots,$$

which yields

$$\sup_{0<s<\infty} s^{\frac{1-\alpha}{2}}\|v_j(s)\|_{\frac{n}{\alpha}} \leq (C_{\alpha,3})^j\left(\sup_{0<s<\infty} s^{\frac{1-\alpha}{2}}\|v_0(s)\|_{\frac{n}{\alpha}}\right) \leq M_{n,\frac{n}{\alpha}}\|a\|_n(C_{\alpha,3})^j. \tag{4.13}$$

Since (4.11) implies $0 < C_{\alpha,3} < 1$ and since $u_j = \sum_{i=0}^{j} v_i$, (4.13) yields a limit $u \in C((0,\infty); L^{n/\alpha}_\sigma)$ with $t^{\frac{1-\alpha}{2}}u(\cdot) \in BC((0,\infty); L^{n/\alpha}_\sigma)$ such that

$$\sup_{0<t<\infty} t^{\frac{1-\alpha}{2}}\|u_j(t) - u(t)\|_{\frac{n}{\alpha}} \to 0 \quad \text{as } j \to \infty. \tag{4.14}$$

Following Kozono-Ogawa [15], we can show $\lim_{t\to+0} t^{\frac{1-\alpha}{2}}\|u(t)\|_{\frac{n}{\alpha}} = 0$. Indeed it follows that

$$\begin{aligned}\sup_{0<t<T} t^{\frac{1-\alpha}{2}}\|e^{-tA}a\|_{\frac{n}{\alpha}} &\leq \sup_{0<t<T} t^{\frac{1-\alpha}{2}}\|e^{-tA}(a-\tilde{a})\|_{\frac{n}{\alpha}} + \sup_{0<t<T} t^{\frac{1-\alpha}{2}}\|e^{-tA}\tilde{a}\|_{\frac{n}{\alpha}} \\ &\leq M_{n,\frac{n}{\alpha}}\|a-\tilde{a}\|_n + M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|\tilde{a}\|_{\frac{n}{\alpha}}T^{\frac{1-\alpha}{2}}\end{aligned} \tag{4.15}$$

for all $\tilde{a} \in L^n_\sigma \cap L^{n/\alpha}_\sigma$ and all $0 < T < \infty$. Since (4.3)–(4.11) hold with $0 < t < \infty$ replaced by $0 < t < T$ for arbitrary $T > 0$ and since $L^n_\sigma \cap L^{n/\alpha}_\sigma$ is dense in L^n_σ, (4.11) with the aid of (4.15) yields

$$\sup_{0<t<T} t^{\frac{1-\alpha}{2}}\|u_j(t)\|_{\frac{n}{\alpha}},\ \sup_{0<t<T} t^{\frac{1-\alpha}{2}}\|u(t)\|_{\frac{n}{\alpha}} \to 0 \quad \text{as } T \to 0. \tag{4.16}$$

We next show $u \in BC([0,\infty)); L^n_\sigma)$ if (4.9) and (4.10) hold for $\alpha = 1/2$. From now on we assume that (4.9) and (4.10) hold for $\alpha = 1/2$. Since $w \in L^\infty(0,\infty; L^{m_1}_\sigma \cap L^{m_2}_\sigma)$, we can take $0 < \gamma < 1$ such that $\alpha + \gamma \geq 1$ and $w \in L^\infty(0,\infty; L^{n/\gamma}_\sigma)$. Then, in the similar way to proving $u_j \in C((0,\infty); L^{n/\alpha}_\sigma)$, by Lemma 3.4 (with $\delta = 1$) and duality, we have $u_j \in C((0,\infty); L^n_\sigma)$. From Lemma 3.1, we obtain

$$\|u_0(t)\|_n \leq M_{n,n}\|a\|_n$$

$$\left|\int_0^t (u_j(s) \cdot \nabla e^{-(t-s)A}\phi, u_j(s))ds\right|$$
$$\leq M'_{\frac{n}{n-1},\frac{n}{n-1}}(k_{\frac{1}{2}})^2 B(\tfrac{1}{2},\tfrac{1}{2})\|\phi\|_{\frac{n}{n-1}},$$
$$\left|\int_0^t (w(s) \cdot \nabla e^{-(t-s)A}\phi, u_j(s))ds\right|$$
$$\leq M'_{\frac{n}{n-1},\frac{2n}{2n-3}}\|w\|_{n,\infty}(k_{\frac{1}{2}})B(\tfrac{1}{4},\tfrac{3}{4})\|\phi\|_{\frac{n}{n-1}},$$
$$\left|\int_0^t (u_j(s) \cdot \nabla e^{-(t-s)A}\phi, w(s))ds\right|$$
$$\leq M'_{\frac{n}{n-1},\frac{2n}{2n-3}}\|w\|_{n,\infty}(k_{\frac{1}{2}})B(\tfrac{1}{4},\tfrac{3}{4})\|\phi\|_{\frac{n}{n-1}},$$

