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A criterion for the existence of subobject classifiers

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Abstract. We give a criterion for the existence of subobject classifiers of cocomplete categories with a small, dense subcategory.

Key words: subobject classifier, cocomplete, dense, topos.

1. Introduction

It often occurs that a cocomplete category $\mathcal{E}$ has a small and dense subcategory $\mathcal{C}$. In this case, there is an adjunction between the category $\mathcal{E}$ and the category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ of presheaves over $\mathcal{C}$, which enables us to construct $\mathcal{E}$ objects from presheaves over $\mathcal{C}$.

In this paper, we give a criterion for the existence of a subobject classifier in $\mathcal{E}$ expressed as a condition in the presheaf category. Moreover we give the subobject classifier concretely by using the presheaves if there exists a subobject classifier.

This criterion is used heavily in the proof of the existence of subobject classifier in the category of functional bisimulations [6]. We expect this applicable to other similar problems.

We proceed as follows. In Section 2 we recall general facts on the adjunction between a cocomplete category $\mathcal{E}$ and the category of presheaves over the small category $\mathcal{C}$ when there is a functor from $\mathcal{C}$ to $\mathcal{E}$. Here, we do not assume that $\mathcal{C}$ is a subcategory of $\mathcal{E}$. Then we introduce the notion of dense functors. When a functor A from $\mathcal{C}$ to $\mathcal{E}$ is dense, the category $\mathcal{E}$ is a reflective subcategory of the presheaf category of $\mathcal{C}$. Hence $\mathcal{E}$ is a complete category since presheaf category is complete, and thus a subobject functor $\mathrm{Sub} : \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ exists.

In Section 3 we give a criterion for the existence of subobject classifier of $\mathcal{E}$, to the effect that the presheaf $\mathrm{Sub}(A(-))$ is in $\mathcal{E}$, i.e., there exists an object $\Omega \in \mathcal{E}$ such that $\mathrm{Sub}(A(-)) \cong \mathcal{E}(A(-), \Omega)$ holds.

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When we are dealing with a concrete category, it is usually difficult to find concretely in $\mathcal{E}$ itself a specific Ω which satisfies the above condition. We explain a method of constructing $\mathcal{E}$ objects via presheaves over $\mathcal{C}$ by making use of the above adjunction, in Section 4. As an example of this construction, we give a terminal object in $\mathcal{E}$. In the case there exists a subobject classifier in $\mathcal{E}$, we give it by using presheaf category. This enables us to modify the criterion slightly, which is more effective for checking it in concrete cases.

2. A basic adjunction

Let $A : \mathcal{C} \to \mathcal{E}$ be a functor from a small category $\mathcal{C}$ to a cocomplete category $\mathcal{E}$. Define a functor $R : \mathcal{E} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ by $R(E) = \mathcal{E}(A(-), E)$.

Proposition 2.1 ([4, pp. 41–42]) *The functor R has a left adjoint $L : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{E}$. The functor L is given for each presheaf P by*

$$LP = \mathrm{Colim}\left(\int P \xrightarrow{\pi_P} \mathcal{C} \xrightarrow{A} \mathcal{E}\right).$$

Recall that the category $\int P$ of elements of a presheaf P is defined as follows.

- Its object is a pair (C, p) of an object $C \in \mathcal{C}$ and $p \in P(C)$.
- An arrow $u : (C, p) \to (C', p')$ is a $\mathcal{C}$ arrow $u : C \to C'$ such that $p' \cdot u = p$, where $p' \cdot u := P(u)(p')$.

The functor $\pi_P : \int P \to \mathcal{C}$ is the projection $(C, p) \mapsto C$, and the composition of functors

$$\int P \xrightarrow{\pi_P} \mathcal{C} \xrightarrow{A} \mathcal{E} \tag{1}$$

is a diagram in $\mathcal{E}$ with the indexing category $\int P$. We often write the diagram (1) simply by $A \circ \pi_P$.

Outline of proof. First we define a bijective correspondence

$$\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(P, R(E)) \cong \mathcal{E}(LP, E)$$

for each presheaf P and an object $E \in \mathcal{E}$.

