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# Moser type theorem for toric hyperKähler quotients

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**Abstract.** We consider the symplectic geometry of toric hyperKähler quotients. Under a mild condition, we obtain that toric hyperKähler quotients have stability about its underlying symplectic structures.

*Key words*: hyperKähler quotients, symplectic manifolds, Hamiltonian torus actions, Moser theorem.

## 1. Introduction

Symplectic manifolds have the properties of both softness and hardness. For softness, there is a classical theorem due to Moser [6].

**Theorem 1.1** (Moser) *Let $M$ be a closed manifold and $\{\omega_t\}_{0\leq t\leq 1}$ a smooth family of cohomologous symplectic forms on $M$. Then there exists $\{\phi_t\}_{0\leq t\leq 1}$ a smooth family of diffeomorphisms of $M$ such that $\phi_t^*\omega_t = \omega_0$ for all $t \in [0, 1]$.*

This theorem is proved by constructing a family of vector fields $\{Z_t\}_{0\leq t\leq 1}$ whose integral flows induce $\{\phi_t\}_{0\leq t\leq 1}$. Therefore the completeness of these vector fields is necessary. But this is automatically satisfied since $M$ is compact. In this paper, we prove that an analog of this theorem holds in the case of not necessarily compact but complete hyperKähler quotients under a mild condition.

Let $(M, g, I, J, K)$ be a complete hyperKähler manifold, i.e. $g$ is a complete Riemannian metric and $I$, $J$, $K$ are almost complex structures of $M$ satisfying

(i) $g$ is Hermitian with respect to $I$, $J$, $K$,
(ii) $I^2 = -1$, $J^2 = -1$, $K^2 = -1$, $IJ = K$, $JK = I$, $KI = J$,
(iii) $\nabla I = 0$, $\nabla J = 0$, $\nabla K = 0$,

where $\nabla$ is the Levi-Civita connection of $g$.

We define the 2-forms $\omega_I$, $\omega_J$, $\omega_K$ by $\omega_I(X, Y) = g(IX, Y)$, etc. From the condition above, it follows that $I$, $J$, $K$ are integrable and $\omega_I$, $\omega_J$, $\omega_K$

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are Kähler with respect to $I$, $J$, $K$ respectively.

Let $G$ be a compact connected Lie group and $\mathfrak{g}$ its Lie algebra. We assume that $G$ acts on $M$ preserving its hyperKähler structure with a moment map

$$\mu = (\mu_I, \mu_J, \mu_K) : M \to \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*,$$

i.e. $G$ preserves the metric $g$ and acts on $M$ in a Hamiltonian way for every symplectic forms $\omega_I$, $\omega_J$, $\omega_K$ with moment maps $\mu_I$, $\mu_J$, $\mu_K$ respectively. Note that $\mu$ is a $G$-equivariant map.

We denote by $\mathcal{C}$ the set of $G$-invariant elements of $\mathfrak{g}^*$. Let $\xi = (\xi_I, \xi_J, \xi_K) \in \mathcal{C} \times \mathcal{C} \times \mathcal{C}$ be a regular value of $\mu$. Then $M^\xi = \mu^{-1}(\xi)$ is a $G$-invariant smooth submanifold of $M$. If $G$ acts on $M^\xi$ freely, the quotient manifold $M^\xi/G$ has the three induced Kähler forms $\omega_I^\xi$, $\omega_J^\xi$, $\omega_K^\xi$ from $\omega_I$, $\omega_J$, $\omega_K$ and the induced metric from $g^\xi$ as the Riemannian submersion $M^\xi \to M^\xi/G$. These define the hyperKähler structure on $M^\xi/G$. This manifold $M^\xi/G$ is called the hyperKähler quotient of $M$ by the moment map $\mu$. For further detail, we refer to Section 3 in [4].

In Section 2, we prove the Moser type theorem for hyperKähler quotients. Roughly speaking, under a mild condition, the symplectic diffeomorphism class of the hyperKähler quotient $(M^\xi/G, \omega_I^\xi)$ is independent of $\xi_J$ and $\xi_K$ (Theorem 2.1). By Duistermaat and Heckman [3], we note that the cohomology class of $\omega_I^\xi$ changes for various choice of $\xi_I$ in general.