for all $\phi \in C_{0,\sigma}^{\infty}$, $t > 0$, which yield the following uniform estimate:

$$\sup_{0<t<\infty}\|u_{j+1}\|_n \leq M_{n,n}\|a\|_n + M'_{\frac{n}{n-1},\frac{n}{n-1}}(k_{\frac{1}{2}})^2 B(\tfrac{1}{2},\tfrac{1}{2}) + 2M'_{\frac{n}{n-1},\frac{2n}{2n-3}}\|w\|_{n,\infty}k_{\frac{1}{2}}B(\tfrac{1}{4},\tfrac{3}{4}).$$

Concerning continuity of u_j at $t = 0$ in L_σ^n, as above we obtain

$$\|u_{j+1}(t) - a\|_n$$
$$\leq \|e^{-tA}a - a\|_n + M'_{\frac{n}{n-1},\frac{n}{n-1}}\left(\sup_{0<s<t} s^{1/4}\|u_j(s)\|_{2n}\right)^2 B(\tfrac{1}{2},\tfrac{1}{2})$$
$$+ 2M'_{\frac{n}{n-1},\frac{2n}{2n-3}}\|w\|_{n,\infty}\left(\sup_{0<s<t} s^{1/4}\|u_j(s)\|_{2n}\right)B(\tfrac{1}{4},\tfrac{3}{4}),$$

which yields with the aid of (4.16) $\lim_{t\to+0}\|u_j(t) - a\|_n = 0$. Concerning $v_j(\equiv u_j - u_{j-1})$, as (4.12) we have

$$|(v_{j+1}(t),\phi)|$$
$$\leq 2M'_{\frac{n}{n-1},\frac{n}{n-1}}k_{1/2}B(\tfrac{1}{2},\tfrac{1}{2})\left(\sup_{0<s<\infty} s^{\frac{1}{4}}\|v_j(s)\|_{2n}\right)\|\phi\|_{\frac{n}{n-1}}$$
$$+ 2M'_{\frac{n}{n-1},\frac{2n}{2n-1}}\|w\|_{n,\infty}B(\tfrac{3}{4},\tfrac{1}{4})\left(\sup_{0<s<\infty} s^{\frac{1}{4}}\|v_j(s)\|_{2n}\right)\|\phi\|_{\frac{n}{n-1}},$$

which implies by duality that

$$\sup_{0<s<\infty}\|v_{j+1}(s)\|_n \leq C(n,w,k_{1/2})\sup_{0<s<\infty} s^{\frac{1}{4}}\|v_j(s)\|_{2n} \quad \text{for } j = 0,1,\dots . \tag{4.17}$$

From this and (4.13) with $\alpha = 1/2$ we obtain

$$\sup_{0<s<\infty} \|u_l(s) - u_m(s)\|_n = \sup_{0<s<\infty} \left\| \sum_{j=m+1}^{l} v_j(s) \right\|_n \leq CM_{n,2n}\|a\|_n \sum_{j=m}^{l-1} (C_{\alpha,3})^j \quad \text{for } l > m \geq 0. \tag{4.18}$$

Hence it follows from (4.18) and $0 < C_{\alpha,3} < 1$ that the limit u belongs to $u \in BC([0,\infty); L^n_\sigma)$.

To see that u is desired mild solution of $(N - S')$ in the class $S_{2n}(0,\infty)$, we need to prove that u satisfies (iii) in Definition 2. By Lemma 3.1 and (4.14), we have