Let $\tau \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(P, R(E))$. Consider the diagram $A \circ \pi_P$ and take a cocone $\tilde{\tau} : A \circ \pi_P \to E$ with $\tilde{\tau}_{(C,p)} = \tau_C(p)$ for each object $(C, p) \in \int P$, where τ_C is the component of the natural transformation τ at $C \in \mathcal{C}$. Since the category $\mathcal{E}$ is cocomplete, there exists a universal cocone $\mu : A \circ \pi_P \to \mathrm{Colim}(A \circ \pi_P) = LP$. By the universality of the cocone μ, we have a unique $\mathcal{E}$ arrow $g : \mathrm{Colim}(A \circ \pi_P) \to E$ satisfying $g \circ \mu_{(C,p)} = \tilde{\tau}_{(C,p)}$ for each object $(C,p) \in \int P$. It is easy to show that the correspondence $\tau \mapsto g$ is bijective and natural for P and E. □

Definition 2.2 For each object $E \in \mathcal{E}$, we call the functor

$$(A(-)/E) \xrightarrow{\partial} \mathcal{C} \xrightarrow{A} \mathcal{E}$$

the **canonical diagram** of E. Here $A(-)/E$ is the comma category. Its object is a pair of $C \in \mathcal{C}$ and an $\mathcal{E}$ arrow $f : A(C) \to E$, and its arrow from $(C, f : A(C) \to E)$ to $(C', g : A(C') \to E)$ is a $\mathcal{C}$ arrow $u : C \to C'$ such that $f = g \circ A(u)$. The functor $\partial : (A(-)/E) \to \mathcal{C}$ is the projection: $(C,f) \mapsto C$.

For each object $E \in \mathcal{E}$, there always exists a cocone $\nu : A \circ \partial \to E$ defined by $\nu_{(C,f)} = f$ for each object $(C,f) \in A(-)/E$, which we call the **canonical cocone** of E.

Definition 2.3 A functor $A : \mathcal{C} \to \mathcal{E}$ is called **dense** if the canonical cocone of E is universal for each object $E \in \mathcal{E}$.

We call a subcategory is dense when the inclusion functor is dense.

Note that the canonical diagram of E is nothing but the diagram $A \circ \pi_{R(E)} : \int R(E) \to \mathcal{E}$. Hence if the functor $A : \mathcal{C} \to \mathcal{E}$ is dense, then

$$\begin{aligned} LR(E) &= \mathrm{Colim}\left(\int R(E) \xrightarrow{\pi_{R(E)}} \mathcal{C} \xrightarrow{A} \mathcal{E}\right) \\ &= \mathrm{Colim}((A(-)/E) \xrightarrow{\partial} \mathcal{C} \xrightarrow{A} \mathcal{E}) \cong E. \end{aligned}$$

The following fact on the above adjunction $L \dashv R$ is known.

Proposition 2.4 (See [2]) *Let $L \dashv R$ be the adjunction of Proposition 2.1. Then the following three conditions are equivalent to each other.*

1. *The functor $A : \mathcal{C} \to \mathcal{E}$ is dense.*
2. *The right adjoint functor R is full and faithful.*
3. *The counit $\varepsilon : LR \to \mathrm{Id}_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ of the adjunction is a natural isomorphism, i.e., the E-component $\varepsilon_E : LR(E) \to E$ is isomorphic for each object $E \in \mathcal{E}$.*

Proof. Because the equivalence between 2 and 3 is well-known, we have only to show the equivalence between 1 and 3. By the adjunction $L \dashv R$ in Proposition 2.1, we have the bijective correspondence

$$\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(R(E), R(E)) \cong \mathcal{E}(LR(E), E)$$

for object $E \in \mathcal{E}$ and $R(E) \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$, and the counit $\varepsilon_E : LR(E) \to E$ corresponds to the identity transformation $\mathrm{id}_{R(E)}$ under this bijective correspondence.