The idea of the proof is as follows. We take an embedded path $\gamma = (\gamma_I, \gamma_J, \gamma_K) : [0,1] \to (\mathcal{C}\times\mathcal{C}\times\mathcal{C}) \cap \mu(M)$ such that $\gamma_I$ is a constant $\xi_I$ which is a regular value of $\mu_I$. We also denote by $\gamma$ its image. Then $M^\gamma = \mu^{-1}(\gamma)$ is a $G$-invariant smooth submanifold of $M$. If $G$ acts on $M^\gamma$ freely, we obtain a family of symplectic manifolds $(M^{\gamma(t)}/G, \omega_I^{\gamma(t)})$, $t \in [0,1]$ in the manifold $M^\gamma/G$. These manifolds are considered as symplectic submanifolds of the symplectic quotient of $(M, \omega_I)$ by the moment map $\mu_I$ at the point $\xi_I$. Because $\gamma_I$ is constant, $M^\gamma/G$ has the induced closed 2-form $\omega_I^\gamma$ from $\omega_I$. This closed 2-form is necessarily degenerate. By using the kernel of $\omega_I^\gamma$, we construct the vector field $Z$ on $M^\gamma/G$. If $Z$ is complete, we can construct symplectic diffeomorphisms between $(M^{\gamma(0)}/G, \omega_I^{\gamma(0)})$ and $(M^{\gamma(t)}/G, \omega_I^{\gamma(t)})$ for all $t \in [0,1]$ by using its integral flows.

The assumptions we used here consist of two parts:

(i) the action of $G$ on $M^\gamma$ is free.

(ii) the vector field $Z$ on $M^\gamma/G$ is complete.

In Section 3, we consider toric hyperKähler linear actions on quarternionic vector spaces with certain moment maps. A sufficient condition for (i) is given by Konno [5]. We prove that the assumption (ii) holds for this case (Theorem 3.1).

This problem was suggested to me by Professor K. Ono. I should like to express my gratitude to him for suggesting this problem.

## 2. Moser type theorem

We use here the same notations in introduction.

Let $(M, g, I, J, K)$ be a complete hyperKähler manifold and $G$ a compact connected Lie group. We denote by $\mathfrak{g}$ its Lie algebra. We assume that $G$ acts on $M$ preserving its hyperKähler structure with a moment map

$$\mu = (\mu_I, \mu_J, \mu_K) : M \to \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*.$$

Let $\xi_I \in \mathcal{C}$ be a regular value of $\mu_I$. We define the space $\mathcal{B}$ by

$$\mathcal{B} = (\xi_I \times \mathcal{C} \times \mathcal{C}) \cap \mu(M).$$

We take an embedded path

$$\gamma = (\gamma_I, \gamma_J, \gamma_K) : [0,1] \to \mathcal{B}.$$

Let $Y^1, \ldots, Y^n$ be a basis of $\mathfrak{g}$. We define the matrix $g(x) = (g_{ij}(x))$ by

$$g_{ij}(x) = g(\underline{Y}^i_x, \underline{Y}^j_x), \qquad x \in M^\gamma,$$

where we denote by $\underline{Y}$ the fundamental vector field associated to $Y \in \mathfrak{g}$. By Lemma 2.2 below, every point of $\mathcal{B}$ is a regular value of $\mu$. So the action of $G$ on $M^\gamma$ is locally free. Hence we have $\det g(x) \neq 0$.
We denote by $g^{-1}(x) = (g^{ij}(x))$ the inverse matrix of $g(x)$. We define the function $\nu : M^\gamma \to \mathbb{R}$ by

$$\nu(x) = |g^{-1}(x)| = \left( \sum_{i,j=1}^{n} |g^{ij}(x)|^2 \right)^{\frac{1}{2}}.$$

Note that the boundedness of $\nu$ does not depend on the choice of a basis of $\mathfrak{g}$.

The main theorem in this section is the following:

**Theorem 2.1** *If $G$ acts on $M^\gamma$ freely and $\nu : M^\gamma \to \mathbb{R}$ is a bounded* *function, then the manifolds* $(M^{\gamma(t)}/G, \omega_I^{\gamma(t)})$, $t \in [0,1]$ *are symplectic diffeomorphic each other.*

This theorem follows from the arguments below.