$$\begin{aligned}
&\left| \int_0^t (u_j(s) \cdot \nabla e^{-(t-s)A}\phi, u_j(s))ds - \int_0^t (u(s) \cdot \nabla e^{-(t-s)A}\phi, u(s))ds \right| \\
&\leq \int_0^t (\|u_j(s)\|_{2n} + \|u(s)\|_{2n})\|u_j(s) - u(s)\|_{2n}\|\nabla e^{-(t-s)A}\phi\|_{\frac{n}{n-1}} ds \\
&\leq 2M'_{\frac{n}{n-1},\frac{n}{n-1}} k_{\frac{1}{2}} \sup_{0<s<\infty} s^{\frac{1}{4}}\|u_j(s) - u_(s)\|_{2n} B(\tfrac{1}{2},\tfrac{1}{2})\|\phi\|_{\frac{n}{n-1}} \\
&\to 0 \text{ as } j \to \infty \quad (\phi \in C^\infty_{0,\sigma}), \\
&\left| \int_0^t (w(s) \cdot \nabla e^{-(t-s)A}\phi, u_j(s))ds - \int_0^t (w(s) \cdot \nabla e^{-(t-s)A}\phi, u(s))ds \right| \\
&\leq M'_{\frac{n}{n-1},\frac{2n}{2n-3}} \|w\|_{n,\infty} \sup_{0<s<\infty} s^{\frac{1}{4}}\|u_j(s) - u_(s)\|_{2n} B(\tfrac{1}{4},\tfrac{3}{4})\|\phi\|_{\frac{n}{n-1}} \\
&\to 0 \text{ as } j \to \infty \quad (\phi \in C^\infty_{0,\sigma}), \\
&\left| \int_0^t (u_j(s) \cdot \nabla e^{-(t-s)A}\phi, w(s))ds - \int_0^t (u(s) \cdot \nabla e^{-(t-s)A}\phi, w(s))ds \right| \\
&\to 0 \text{ as } j \to \infty \quad (\phi \in C^\infty_{0,\sigma}),
\end{aligned}$$

which yield (iii) in Definition 2. Obviously we can choose λ_1, λ_2 such that (2.3) and (2.4) imply (4.9) and (4.10) with $\alpha = 1/2$: $\lambda_1 \geq C_2(\frac{1}{2}, m_1, m_2, n)$, $\lambda_2 \leq \frac{1}{4C_1(\frac{1}{2},n)M_{n,2n}}$. This proves the existence of a mild solution of $(N - S')$ in the class $S_{2n}(0,\infty)$ under the conditions (2.3) and (2.4).

Now it remains to show (2.5) and (2.6). Since $u \in L^\infty(0,\infty; L^n)$ and $t^{1/4}u(\cdot) \in L^\infty(0,\infty; L^{2n})$, we get (2.5) by the Hölder inequality.

For the proof of (2.6), we first consider the case $a \in L^n_\sigma \cap L^{3n/4}_\sigma$. Let us prove $u \in BC((0,\infty); L_\sigma^{3n/4})$ when $a \in L_\sigma^n \cap L_\sigma^{3n/4}$. To prove this, we need to show

$$\sup_{0<t<\infty} \|u_j(t)\|_{\frac{3n}{4}} \le N_j < \infty \quad j = 0, 1, 2, \ldots$$

under the condition $a \in L_\sigma^n \cap L_\sigma^{3n/4}$. Set $m = m_1$ in (Case 1) and $m = \max\{m_1, \frac{3n}{3n-7}\}$ in (Case 2). From Lemma 3.1 and Lemma 3.3 (ii) with $\beta = 4/3$, we obtain

$$\begin{aligned}
&|(u_0(t), \phi)| \le M_{\frac{3n}{4},\frac{3n}{4}} \|a\|_{\frac{3n}{4}} \|\phi\|_{\frac{3n}{3n-4}},\\
&\left|\int_0^t (u_j(s)\cdot\nabla e^{-(t-s)A}\phi, u_j(s))ds\right|\\
&\qquad \le M'_{\frac{3n}{3n-4},\frac{6n}{6n-11}} B(\tfrac{1}{4}, \tfrac{3}{4}) k_{\frac{1}{2}} N_j \|\phi\|_{\frac{3n}{3n-4}},\\
&\left|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u_j(s))ds\right|\\
&\qquad \le C(m_1, m_2, n)(\|w\|_{m,\infty} + \|w\|_{m_2,\infty}) N_j \|\phi\|_{\frac{3n}{3n-4}}\\
&\qquad \le 2C(m_1, m_2, n)|||w||| N_j \|\phi\|_{\frac{3n}{3n-4}},\\
&\left|\int_0^t (u_j(s)\cdot\nabla e^{-(t-s)A}\phi, w(s))ds\right|\\
&\qquad \le 2C(m_1, m_2, n)|||w||| N_j \|\phi\|_{\frac{3n}{3n-4}}
\end{aligned}$$

for all $\phi \in C_{0,\sigma}^\infty$, $t > 0$. By duality we may take N_{j+1} as

$$N_{j+1} = M_{\frac{3n}{4},\frac{3n}{4}} \|a\|_{\frac{3n}{4}} + \{B_1(n) k_{\frac{1}{2}} + B_2(m_1, m_2, n)|||w|||\} N_j.$$