If we assume A is dense, then we have ε_E is isomorphic. Conversely if we assume ε_E is isomorphic, then the canonical cocone has the universality. Hence 1 and 3 are equivalent. □

When the functor A is dense, the functor $R : \mathcal{E} \to \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is full and faithful by Proposition 2.4. Consequently the category $\mathcal{E}$ is a reflective subcategory of $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$. Since the category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ is complete, we obtain by using [1, Proposition 3.5.3]:

Proposition 2.5 *If the functor $A : \mathcal{C} \to \mathcal{E}$ is dense from a small category $\mathcal{C}$ to a cocomplete category $\mathcal{E}$, then the category $\mathcal{E}$ is complete. Hence a cocomplete category with a small, dense subcategory is complete.*

The monic arrows in $\mathcal{E}$ are preserved by the functor R.

Lemma 2.6 *The $\mathcal{E}$ arrow $m : E_1 \to E_2$ is monic if and only if the arrow $R(m) : R(E_1) \to R(E_2)$ is monic in $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.*

Proof. First we assume $m : E_1 \rightarrowtail E_2$ be a monic arrow in $\mathcal{E}$. Let $\alpha, \beta : P \to R(E_1)$ be $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ arrows. Suppose that $R(m) \circ \alpha = R(m) \circ \beta$. We must show $\alpha_C = \beta_C$ at each object $C \in \mathcal{C}$. Let x be any element of $P(C)$. Then we have $R(m)_C(\alpha_C(x)) = R(m)_C(\beta_C(x))$. Since $R(m)_C(\alpha_C(x)) = m \circ \alpha_C(x)$ and $R(m)_C(\beta_C(x)) = m \circ \beta_C(x)$, we have $m \circ \alpha_C(x) = m \circ \beta_C(x)$, whence $\alpha_C(x) = \beta_C(x)$ because m is monic. It follows then $\alpha_C = \beta_C$ at each $C \in \mathcal{C}$ since x was arbitrary. Thereby we have $\alpha = \beta$ and conclude that $R(m)$ is monic in $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

The reverse direction follows from the fact that a faithful functor reflects monics. □

3. The subobject classifier

Let $\mathcal{E}$ be a cocomplete category and assume, throughout this section, the existence of a dense functor $A : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$ to $\mathcal{E}$.

We consider the condition under which such $\mathcal{E}$ has a subobject classifier. As we have seen in the previous section, the category $\mathcal{E}$ is complete by Proposition 2.5. Hence we have the following two properties for the category $\mathcal{E}$:

- There exists a terminal object 1 in $\mathcal{E}$.
- The pullbacks exist in $\mathcal{E}$. In particular, the pullbacks of monic arrows exist in $\mathcal{E}$. Hence there exists a subobject functor $\mathrm{Sub} : \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$.

We first define the $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ arrows.

Let $j : E_0 \rightarrowtail E$ be any subobject of $E \in \mathcal{E}$. We define a $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ arrow $\chi_j : R(E) \to \mathrm{Sub}(A(-))$ whose C component $\chi_{jC} : R(E)(C) \to \mathrm{Sub}(A(C))$ is given by

$$\chi_{jC}(f) = f^{-1}(j)$$

for $f \in R(E)(C) = \mathcal{E}(A(C), E)$. In the above definition, $f^{-1}(j)$ is a subobject of $A(C)$ defined by the following pullback:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, tail, "f^{-1}(j)"'] & E_0 \arrow[d, tail, "j"] \\ A(C) \arrow[r, "f"'] & E \end{tikzcd}$$

Next we define a $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ arrow $\xi : R(1) \to \mathrm{Sub}(A(-))$ whose C component ξ_C is given by

$$\xi_C(!_{A(C)}) = \mathrm{id}_{A(C)},$$

where $!_E$ denote the unique $\mathcal{E}$ arrow from E to 1.

Then we have the following result.

Lemma 3.1 *For each subobject $j : E_0 \rightarrowtail E$ of $E \in \mathcal{E}$, the following diagram is a pullback in* $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$:

$$\begin{tikzcd}
R(E_0) \arrow[r, "R(!_{E_0})"] \arrow[d, "R(j)"'] & R(1) \arrow[d, "\xi"] \\
R(E) \arrow[r, "\chi_j"'] & \mathrm{Sub}(A(-))
\end{tikzcd}$$

Proof. We examine only the commutativity of the above diagram, since the other assertion is shown by routine arguments.