First of all, we shall review the tangent spaces of the manifolds $M^\gamma$ and $M^\xi$. We set $\xi = \mu(x) = \gamma(t)$. The tangent space of $M^\gamma$ at $x$ is the inverse image of $T_{\mu(x)}\gamma$ by $(d\mu)_x$. The tangent vector $X_x \in T_xM$ belongs to $T_xM^\gamma$ if and only if there exists $s \in \mathbb{R}$ and the following equations are satisfied:

$$\begin{aligned} g(I\underline{Y}_x, X_x) &= 0 \\ g(J\underline{Y}_x, X_x) &= s\langle \dot{\gamma}_J(t), Y\rangle \\ g(K\underline{Y}_x, X_x) &= s\langle \dot{\gamma}_K(t), Y\rangle \qquad \text{for all } \; Y \in \mathfrak{g}. \end{aligned} \tag{2.1}$$

The tangent space of $M^\xi$ at $x$ is the kernel of $(d\mu)_x$. The tangent vector $X_x \in T_xM$ belongs to $T_xM^\xi$ if and only if we can take $s = 0$ in the above condition. The tangent space of $G$-orbit at $x$ is generated by the fundamental vector fields. Every element of $T_x(Gx)$ is represented by $\underline{Y}_x$ for some $Y \in \mathfrak{g}$. It is easy to see that $T_x(Gx) \subset T_xM^\xi \subset T_xM^\gamma$.

**Lemma 2.2** *The vector subspaces of* $T_xM$

$$T_x(Gx),\ IT_x(Gx),\ JT_x(Gx) \ \textit{ and } \ KT_x(Gx)$$

*are mutually orthogonal with respect to* $g$.

*Proof.* Since $\mu_I(hx) = \mu_I(x)$ for all $h \in G$, it follows that $(d\mu_I)_x\underline{X}_x = 0$ for all $X \in \mathfrak{g}$. By the property of moment maps, for every $X, Y \in \mathfrak{g}$, we have $g(I\underline{Y}_x, \underline{X}_x) = \langle (d\mu_I)_x\underline{X}_x, Y\rangle = 0$. Hence $T_x(Gx)$ and $IT_x(Gx)$ are mutually orthogonal. The other cases are proved in the same way. □

**Lemma 2.3** *Every point of* $\mathcal{B}$ *is a regular value of* $\mu$.

*Proof.* Take an arbitrary $(\xi_I, \xi_J, \xi_K) \in \mathcal{B}$ and an arbitrary $x \in \mu^{-1}(\xi_I, \xi_J, \xi_K)$. Because $x$ is a regular value of $\mu_I$, we have $g_{ii}(x) \neq 0$. For every $(\xi'_I, \xi'_J, \xi'_K) \in \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$, we define the tangent vector in $T_xM$

$$X_x = \sum_{i=1}^{n} \frac{1}{g_{ii}(x)} \left( \langle \xi'_I, Y^i\rangle I\underline{Y}^i_x + \langle \xi'_J, Y^i\rangle J\underline{Y}^i_x + \langle \xi'_K, Y^i\rangle K\underline{Y}^i_x \right).$$

By using Lemma 2.2, it is easy to see that $(d\mu)_xX_x = (\xi'_I, \xi'_J, \xi'_K)$. Therefore $(d\mu)_x$ is surjective. □

We assume that $G$ acts on $M^\gamma$ freely. Then the canonical projection

$$p : M^\gamma \to M^\gamma/G$$

is submersion.

**Lemma 2.4** *The closed 2-form $\omega_I$ on $M^\gamma$ induces the closed 2-form $\omega_I^\gamma$ on $M^\gamma/G$ by*

$$(\omega_I^\gamma)_{p(x)}(X_{p(x)}, Y_{p(x)}) = \omega_I(X_x, Y_x), \quad X_{p(x)}, Y_{p(x)} \in T_{p(x)}(M^\gamma/G), \tag{2.2}$$

*where $X_x$ and $Y_x$ are any tangent vectors in $T_xM^\gamma$ which project to $X_{p(x)}$ and $Y_{p(x)}$ respectively.*

*Proof.* First, we show that the definition (2.2) is independent of the choice of a point in the $G$-orbit $p(x)$. From (2.1), we obtain that

$$\omega_I(T_xM^\gamma, T_x(Gx)) = 0. \tag{2.3}$$

For any $h \in G$, we take tangent vectors $X'_{hx}$ and $Y'_{hx}$ in $T_{hx}M^\gamma$ which project to $X_{p(x)}$ and $Y_{p(x)}$ respectively. Because of $p(hx) = p(x)$, it follows that both $(dh)_xX_x - X'_{hx}$ and $(dh)_xY_x - Y'_{hx}$ belong to $T_{hx}(Gx)$, where we identify the element $h \in G$ and the diffeomorphism $M^\gamma \to M^\gamma$, $x \mapsto hx$. By (2.3) and the $G$-invariance of $\omega_I$, we have $\omega_I(X'_{hx}, Y'_{hx}) = \omega_I(X_x, Y_x)$. So (2.2) is well-defined.