Since

$$k_{\frac{1}{2}} \le \frac{2M_{n,2n}\|a\|_n}{1 - C_2|||w|||} \le \frac{2M_{n,2n}\|a\|_n}{(1 - C_2|||w|||)^2} \le \frac{2M_{n,2n}\|a\|_n}{(1 - \lambda_1|||w|||)^2} < 2M_{n,2n}\lambda_2,$$

we see that the conditions (2.3), (2.4) imply $B_1(n)k_{\frac{1}{2}} + B_2(m_1, m_2, n)|||w||| < 1$, by arranging λ_1 larger and λ_2 smaller if necessary. By standard argument, we have $u \in BC((0,\infty); L_\sigma^{3n/4})$ if $a \in L_\sigma^n \cap L_\sigma^{3n/4}$. This and (2.5) imply (2.6).

We next show that (2.6) is true in general. Set $U \equiv \{a \in L_\sigma^n; \|a\|_n < \lambda_2(1 - \lambda_1|||w|||)^2\}$. Obviously we can define a map F by

$$F : a \in U \mapsto u = Fa \in BC([0,\infty); L_\sigma^n)$$

where u is the unique mild solution of $(N-S')$ in the class $S_{2n}(0,\infty)$ with $u(0)=a$. As we have seen, there holds

$$\lim_{t\to\infty}\|(F\tilde{a})(t)\|_n=0 \quad \text{for each} \;\; \tilde{a}\in U\cap L^{3n/4}_\sigma. \tag{4.19}$$

We can show that

$$\sup_{0<t<\infty}\|(Fa)(t)-(F\tilde{a})(t)\|_n\le C\|a-\tilde{a}\|_n \quad \text{for each} \;\; a,\tilde{a}\in U, \tag{4.20}$$

where C is independent of a and $\tilde{a}$. Indeed by Definition 2, Lemma 3.3 (ii) and duality we have

$$\begin{aligned}
&\sup_{0<t<\infty}\|(Fa)(t)-(F\tilde{a})(t)\|_n\\
&\quad\le M_{n,n}\|a-\tilde{a}\|_n+M'_{\frac{n}{n-1},\frac{2n}{2n-3}}\\
&\qquad\cdot\left(\sup_{0<t<\infty}t^{1/4}\|(Fa)(t)\|_{2n}+\sup_{0<t<\infty}t^{1/4}\|(F\tilde{a})(t)\|_{2n}\right)\\
&\qquad\times\sup_{0<t<\infty}\|(Fa)(t)-(F\tilde{a})(t)\|_n\\
&\quad\;+C(m_1,m_2,n)(\|w\|_{m,\infty}+\|w\|_{m_2,\infty})\\
&\qquad\times\sup_{0<t<\infty}\|(Fa)(t)-(F\tilde{a})(t)\|_n.
\end{aligned} \tag{4.21}$$

Since $\sup_{0<t<\infty}t^{1/4}\|(Fa)(t)\|_{2n}\le k_{\frac{1}{2}}\le 2M_{n,2n}\lambda_2$ and $\|w\|_{m,\infty}+\|w\|_{m_2,\infty}\le 2|||w|||\le 2/\lambda_1$, we see that (4.21) implies (4.20), by arranging λ_1 larger and λ_2 smaller if necessary. Hence, as [15, p.29], it is follows from (4.19) and (4.20) that

$$\limsup_{t\to\infty}\|(Fa(t)\|_n=0 \quad \text{for each} \;\; a\in U.$$

This proves Theorem 2.1. □

As for the proof of Theorem 2.2, we have $t^{\frac{1-n/r}{2}}u(\cdot)\in L^\infty(0,\infty;L^r_\sigma)$, provided (4.9) and (4.10) hold for $\alpha=n/r$. By Hölder inequality we have (2.9). This proves Theorem 2.2.