We fix a subobject $j : E_0 \rightarrowtail E$. In order to show the commutativity of the above diagram, we check the commutativity at each C component. Take any $f \in R(E_0)(C) = \mathcal{E}(A(C), E_0)$. Then we have

$$\chi_{j_C}(R(j)_C(f)) = \chi_{j_C}(j \circ f) = \mathrm{id}_{A(C)}$$

since $(j \circ f)^{-1}(j) = \mathrm{id}_{A(C)}$, and

$$\xi_C(R(!_{E_0})_C(f)) = \xi_C(!_{E_0} \circ f) = \xi_C(!_{A(C)}) = \mathrm{id}_{A(C)}.$$

Then $\chi_{j_C} \circ R(j)_C = \xi_C \circ R(!_{E_0})_C$ since f was arbitrary. Hence the commutativity $\chi_j \circ R(j) = \xi \circ R(!_{E_0})$ because C was also arbitrary. □

Recall the condition that the category $\mathcal{E}$ has a subobject classifier is given by the statement that the subobject functor Sub : $\mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ is representable. Here, the assumption on $\mathcal{E}$, that it has a dense functor $A : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$, cut down the condition as follows.

Theorem 3.2 *Let $A : \mathcal{C} \to \mathcal{E}$ be a dense functor from a small category $\mathcal{C}$ to a cocomplete category $\mathcal{E}$. Then the following two conditions are equivalent:*

1. *The presheaf* $\mathrm{Sub}(A(-)) \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ *is in $\mathcal{E}$, i.e., there exists an object $\Omega \in \mathcal{E}$ such that the following isomorphism holds.*

$$\mathrm{Sub}(A(-)) \cong \mathcal{E}(A(-), \Omega) = R(\Omega) \tag{2}$$

2. *There exists a subobject classifier in $\mathcal{E}$.*

Proof. 1. ⇒ 2.

Assume that the presheaf $\mathrm{Sub}(A(-))$ is in $\mathcal{E}$. Consider a $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ arrow from $R(1)$ to $R(\Omega)$ given by the composition of the arrow ξ and the isomorphism of the assumption (2). Since the functor R is full and faithful, this

$\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ arrow determines a unique $\mathcal{E}$ arrow: $\mathbf{t} : 1 \to \Omega$. This $\mathbf{t}$ is obviously monic by Lemma 2.6.

Now we show the following to show the existence of the subobject classifier in $\mathcal{E}$:

(i) For each object $E \in \mathcal{E}$, there is a bijection

$$\mathrm{Sub}(E) \cong \mathcal{E}(E, \Omega),$$

(ii) This bijection is natural for $E \in \mathcal{E}$.

First we show the existence of bijections. Define a map $\Phi_E : \mathrm{Sub}(E) \to \mathcal{E}(E, \Omega)$ by sending a subobject $j \in \mathrm{Sub}(E)$ to an $\mathcal{E}$ arrow $g : E \to \Omega$ such that $R(g)$ is equal to the composite of $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ arrow χ_j and the isomorphism of the assumption (2), where the well-definedness of this map Φ_E is assured by the fullness and the faithfulness of the functor R.

Then, for the above subobject j of E and the arrow $g = \Phi_E(j)$, the following diagram is a pullback in $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ by Lemma 3.1.

$$\begin{tikzcd}
R(E_0) \arrow[r] \arrow[d, "R(j)"'] & R(1) \arrow[d, "R(\mathbf{t})"] \\
R(E) \arrow[r, "R(g)"'] & R(\Omega)
\end{tikzcd}$$

Hence we obtain the following pullback in $\mathcal{E}$ since R is full and faithful.

$$\begin{tikzcd}
E_0 \arrow[r] \arrow[d, tail, "j"'] & 1 \arrow[d, tail, "\mathbf{t}"] \\
E \arrow[r, "g"'] & \Omega
\end{tikzcd}$$

In particular, for each $C \in \mathcal{C}$ the map $\Phi_{A(C)}$ coincides with the C-component of the natural isomorphism (2). Moreover the following pullback holds for each $r \in \mathrm{Sub}(A(C))$:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "r"'] & 1 \arrow[d, "\mathbf{t}"] \\ A(C) \arrow[r, "\Phi_{A(C)}(r)"'] & \Omega \end{tikzcd}$$

This means that each correspondence under each C component $\mathrm{Sub}(A(C)) \cong \mathcal{E}(A(C), \Omega)$ of (2) gives a pullback as above.