Next, we show that $\omega_I^\gamma$ is closed. By construction, we have $p^*(d\omega_I^\gamma) = d(p^*\omega_I^\gamma) = d\omega_I$. Since $p$ is submersion and $\omega_I$ is closed, so is $\omega_I^\gamma$. □

Because the manifold $(M^\xi/G, \omega_I^\xi)$ has codimension one in $(M^\gamma/G, \omega_I^\gamma)$ and the 2-form $\omega_I^\xi$ is non-degenerate, the 2-form $\omega_I^\gamma$ has 1-dimensional kernel.

We define the submersion

$$\overline{\mu} : M^\gamma/G \to \gamma$$

by

$$\overline{\mu}(p(x)) = \mu(x).$$

**Lemma 2.5** *The restriction of the differential of $\overline{\mu}$*

$$(d\overline{\mu})_{p(x)} : \ker(\omega_I^\gamma)_{p(x)} \to T_{\mu(x)}\gamma$$

*is a linear isomorphism.*

*Proof.* Because $(d\overline{\mu})_{p(x)}$ is a linear map between 1-dimensional vector spaces, it is enough to show that $(d\overline{\mu})_{p(x)}$ is non-trivial. Let $X_{p(x)} \in \ker(\omega_I^\gamma)_{p(x)}$ be a non-zero element. We take a tangent vector $X_x \in T_x M^\gamma$ which project to $X_{p(x)}$. If $X_x$ belongs to $T_x M^\xi$, it follows that $X_x = 0$ from the non-degeneracy of $\omega_I$ on $T_x M^\xi$. This contradicts that $X_{p(x)}$ is non-zero. Hence $X_x$ does not belong to $T_x M^\xi$. This means that $(d\mu)_x X_x \neq 0$. Hence we conclude that $(d\overline{\mu})_{p(x)} X_{p(x)} \neq 0$. □

We define the vector field $Z$ on $M^\gamma/G$ by the following conditions:

$$\begin{aligned} (d\overline{\mu})_{p(x)} Z_{p(x)} &= \dot{\gamma}(t) \\ Z_{p(x)} &\in \ker(\omega_I^\gamma)_{p(x)} \\ \mu(x) &= \gamma(t). \end{aligned} \tag{2.4}$$

By Lemma 2.5, $Z$ is uniquely determined by these conditions. Because $\dot{\gamma}(t)$ is non-zero, $Z$ is a nowhere vanishing vector field.

We shall consider the submersion

$$p : M^\gamma \to M^\gamma/G.$$

The vertical subspace $V_x$ of $T_x M^\gamma$ is defined by $T_x(Gx)$. The horizontal subspace $H_x$ of $T_x M^\gamma$ is defined by the orthogonal complement of $T_x(Gx)$ in $T_x M^\gamma$. The restriction of $(dp)_x$ to the horizontal subspace $H_x$ is a linear isomorphism between $H_x$ and $T_{p(x)}(M^\gamma/G)$. Therefore any tangent vector $X_{p(x)} \in T_{p(x)}(M^\gamma/G)$ has a unique horizontal lift $\widetilde{X}_x \in H_x$.

**Lemma 2.6** *The horizontal lift of $Z_{p(x)}$ is given by*

$$\widetilde{Z}_x = \sum_{i=1}^{n} a_J^i(x) J \underline{Y}^i_x + a_K^i(x) K \underline{Y}^i_x,$$

*where $a_J^i(x)$ and $a_K^i(x)$ are uniquely determined by the following equation*

$$\begin{pmatrix} g_{11}(x) & \cdots & g_{1n}(x) \\ \vdots & & \vdots \\ g_{n1}(x) & \cdots & g_{nn}(x) \end{pmatrix} \begin{pmatrix} a_J^1(x) & a_K^1(x) \\ \vdots & \vdots \\ a_J^n(x) & a_K^n(x) \end{pmatrix}$$

$$= \begin{pmatrix} \langle \dot\gamma_J(t), Y^1 \rangle & \langle \dot\gamma_K(t), Y^1 \rangle \\ \vdots & \vdots \\ \langle \dot\gamma_J(t), Y^n \rangle & \langle \dot\gamma_K(t), Y^n \rangle \end{pmatrix} \tag{2.5}$$