5. Proof of Theorem 2.3

Let $L^\infty_{loc}([0,\infty);L^n)$ denote the set of all measurable functions u such that $u\in L^\infty(0,T;L^n)$ for all $T>0$. To prove Theorem 2.3, we need the following local existence theorem:

Theorem 5.1 (Local existence) *Let $a \in L^n_\sigma \cap L^{n/\alpha}_\sigma$ for some $\alpha \in (0,1)$ and let w be a measurable function on $(0,\infty)$ with $w \in L^\infty(0,\infty; L^m_\sigma)$ for some $m > n$ and $t^{1/2}\nabla w(\cdot) \in L^\infty_{loc}([0,\infty); L^n)$. Then there exists a mild solution u of $(N-S')$ in the class $S_{n/\alpha}(0,T^*)$ satisfying*

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A}P(u\cdot\nabla u + w\cdot\nabla u + u\cdot\nabla w)(s)ds \quad in \quad L^n_\sigma$$

where

$$T^* = \min\left\{\left[\frac{1}{16(C_1+C_4)M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|a\|_{\frac{n}{\alpha}}}\right]^{\frac{2}{1-\alpha}}, \left(\frac{1}{2(C_4+C_5)\|w\|_{m,\infty}}\right)^{\frac{2m}{m-n}}\right\},$$

$$C_1 = C_1(\alpha,n) = M'_{\frac{n}{n-\alpha},\frac{n}{n-2\alpha}}B(\alpha,\tfrac{1-\alpha}{2}),$$

$$C_4 = Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}B(\tfrac{1-\alpha}{2},\tfrac{\alpha}{2}) + Q_{\frac{nm}{n+m}}M'_{\frac{nm}{n+m},n}B(\tfrac{1}{2}(1-\tfrac{n}{m}),\tfrac{1}{2}),$$

$$C_5 = 2M'_{\frac{n}{n-\alpha},\frac{mn}{mn-m\alpha-n}}B(\tfrac{\alpha+1}{2},\tfrac{1}{2}(1-\tfrac{n}{m})), \quad Q_l = \|P_l\|_{B(L^l,L^l_\sigma)}.$$

Moreover if there is positive number $\kappa \in (0,1)$ such that

$$w \in C^\kappa([\xi,T^*];L^\infty), \quad \nabla w \in C^\kappa([\xi,T^*];L^n)$$

for all $\xi \in (0,T^*)$, then u is also a strong solution of $(N-S')$ on $(0,T^*)$.

Remark. In case $w \equiv 0$, the existence interval T^* was obtained by Giga [8].

Proof of Theorem 5.1. Let us construct the strong solution according to the following scheme:

$$u_0(t) = e^{-tA}a, \tag{5.1}$$

$$\begin{aligned} u_{j+1}(t) = {} & e^{-tA}a - \int_0^t e^{-(t-s)A}P(u_j\cdot\nabla u_j)(s)ds \\ & - \int_0^t e^{-(t-s)A}P(w\cdot\nabla u_j)(s)ds \\ & - \int_0^t e^{-(t-s)A}P(u_j\cdot\nabla w)(s)ds. \end{aligned} \tag{5.2}$$

Then we can see that for $0 < T < \infty$

$$\sup_{0<t<T} t^{\frac{1-\alpha}{2}}\|u_j(t)\|_{n/\alpha} \le K^T_{\alpha,j} < \infty, \quad j = 0,1,\ldots, \tag{5.3}$$

$$\sup_{0<t<T} t^{\frac{1}{2}}\|\nabla u_j(t)\|_n \le L_j^T < \infty, \quad j=0,1,\dots . \tag{5.4}$$

Suppose that (5.3) and (5.4) are true. Then, multiplying (5.2) by ϕ and integrating by parts, we obtain the identity (4.2). We have by Lemma 3.1 and the Hölder inequality that

$$\begin{aligned}&\left|\int_0^t (w(s)\cdot\nabla e^{-(t-s)A}\phi, u_j(s))ds\right| + \left|\int_0^t (u_j(s)\cdot\nabla e^{-(t-s)A}\phi, w(s))ds\right| \\ &\quad \le C_5\|w\|_{m,\infty;T}\Big(\sup_{0<s<t} s^{\frac{1-\alpha}{2}}\|u_j(s)\|_{n/\alpha}\Big)t^{\frac{\alpha-1}{2}}T^{\frac{1}{2}(1-\frac{n}{m})}\|\phi\|_{\frac{n}{n-\alpha}}.\end{aligned} \tag{5.5}$$

As in the proof of Theorem 2.1, by (4.4) and (5.5) we have that

$$K_{\alpha,j+1}^T \le K_{\alpha,0}^T + C_1(\alpha,n)(K_{\alpha,j}^T)^2 + C_5\|w\|_{m,\infty;T}T^{\frac{1}{2}(1-\frac{n}{m})}K_{\alpha,j}^T. \tag{5.6}$$