Now we show that Φ_E is injective. Suppose $i, j \in \mathrm{Sub}(E)$ and satisfies $\Phi_E(i) = \Phi_E(j) = g$. Then there exist two pullback diagrams: One is a diagram which pulls back $\mathbf{t}$ along g producing i, and the other is pulling back $\mathbf{t}$ along g producing j. Hence $i \cong j$ and Φ_E is injective.

Next we show that Φ_E is surjective. Fix any $g \in \mathcal{E}(E, \Omega)$ and take a subobject j of E given by pulling back $\mathbf{t}$ along g. Put $h = \Phi_E(j)$ and we show $g = h$ for this h. By the previous observation for Φ_E, we have the following pullback diagram:

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d, tail, "j"'] & 1 \arrow[d, tail, "\mathbf{t}"] \\ E \arrow[r, "h"'] & \Omega \end{tikzcd}$$

Remember that each object $E \in \mathcal{E}$ is given by the colimit of the canonical diagram

$$A(-)/E \xrightarrow{\partial} \mathcal{C} \xrightarrow{A} \mathcal{E}$$

since the functor $A : \mathcal{C} \to \mathcal{E}$ is dense. Let $\mu : A \circ \partial \to E$ be the canonical cocone of above diagram. Consider the subobjects given by pulling back j along the coprojections of μ. Here each subobject determines an $\mathcal{E}$ arrow with codomain Ω by the isomorphism of assumption (2), which amounts to determine a cocone $\nu : A \circ \partial \to \Omega$ of a canonical diagram of E with the vertex Ω.

As we have seen in the previous observation, each pair of a subobject and an arrow under the bijective correspondence given by the C-component of (2) constitute a pullback diagram. Hence we have

$$h \circ \mu_{(C,f)} = \nu_{(C,f)} = g \circ \mu_{(C,f)}$$

for each object $(C, f) \in A(-)/E$ because $\nu_{(C,f)}$ is the unique arrow which corresponds both to the subobject pulling back $\mathbf{t}$ along $g \circ \mu_{(C,f)}$ and the one pulling back $\mathbf{t}$ along $h \circ \mu_{(C,f)}$.

Since the colimit of the canonical diagram $A \circ \partial$ is E, the cocone ν determines a unique arrow from E to Ω. Hence $g = h$ and we have shown that Φ_E is surjective.

The naturality of the bijection follows by routine diagram chasing.

2. $\Rightarrow$ 1.

The reverse direction is obvious. □

Note that when the presheaf $\mathrm{Sub}(A(-))$ is in $\mathcal{E}$, namely the isomorphism $\mathrm{Sub}(A(-)) \cong R(\Omega)$ holds, then the subobject classifier is given by the $\mathcal{E}$ arrow $\mathbf{t} : 1 \to \Omega$ such that $R(\mathbf{t})$ is equal to the composite of $\xi : R(1) \to \mathrm{Sub}(A(-))$ and the isomorphism.

By Theorem 3.2, we obtain

Corollary 3.3 *Under the same assumption as in Theorem* 3.2, *the unit* $\eta_{\mathrm{Sub}(A(-))} : \mathrm{Sub}(A(-)) \to RL\,\mathrm{Sub}(A(-))$ *of the adjunction has a retraction if and only if there exists a subobject classifier in* $\mathcal{E}$.

Proof. First we assume $\eta_{\mathrm{Sub}(A(-))}$ has a retraction. Then the unit $\eta_{\mathrm{Sub}(A(-))}$ becomes an isomorphism since R is full and faithful. By using the Theorem 3.2, the category $\mathcal{E}$ has a subobject classifier.