*Proof.* From (2.1), (2.5) and Lemma 2.2, it is easy to check that $(dp)_x \widetilde{Z}_x \in \ker(\omega_I^\gamma)_{p(x)}$, $\widetilde{Z}_x \in H_x$ and $(d\overline{\mu})_{p(x)}(dp)_x \widetilde{Z}_x = (d\mu)_x \widetilde{Z}_x = \dot\gamma(t)$. Because the restriction of $(d\overline{\mu})_{p(x)}$ to $\ker(\omega_I^\gamma)_{p(x)}$ is a linear isomorphism, we conclude that $(dp)_x \widetilde{Z}_x = Z_{p(x)}$. Therefore $\widetilde{Z}_x$ is the horizontal lift of $Z_{p(x)}$. □

**Lemma 2.7** *There exists some constant $K > 0$ such that*

$$g(\widetilde{Z}_x, \widetilde{Z}_x) \leq K\nu(x) \qquad \textit{for all } x \in M^\gamma.$$

*Proof.* We put

$$K_1 = \max_{t \in [0,1]} \left( \sum_{i=1}^n \langle \dot\gamma_J(t), Y^i \rangle^2 + \langle \dot\gamma_K(t), Y^i \rangle^2 \right)^{\frac{1}{2}}.$$

It follows from (2.5) that

$$|a_J^i(x)|, |a_K^i(x)| \leq K_1 \nu(x).$$

By using Lemma 2.2 and (2.5), $g(\widetilde{Z}_x, \widetilde{Z}_x)$ is estimated as follows:

$$\begin{aligned} g(\widetilde{Z}_x, \widetilde{Z}_x) &= \sum_{i,j=1}^n a_J^i(x) a_J^j(x) g_{ij}(x) + a_K^i(x) a_K^j(x) g_{ij}(x) \\ &= \sum_{i=1}^n a_J^i(x) \langle \dot\gamma_J(t), Y^i \rangle + a_K^i(x) \langle \dot\gamma_K(t), Y^i \rangle \\ &\leq 2K_1^2 \nu(x). \end{aligned}$$

□

**Proposition 2.8** *If $G$ acts on $M^\gamma$ freely and $\nu : M^\gamma \to \mathbb{R}$ is a bounded function, then the vector field $Z$ is complete. Its integral curves induce the symplectic diffeomorphism*

$$\phi_t : (M^{\gamma(0)}/G, \omega_I^{\gamma(0)}) \to (M^{\gamma(t)}/G, \omega_I^{\gamma(t)}) \quad \textit{for every } t \in [0,1].$$

*Proof.* By the boundedness of $\nu$ and Lemma 2.7, there exists some constant $K'$ such that

$$g(\widetilde{Z}_x, \widetilde{Z}_x) \leq K' \qquad \text{for all } \ x \in M^\gamma.$$

From this estimate and the completeness of $M$, it follows that $\widetilde{Z}$ is complete. By using its integral flows and the formulation (2.4), we can construct canonical identifications

$$\phi_t : M^{\gamma(0)}/G \to M^{\gamma(t)}/G.$$

From (2.4) and the closedness of $\omega_I^\gamma$ from Lemma 2.4, we have

$$\frac{d}{dt}(\iota_t \circ \phi_t)^* \omega_I^\gamma = (\iota_t \circ \phi_t)^* \mathcal{L}_Z \omega_I^\gamma = (\iota_t \circ \phi_t)^* (d i_Z \omega_I^\gamma + i_Z d\omega_I^\gamma) = 0,$$

where $\iota_t$ denotes the inclusion map $M^{\gamma(t)}/G \to M^\gamma/G$. Therefore we conclude that $\phi_t^* \omega_I^{\gamma(t)} = \omega_I^{\gamma(0)}$. □

## 3. Toric hyperKähler linear actions

In this section, we consider toric hyperKähler linear actions on quarternionic vector spaces.

Let $\mathbb{H} = \{a + bI + cJ + dK : a, b, c, d \in \mathbb{R}\}$ be the quarternion algebra and $\operatorname{Im}\mathbb{H}$ the purely quarternions in $\mathbb{H}$. The right $\mathbb{H}$-linear vector space

$$\mathbb{H}^N = \{x = (x_1, \ldots, x_N) : x_j \in \mathbb{H}\}$$

has the Euclidean metric $g$ of $\mathbb{R}^{4N}$ and the three complex structures $I$, $J$, $K$. These define the hyperKähler structure on $\mathbb{H}^N$. We denote by $\omega_I$, $\omega_J$, $\omega_K$ the associated Kähler forms.