Concerning (5.4), we have

$$\begin{aligned}&\|\nabla u_0(t)\|_n \le M'_{n,n}\|a\|_n t^{-1/2}, \\ &\left\|\nabla\int_0^t e^{-(t-s)A}P(u_j\cdot\nabla u_j)(s)ds\right\|_n \\ &\qquad \le Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}K_{\alpha,j}^T L_j^T B(\tfrac{1-\alpha}{2},\tfrac{\alpha}{2})t^{-1/2}, \\ &\left\|\nabla\int_0^t e^{-(t-s)A}P(w\cdot\nabla u_j)(s)ds\right\|_n \\ &\qquad \le Q_{\frac{nm}{n+m}}M'_{\frac{nm}{n+m},n}\|w\|_{m,\infty}L_j^T B(\tfrac{1}{2}(1-\tfrac{n}{m}),\tfrac{1}{2})t^{-\frac{n}{2m}}, \\ &\left\|\nabla\int_0^t e^{-(t-s)A}P(u_j\cdot\nabla w)(s)ds\right\|_n \\ &\qquad \le Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}K_{\alpha,j}^T\|(\cdot)^{1/2}\nabla w\|_{n,\infty;T}B(\tfrac{1-\alpha}{2},\tfrac{\alpha}{2})t^{-1/2},\end{aligned}$$

where $Q_r = \|P_r\|_{B(L^r,L^r_\sigma)}$. Hence (5.4) is true with j replaced by $j+1$, with

$$\begin{aligned}L_{j+1}^T \equiv\ & M'_{n,n}\|a\|_n + Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}B(\tfrac{1-\alpha}{2},\tfrac{\alpha}{2})K_{\alpha,j}^T\|(\cdot)^{1/2}\nabla w\|_{n,\infty;T} \\ &+ C_4(K_{\alpha,j}^T + \|w\|_{m,\infty}T^{\frac{1}{2}(1-\frac{n}{m})})L_j^T,\end{aligned} \tag{5.7}$$

where $C_4 = Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}B(\frac{1-\alpha}{2},\frac{\alpha}{2}) + Q_{\frac{nm}{n+m}}M'_{\frac{nm}{n+m},n}B(\frac{1}{2}(1-\frac{n}{m}),\frac{1}{2})$. Therefore by induction, we get (5.3) and (5.4) for $j=0,1,\dots$. Let $C_6(T) =$

$1 - C_5\|w\|_{m,\infty}T^{\frac{1}{2}(1-\frac{n}{m})}$. Since we may take $K^T_{\alpha,0} \equiv M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|a\|_{\frac{n}{\alpha}}T^{\frac{1-\alpha}{2}}$, by (5.6) we have

$$K^T_{\alpha,j} < \frac{C_6(T) - \sqrt{(C_6(T))^2 - 4C_1M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|a\|_{\frac{n}{\alpha}}T^{\frac{1-\alpha}{2}}}}{2C_1} \equiv k^T_\alpha, \quad j = 0, 1, \ldots, \tag{5.8}$$

provided

$$C_6(T) = 1 - C_5\|w\|_{m,\infty}T^{\frac{1}{2}(1-\frac{n}{m})} > 0, \tag{5.9}$$

$$4C_1M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|a\|_{\frac{n}{\alpha}}T^{\frac{1-\alpha}{2}} < (1 - C_5\|w\|_{m,\infty}T^{\frac{1}{2}(1-\frac{n}{m})})^2. \tag{5.10}$$

Since $C_6(T^*) = 1-C_5\|w\|_{m,\infty}T^{*\frac{1}{2}(1-\frac{n}{m})} > 1/2$, $4C_1M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|a\|_{\frac{n}{\alpha}}T^{*\frac{1-\alpha}{2}} < 1/4$, T^* satisfies (5.9) and (5.10). Hence, as in the proof of Theorem 2.1, we obviously see that there is a limit $u \in C((0,T^*); L^{n/\alpha}_\sigma)$ with $t^{\frac{1-\alpha}{2}}u(\cdot) \in BC([0,T^*); L^{n/\alpha}_\sigma)$ satisfying

$$\sup_{0<t<T^*} t^{\frac{1-\alpha}{2}}\|u_j(t) - u(t)\|_{n/\alpha} \to 0 \quad \text{as } j \to \infty,$$

$$\sup_{0<t<T} t^{\frac{1-\alpha}{2}}\|u(t)\|_{n/\alpha} \to 0 \quad \text{as } T \to +0.$$