Next we assume that $\mathcal{E}$ has a subobject classifier. Then there exists an object $\Omega \in \mathcal{E}$ with an isomorphism $\mathrm{Sub}(A(-)) \cong R(\Omega)$ by Theorem 3.2. Because the unit η is a natural transformation, we have the following commutative diagram for the isomorphism:

$$\begin{tikzcd}
\mathrm{Sub}(A(-)) \arrow[r] \arrow[d, "\eta_{\mathrm{Sub}(A(-))}"'] & R(\Omega) \arrow[d, "\eta_{R(\Omega)}"] \\
RL\,\mathrm{Sub}(A(-)) \arrow[r] & RLR(\Omega)
\end{tikzcd}$$

In the above diagram, both the top and the bottom horizontal arrows are isomorphisms. Moreover the right vertical $\eta_{R(\Omega)}$ is an isomorphism because the counit $\varepsilon_\Omega : LR(\Omega) \to \Omega$ is an isomorphism by Proposition 2.4. This implies $\eta_{\mathrm{Sub}(A(-))}$ is an isomorphism, and hence has a retraction. □

4. The construction of $\mathcal{E}$ objects via presheaf category

4.1. Preliminary

Now we study the construction of $\mathcal{E}$ objects from the category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ of presheaves by using the left adjoint functor $L : \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{E}$.

To begin with, we introduce a notation for the "components" of the unit η of the adjunction $L \dashv R$. Because the P component of the unit η is a natural transformation $\eta_P : P \to RLP$, the component of η_P at object $C \in \mathcal{C}$ is a map $\eta_{PC} : P(C) \to RLP(C)(=\mathcal{E}(A(C), LP))$. Define an $\mathcal{E}$ arrow $\kappa_p^{LP} : A(C) \to LP$ by

$$\kappa_p^{LP} := \eta_{PC}(p)$$

for each object $C \in \mathcal{C}$ and $p \in P(C)$. Then we have the following Corollary to Proposition 2.1.

Corollary 4.1 *Let P be a presheaf over $\mathcal{C}$. The collection of $\mathcal{E}$ arrows*

$$\left\{\kappa_p^{LP} : A(C) \to LP \mid (C,p) \in \int P\right\} \tag{3}$$

constitutes a universal cocone of the diagram $A \circ \pi_P$.

Proof. By the naturality condition on the unit $\eta_P : P \to RLP$, the collection (3) is a cocone of $A \circ \pi_P$. From Proposition 2.1, we have a bijective correspondence

$$\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}(P, RLP) \cong \mathcal{E}(LP, LP) \tag{4}$$

for object $P \in \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ and $LP \in \mathcal{E}$. The identity arrow id_{LP} on $LP \in \mathcal{E}$ corresponds to η_P under the correspondence (4), which means that the cocone (3) is universal. □

For each $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ arrow $\alpha : P \to Q$, the $\mathcal{E}$ arrow $L\alpha : LP \to LQ$ has a following property.

Lemma 4.2 *Let $\alpha : P \to Q$ be a $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ arrow. For each $p \in P(C)$ with*

$C \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} & A(C) \arrow[dl, "\kappa_p^{LP}"'] \arrow[dr, "\kappa_{\alpha_C(p)}^{LQ}"] & \\ LP \arrow[rr, "L\alpha"'] & & LQ \end{tikzcd}$$

Proof. From the naturality condition on the unit $\eta : \mathrm{id}_{\mathbf{Set}^{\mathcal{C}^{op}}} \to RL$ for $\mathbf{Set}^{\mathcal{C}^{op}}$ arrow α, we have the following commutative diagram:

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "\eta_P"'] & Q \arrow[d, "\eta_Q"] \\ RLP \arrow[r, "R(L\alpha)"'] & RLQ \end{tikzcd}$$

By considering the C-component of the above diagram, we get

$$\kappa_{\alpha_C(p)}^{LQ} = L\alpha \circ \kappa_p^{LP}$$

for each $p \in P(C)$. □

The counit ε_E has a following property.