The real torus

$$T^N = \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N : |z_j| = 1\}$$

acts on $\mathbb{H}^N$ by

$$z \cdot x = (z_1 x_1, \ldots, z_N x_N).$$

This action preserves the hyperKähler structure on $\mathbb{H}^N$. We denote by $\mathfrak{t}^N$ the Lie algebra of $T^N$. Let $X^1, \ldots, X^N$ be a basis of $\mathfrak{t}^N$ satisfying

$$\exp\left(\sum_{j=1}^N t_j X^j\right) = (e^{2\pi\sqrt{-1}t_1}, \ldots, e^{2\pi\sqrt{-1}t_N}),$$

where $\exp : \mathfrak{t}^N \to T^N$ is the exponential map. We denote by $\mathfrak{t}^N_{\mathbb{Z}}$ the kernel of exp. Let $u_1, \dots, u_N$ be the dual basis of $X^1, \dots, X^N$. We identify $(\mathfrak{t}^N)^*$ with $\mathbb{R}^N$ by using this basis. The moment map

$$\mu_0 : \mathbb{H}^N \to (\mathfrak{t}^N)^* \otimes \operatorname{Im} \mathbb{H}$$

is given by

$$\mu_0(x) = \pi\left(\overline{x_1} I x_1, \dots, \overline{x_N} I x_N\right),$$

where $\overline{x}$ denotes the quarternionic conjugate of $x$.

Let $G$ be an $n$-dimensional subtorus of $T^N$ and $\mathfrak{g}$ its Lie algebra. We define the lattice $\mathfrak{g}_{\mathbb{Z}} = \mathfrak{g} \cap \mathfrak{t}^N_{\mathbb{Z}}$. The basis $Y^1, \dots, Y^n$ of $\mathfrak{g}$ is represented by

$$Y^i = \sum_{j=1}^{N} a_{ij} X^j, \qquad i = 1, \dots, n,$$

where the matrix

$$A = (\mathbf{a}_1, \dots, \mathbf{a}_N) = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nN} \end{pmatrix}$$

is a rational matrix of maximal rank. We identify $\mathfrak{g}^*$ with $\mathbb{R}^n$ by using the dual basis of $Y^1, \dots, Y^n$. We denote by $\iota^*$ the dual of the inclusion map $\mathfrak{g} \to \mathfrak{t}^N$. Note that

$$\iota^* u_j = {}^t\mathbf{a}_j, \qquad j = 1, \dots, N. \tag{3.1}$$

The group $G$ also acts on $\mathbb{H}^N$ preserving its hyperKähler structure. The moment map

$$\mu : \mathbb{H}^N \to \mathfrak{g}^* \otimes \operatorname{Im} \mathbb{H}$$

is given by

$$\mu(x) = (\iota^* \circ \mu_0)(x) = \pi\left(\sum_{j=1}^{N} a_{1j} \overline{x_j} I x_j, \dots, \sum_{j=1}^{N} a_{nj} \overline{x_j} I x_j\right).$$

It is easy to see that $\mu$ is surjective. We put $M = \mathbb{H}^N$ and use the same notations $M^\xi$ and $M^\gamma$ in the preceding sections.

The main theorem in this section is the following:

**Theorem 3.1** *Suppose a subtorus $G$ of $T^N$ satisfies the condition* (ii) *of Proposition* 3.2 *below. For an arbitrary $\xi = (\xi_I, \xi_J, \xi_K) \in \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$ such that $\xi_I$ is a regular value of $\mu_I$, the toric hyperKähler quotient $M^\xi/G$ is smooth and the symplectic diffeomorphism class of $(M^\xi/G, \omega_I^\xi)$ is independent of $\xi_J$ and $\xi_K$.*

This theorem follows from Theorem 2.1 and Propositions 3.2 and 3.5 below.