Moreover we shall show $t^{1/2}\nabla u(\cdot) \in L^\infty(0,T^*;L^n)$. (5.7) and (5.8) yield

$$\begin{aligned} L^{T^*}_{j+1} \le\ & M'_{n,n}\|a\|_n + Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}B(\tfrac{1-\alpha}{2},\tfrac{\alpha}{2})k^{T^*}_\alpha\|(\cdot)^{1/2}\nabla w\|_{n,\infty;T^*} \\ & + C_4(k^{T^*}_\alpha + \|w\|_{m,\infty}T^{*\frac{1}{2}(1-\frac{n}{m})})L^{T^*}_j. \end{aligned} \tag{5.11}$$

It follows from $C_6(T^*) > 1/2$ and $k^{T^*}_\alpha < \frac{2M_{\frac{n}{\alpha},\frac{n}{\alpha}}\|a\|_{\frac{n}{\alpha}}T^{*\frac{1-\alpha}{2}}}{C_6(T^*)}$ that $C_4k^{T^*}_\alpha < 1/2$. By the definition of T^* we obviously have $C_4\|w\|_{m,\infty}T^{*\frac{1}{2}(1-\frac{n}{m})} < 1/2$. Thus we obtain

$$C_4(k^{T^*}_\alpha + \|w\|_{m,\infty}T^{*\frac{1}{2}(1-\frac{n}{m})}) < 1. \tag{5.12}$$

Hence from (5.11) we see that the sequence $\{L^{T^*}_j\}^\infty_{j=0}$ is bounded with

$$\begin{aligned} L^{T^*}_j &< \frac{M'_{n,n}\|a\|_n + Q_{\frac{n}{\alpha+1}}M'_{\frac{n}{\alpha+1},n}B(\frac{1-\alpha}{2},\frac{\alpha}{2})\|(\cdot)^{1/2}\nabla w\|_{n,\infty;T^*}k^{T^*}_\alpha}{1 - C_4(k^{T^*}_\alpha + \|w\|_{m,\infty}T^{*\frac{1}{2}(1-\frac{n}{m})})} \\ &\equiv L^{T^*}. \end{aligned} \tag{5.13}$$

By standard argument, such a bound yields

$$t^{1/2}\nabla u(\cdot) \in L^\infty(0,T^*;L^n).$$

By (5.1) and (5.2) we easily show that $u_j \in C([0,T^*);L^n_\sigma)$ for $j=0,1,\dots$. In the similar way to proving (5.11), we have

$$\begin{aligned}\sup_{0<t<T^*}\|u_{j+1}\|_n &\le M_{n,n}\|a\|_n \\ &\quad + Q_{\frac{n}{\alpha+1}}M_{\frac{n}{\alpha+1},n}B(1-\tfrac{\alpha}{2},\tfrac{\alpha}{2})k_\alpha^{T^*}(L^{T^*}+\|(\cdot)^{\frac12}\nabla w\|_{n,\infty;T^*}) \\ &\quad + Q_{\frac{mn}{m+n}}M_{\frac{mn}{m+n},n}B(1-\tfrac{n}{2m},\tfrac12)\|w\|_{m,\infty;T^*}L^{T^*}T^{*\frac12(1-\frac{n}{m})}\end{aligned}$$

for $j=0,1,\dots$, which yields $u \in BC([0,T^*);L^n_\sigma)$. Hence as in the proof of Theorem 2.1, we see that u is a unique mild solution of $(N-S')$ in the class $S_{n/\alpha}(0,T^*)$. It follows from (iii) of Definition 2 and integration by parts that

$$\begin{aligned}(u(t),\phi) &= (e^{-tA}a,\phi)-\int_0^t(e^{-(t-s)A}P(u\cdot\nabla u)(s),\phi)ds \\ &\quad -\int_0^t(e^{-(t-s)A}P(w\cdot\nabla u)(s),\phi)ds \\ &\quad -\int_0^t(e^{-(t-s)A}P(u\cdot\nabla w)(s),\phi)ds,\end{aligned}$$

for all $\phi \in C^\infty_{0,\sigma}$, all $0<t<T^*$. It is easily shown that $\int_0^t e^{-(t-s)A}P(u\cdot\nabla u)(s)ds$, $\int_0^t e^{-(t-s)A}P(w\cdot\nabla u)(s)ds$ and $\int_0^t e^{-(t-s)A}P(u\cdot\nabla w)(s)ds$ belong to L^n_σ for all $0<t<T^*$. Thus we obtain