Lemma 4.3 *The following diagram commutes for each $\mathcal{E}$ arrow $f : A(C) \to E$ with $C \in \mathcal{C}$:*

$$\begin{tikzcd} & A(C) \arrow[dl, "\kappa_f^{LR(E)}"'] \arrow[dr, "f"] & \\ LR(E) \arrow[rr, "\varepsilon_E"'] & & E \end{tikzcd}$$

Proof. From one of the triangular identities of the adjunction $L \dashv R$, we have the following commutative diagram.

$$\begin{tikzcd} R(E) \arrow[r, "\eta_{R(E)}"] \arrow[dr, "\mathrm{id}_{R(E)}"'] & RLR(E) \arrow[d, "R(\varepsilon_E)"] \\ & R(E) \end{tikzcd}$$

The C component of above diagram gives

$$f = \varepsilon_E \circ \kappa_f^{LR(E)}$$

for every $f \in \mathcal{E}(A(C), E)$. $\square$

4.2. The terminal object

We can give the terminal object in $\mathcal{E}$ as follows.

Lemma 4.4 *Under the same assumption as in Theorem 3.2, let* $\mathbf{1}$ *be a terminal object in* $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$. *Then the object* $L\mathbf{1} \in \mathcal{E}$ *is a terminal object in* $\mathcal{E}$.

Proof. Let 1 be a terminal object in $\mathcal{E}$. Then $\mathbf{1} \cong R(1)$. By applying the functor L and by using denseness of A, we have $L\mathbf{1} \cong LR(1) \cong 1$. Hence $L\mathbf{1}$ is a terminal object in $\mathcal{E}$. $\square$

4.3. The subobject classifier

In case there exists a subobject classifier in $\mathcal{E}$, we can give it by using the presheaves. This enables us to restate the Theorem 3.2 in a bit different form, which is more useful for us to check its existence in a concrete category.

Corollary 4.5 *Under the same assumption as in Theorem 3.2, the* $\mathcal{E}$ *arrow* $L\xi : LR(1) \to L\,\mathrm{Sub}(A(-))$ *is a subobject classifier if the category* $\mathcal{E}$ *has a subobject classifier.*

Proof. Suppose that we have a subobject classifier in $\mathcal{E}$. By using Corollary 3.3, the unit $\eta_{\mathrm{Sub}(A(-))} : \mathrm{Sub}(A(-)) \to RL\,\mathrm{Sub}(A(-))$ is an isomorphism. Now recall the construction of the subobject classifier in the proof of Theorem 3.2, according to which, the subobject classifier is given by the $\mathcal{E}$ arrow $\mathbf{t} : 1 \to L\,\mathrm{Sub}(A(-))$ which satisfies

$$R(\mathbf{t}) = \eta_{\mathrm{Sub}(A(-))} \circ \xi. \tag{5}$$

Since R is full and faithful, both the existence and the uniqueness of such an arrow are guaranteed.

By the naturality diagram of the unit η for the $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ arrow ξ and by the equation (5), we have the following commutative diagram:

$$\begin{tikzcd}
R(1) \arrow[r, "\eta_{R(1)}"] \arrow[dr, "R(\mathbf{t})"'] & RLR(1) \arrow[d, "R(L\xi)"] \\
 & RL\,\mathrm{Sub}(A(-))
\end{tikzcd}$$

Because the unit $\eta_{R(1)}$ is an isomorphism and the functor R is full and faithful, the underlying subobjects $\mathbf{t}$ and $L\xi$ of $L\,\mathrm{Sub}(A(-))$ are isomorphic. Hence we conclude that the $\mathcal{E}$ arrow $L\xi : LR(1) \to L\,\mathrm{Sub}(A(-))$ is a subobject classifier. $\square$

By Corollary 3.3, if there exists a subobject classifier in $\mathcal{E}$, then the unit $\eta_{\mathrm{Sub}(A(-))}$ has a retraction. Now we characterize this retraction.

Corollary 4.6 *Under the same assumptions as in Theorem 3.2, there exists a subobject classifier in $\mathcal{E}$ if and only if $\chi_{L\xi}$ is a retraction of the unit $\eta_{\mathrm{Sub}(A(-))} : \mathrm{Sub}(A(-)) \to RL\,\mathrm{Sub}(A(-))$.*