**Proposition 3.2** (Konno) *Let $\xi$ be a regular value of $\mu$. Then following* (i) *and* (ii) *are equivalent:*

(i) *The action of $G$ on $M^\xi$ is free.*

(ii) *For every $\mathcal{J} \subset \{1, 2, \ldots, N\}$ such that $\{\iota^* u_j\}_{j \in \mathcal{J}}$ forms a basis of $\mathfrak{g}^*$,*

$$\mathfrak{t}^N_{\mathbb{Z}} = \mathfrak{g}_{\mathbb{Z}} \oplus \bigoplus_{j \in \mathcal{J}^c} \mathbb{Z} X^j$$

*holds as a $\mathbb{Z}$-module, where $\mathcal{J}^c$ denotes $\{1, \ldots, N\} - \mathcal{J}$.*

**Proposition 3.3** (Konno) *Fix an element $\xi = (\xi_I, \xi_J, \xi_K) \in \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$. Then the following* (i) *and* (ii) *are equivalent:*

(i) *$\xi$ is a regular value of $\mu$.*

(ii) *For any $\mathcal{J} \subset \{1, \ldots, N\}$, whose number of elements is less than $n$, $\xi_I$, $\xi_J$, $\xi_K$ are not simultaneously contained in the linear subspace of $\mathfrak{g}^*$ spanned by $\{\iota^* u_j\}_{j \in \mathcal{J}}$.*

For the proofs of these propositions, we refer to [5].

The fundamental vector field $\underline{Y^i}$ associated to $Y^i \in \mathfrak{g}$ is

$$\underline{Y^i_x} = (2\pi a_{i1} I x_1, \ldots, 2\pi a_{iN} I x_N) \in \mathbb{H}^N, \qquad i = 1, \ldots, n.$$

So we have

$$g_{ij}(x) = g(\underline{Y^i_x}, \underline{Y^j_x}) = 4\pi^2 \sum_{k=1}^{N} a_{ik} a_{jk} |x_k|^2.$$

We denote by $\widetilde{g_{ij}}(x)$ the cofactor of $g(x)$ associated to $g_{ij}(x)$. For further discussions, we shall calculate $\det g(x)$ and $\widetilde{g_{ij}}(x)$ explicitly.

**Lemma 3.4** *The determinant and the cofactor of $g(x)$ are calculated as follows:*

(i) $$\det g(x) = (4\pi^2)^n \sum_{1 \le l_1 < \cdots < l_n \le N} (\det(\mathbf{a}_{l_1} \cdots \mathbf{a}_{l_n}))^2 \, |x_{l_1}|^2 \cdots |x_{l_n}|^2.$$

(ii) $$\widetilde{g_{ij}}(x) = (4\pi^2)^{n-1} \sum_{1 \le l_1 < \cdots < l_{n-1} \le N} L_{ij}^{l_1 \cdots l_{n-1}} |x_{l_1}|^2 \cdots |x_{l_{n-1}}|^2,$$

*where the constant* $L_{ij}^{l_1 \cdots l_{n-1}}$ *vanishes if* $\mathbf{a}_{l_1}, \ldots, \mathbf{a}_{l_{n-1}}$ *are linearly dependent.*

*Proof.* (i) is followed by a direct computation. So we prove (ii). We denote by $\mathfrak{S}_n$ the symmetric group of order $n$. The matrix $g(x)$ can be written as

$$g(x) = 4\pi^2 \sum_{k=1}^{N} A^k |x_k|^2,$$

where $A^k = \mathbf{a}_k{}^t\mathbf{a}_k$ is $n \times n$-matrix. We denote by $A^k_{ij}$ the minor matrix obtained by deleting both the $i$-th row and the $j$-th column from $A^k$. By the definition of the cofactor $\widetilde{g_{ij}}(x)$, we have

$$\widetilde{g_{ij}}(x) = (-1)^{i+j} (4\pi^2)^{n-1} \det \left( \sum_{k=1}^{N} A^k_{ij} |x_k|^2 \right). \tag{3.2}$$

We calculate the constant $L_{ij}^{l_1 \cdots l_{n-1}}$ in (ii). For $\sigma \in \mathfrak{S}_{n-1}$, we denote by $A_{ij}^{\sigma, l_1 \cdots l_{n-1}}$ the matrix whose $k$-th column consists of the $\sigma(k)$-th column of $A^{l_k}_{ij}$. From (3.2), we have

$$L_{ij}^{l_1 \cdots l_{n-1}} = (-1)^{i+j} \sum_{\sigma \in \mathfrak{S}_{n-1}} \det(A_{ij}^{\sigma, l_1 \cdots l_{n-1}}).$$

Note that every column of $A_{ij}^{\sigma, l_1 \cdots l_{n-1}}$ is a constant multiple of

$${}^t( a_{1l_k} \;\cdots\; a_{i-1 l_k} \;\; a_{i+1 l_k} \;\cdots\; a_{n l_k} )$$

for some $k = 1, 2, \ldots, n-1$. Therefore if $\mathbf{a}_{l_1}, \ldots, \mathbf{a}_{l_{n-1}}$ are linearly dependent, $\det A_{ij}^{\sigma, l_1 \cdots l_{n-1}}$ vanishes. In particular, if at least two of $l_1, \ldots, l_{n-1}$ are equal, we have $L_{ij}^{l_1 \cdots l_{n-1}} = 0$. □