$$\begin{aligned}u(t) &= e^{-tA}a-\int_0^t e^{-(t-s)A}P(u\cdot\nabla u)(s)ds \\ &\quad -\int_0^t e^{-(t-s)A}P(w\cdot\nabla u)(s)ds \\ &\quad -\int_0^t e^{-(t-s)A}P(u\cdot\nabla w)(s)ds \ \text{ in } \ L^n_\sigma,\end{aligned} \tag{5.14}$$

for $0<t<T^*$. Next we shall show that this mild solution u is actually a strong solution if w satisfies, for some $\kappa \in (0,1)$, $w \in C^\kappa([\xi,T^*];L^\infty)$ and $\nabla w \in C^\kappa([\xi,T^*];L^n)$ for all $\xi \in (0,T^*)$.

Since $w \in L^\infty(0,T^*;L^m)$ implies that $\sup_{0<s<T^*} s^{\frac{1-\delta}{2}}\|w(s)\|_{n/\delta}<\infty$ for $\delta = n/m$, by (5.14) we have $\sup_{0<s<T^*} s^{\frac{1-\delta}{2}}\|u(s)\|_{n/\delta}<\infty$. As in

[14, Lemma A.4], from Lemmas 3.1 and 3.2 we obtain for $\kappa' > 0$ with $0 < \delta/2 + \kappa' < 1/2$,

$$\|u(t+h) - u(t)\|_\infty \le M(h^{\kappa'} t^{-\frac{1}{2}-\kappa'} + h^{\frac{1}{2}-\frac{\delta}{2}} t^{-1+\frac{\delta}{2}}), \tag{5.15}$$

$$\|\nabla u(t+h) - \nabla u(t)\|_n \le M(h^{\kappa'} t^{-\frac{1}{2}-\kappa'} + h^{\frac{1}{2}-\frac{\delta}{2}} t^{-1+\frac{\delta}{2}}), \tag{5.16}$$

for all $0 < t < t+h < T^*$. From these estimates and the hypotheses on w it follows that, for some $\kappa_0 > 0$,

$$u \cdot \nabla u, \quad w \cdot \nabla u, \quad u \cdot \nabla w \in C^{\kappa_0}([\xi, T^*]; L^n)$$

for all $\xi \in (0, T^*)$. Then a well-known theory of holomorphic semigroup states that u is a strong solution of $(N - S')$ on $(0, T^*)$ (see, e.g., Tanabe [20, Theorem 3.3.4]). This completes the proof of Theorem 5.1. □

Proof of Theorem 2.3. Let w be a strong solution of $(N - S)$ for some $f \in C(0, \infty; L^n_\sigma)$. By the definition of strong solutions of $(N - S)$ we have $\nabla w \in L^\infty(\epsilon, T; L^n)$ for all $0 < \epsilon < T < \infty$, which implies

$$t^{1/2} \nabla w(\cdot + \epsilon) \in L^\infty_{loc}([0, \infty); L^n).$$

Moreover, as in [14, Lemma A.4], from Lemmas 3.1 and 3.2 we obtain for some $\kappa \in (0, 1)$,

$$w \in C^\kappa([\xi, T]; L^\infty), \quad \nabla w \in C^\kappa([\xi, T]; L^n) \tag{5.17}$$

for all $0 < \epsilon < \xi < T < \infty$. Since u is the mild solution in the class $S_{2n}(0, \infty)$, we have

$$\sup_{s \ge \epsilon} \|u(s)\|_{2n} \le A_\epsilon < \infty \quad \text{for} \quad \epsilon > 0.$$

Letting $\alpha = 1/2$ and

$$T^*_\epsilon = \min\left\{ \left[\frac{1}{16(C_1 + C_4) M_{2n,2n} A_\epsilon} \right]^4, \left(\frac{1}{2(C_4 + C_5)\|w\|_{m_2,\infty}} \right)^{\frac{2m_2}{m_2 - n}} \right\},$$

by Lemma 3.5, Lemma 3.6 and Theorem 5.1 we see that u is a strong solution on all interval $(t, t + T^*_\epsilon) \subset (\epsilon, \infty)$. Hence we conclude by standard argument that u is a strong solution on (ϵ, ∞). Since $\epsilon > 0$ is arbitrary, this completes the proof of Theorem 2.3. □

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