Proof. Assume the existence of a subobject classifier in $\mathcal{E}$. Then the unit $\eta_{\mathrm{Sub}(A(-))} : \mathrm{Sub}(A(-)) \to RL\,\mathrm{Sub}(A(-))$ is an isomorphism by Corollary 3.3. Since the unit η is a natural transformation, the following diagram commutes, and since both $\eta_{R(1)}$ and $\eta_{\mathrm{Sub}(A(-))}$ are isomorphic, the diagram is a pullback in $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

$$\begin{tikzcd}
R(1) \arrow[r, "\eta_{R(1)}"] \arrow[d, "\xi"'] & RLR(1) \arrow[d, "R(L\xi)"] \\
\mathrm{Sub}(A(-)) \arrow[r, "\eta_{\mathrm{Sub}(A(-))}"'] & RL\,\mathrm{Sub}(A(-))
\end{tikzcd}$$

By using Lemma 3.1 for monic arrow $L\xi$ we have the following pullback diagram:

$$\begin{tikzcd} RLR(1) \arrow[r] \arrow[d, "R(L\xi)"'] & R(1) \arrow[d, "\xi"] \\ RL\,\mathrm{Sub}(A(-)) \arrow[r, "\chi_{L\xi}"'] & \mathrm{Sub}(A(-)) \end{tikzcd}$$

By the above two diagrams and the pullback lemma, we get the following pullback diagram:

$$\begin{tikzcd} RLR(1) \arrow[r] \arrow[d, "R(L\xi)"'] & RLR(1) \arrow[d, "R(L\xi)"] \\ RL\,\mathrm{Sub}(A(-)) \arrow[r, "\eta_{\mathrm{Sub}(A(-))} \circ \chi_{L\xi}"'] & RL\,\mathrm{Sub}(A(-)) \end{tikzcd}$$

Because the functor R is full and faithful, we have the following pullback diagram, where $h : L\,\mathrm{Sub}(A(-)) \to L\,\mathrm{Sub}(A(-))$ is a unique $\mathcal{E}$ arrow satisfying $R(h) = \eta_{\mathrm{Sub}(A(-))} \circ \chi_{L\xi}$:

$$\begin{tikzcd} LR(1) \arrow[r] \arrow[d, "L\xi"'] & LR(1) \arrow[d, "L\xi"] \\ L\,\mathrm{Sub}(A(-)) \arrow[r, "h"'] & L\,\mathrm{Sub}(A(-)) \end{tikzcd}$$

Since $L\xi$ is a subobject classifier by Lemma 4.5, and since the characteristic arrow of the subobject classifier $L\xi$ is an identity on $L\,\mathrm{Sub}(A(-))$, we have $h = \mathrm{id}_{L\,\mathrm{Sub}(A(-))}$. Hence we obtain $\eta_{\mathrm{Sub}(A(-))} \circ \chi_{L\xi} = \mathrm{id}_{RL\,\mathrm{Sub}(A(-))}$, i.e., $\chi_{L\xi}$ is an inverse of $\eta_{\mathrm{Sub}(A(-))}$. Thereby $\chi_{L\xi}$ is a retraction of $\eta_{\mathrm{Sub}(A(-))}$.

The reverse direction follows by Theorem 3.3. □

The above Corollary 4.6 enable us to check effectively the existence of subobject classifier in concrete cases. In order to show the existence of subobject classifier in $\mathcal{E}$, we have only to check the following equation holds in $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$.

$$\chi_{L\xi} \circ \eta_{\mathrm{Sub}(A(-))} = \mathrm{id}_{\mathrm{Sub}(A(-))} \tag{6}$$

Here the equation (6) is equivalent to the following condition: When the universal cocone of the diagram $\int \mathrm{Sub}(A(-)) \to \mathcal{C} \to \mathcal{E}$ is given, in accordance with the notation in Section 4.1, by the collection

$$\left\{ \kappa_r^{L\,\mathrm{Sub}(A(-))} : A(C) \to L\,\mathrm{Sub}(A(-)) \mid (C,r) \in \int \mathrm{Sub}(A(-)) \right\},$$

the following diagram is a pullback for each coprojection $\kappa_r^{L\,\mathrm{Sub}(A(-))}$ of the cocone.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "r"'] & LR(1) \arrow[d, "L\xi"] \\ A(C) \arrow[r, "\kappa_r^{L\,\mathrm{Sub}(A(-))}"'] & L\,\mathrm{Sub}(A(-)) \end{tikzcd}$$

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