**Proposition 3.5** *Let* $\xi_I \in \mathfrak{g}^*$ *be a regular value of* $\mu_I$. *For every embedded path*

$$\gamma = (\gamma_I, \gamma_J, \gamma_K) : [0,1] \to \{\xi_I\} \times \mathfrak{g}^* \times \mathfrak{g}^*,$$

*the function* $\nu : M^\gamma \to \mathbb{R}$ *is bounded.*

*Proof.* Because of Lemma 3.4 (ii), for an arbitrary path

$$p = (p_1, \dots, p_N) : [0,1) \to M^\gamma,$$

it is enough to show the boundedness for

$$H(x) = \frac{(4\pi^2)^{n-1} L_{ij}^{l_1 \cdots l_{n-1}} |x_{l_1}|^2 \cdots |x_{l_{n-1}}|^2}{\det g(x)}$$

along $p$. Without loss of generality, we may assume that $l_k = k$ for $k = 1, 2, \dots, n-1$. We define

$$q_k(s) = \frac{p_k(s)}{|p(s)|}, \quad r(s) = |p(s)| \qquad \text{for all } \ s \in [0,1).$$

Since $M^\gamma$ do not contain the origin $0 \in \mathbb{H}^N$, $q_k(s)$ is well-defined. We consider the limit of

$$H(p(s)) = \frac{(4\pi^2)^{n-1} L_{ij}^{1 \cdots n-1} |q_1(s)|^2 \cdots |q_{n-1}(s)|^2}{r(s)^2 \det g(q(s))}$$

as $s \to 1$. If $\mathbf{a}_1, \dots, \mathbf{a}_{n-1}$ are linearly dependent, by Lemma 3.4 (ii), we have $L_{ij}^{1 \cdots n-1} = 0$. So we may assume that $\mathbf{a}_1, \dots, \mathbf{a}_{n-1}$ are linearly independent and $L_{ij}^{1 \cdots n-1} = 1$. The equation of $M^\gamma$ is given by

$$\pi r(s)^2 (\mathbf{a}_1 \cdots \mathbf{a}_N) \begin{pmatrix} \overline{q_1(s)} I q_1(s) \\ \vdots \\ \overline{q_N(s)} I q_N(s) \end{pmatrix} = \begin{pmatrix} \gamma^1(t) \\ \vdots \\ \gamma^n(t) \end{pmatrix},$$

where $\gamma$ is considered as

$$\gamma = (\gamma^1, \dots, \gamma^n) : [0,1] \to \operatorname{Im}\mathbb{H} \times \cdots \times \operatorname{Im}\mathbb{H}.$$

By taking some regular $n \times n$-matrix $P = (p_{ij})$, we obtain that

$$P(\mathbf{a}_1 \cdots \mathbf{a}_{n-1} \mathbf{a}_n \cdots \mathbf{a}_N) = \begin{pmatrix} a'_{11} & \cdots & a'_{1n} & a'_{1n+1} & \cdots & a'_{1N} \\ & \ddots & \vdots & \vdots & & \vdots \\ 0 & & a'_{nn} & a'_{nn+1} & \cdots & a'_{nN} \end{pmatrix}, \tag{3.3}$$

where at least one of the entries $a'_{nn}, \dots, a'_{nN}$ is non-zero because $A$ has the maximal rank. Therefore we have an equation

$$\pi r(s)^2 \left(a'_{nn}\overline{q_n(s)}Iq_n(s) + \cdots + a'_{nN}\overline{q_N(s)}Iq_N(s)\right) = p_{n1}\gamma^1(t) + \cdots + p_{nn}\gamma^n(t). \tag{3.4}$$

We define

$$\delta = \min_{t\in[0,1]} |p_{n1}\gamma^1(t) + \cdots + p_{nn}\gamma^n(t)|.$$

Suppose that $\delta > 0$. From (3.4), we obtain an estimate

$$\frac{1}{r(s)^2} \leq \frac{\pi}{\delta}\left(|a'_{nn}||q_n(s)|^2 + \cdots + |a'_{nN}||q_N(s)|^2\right).$$

From this and Lemma 3.4 (i), it follows that $H(p(s))$ is less than or equal to

$$\frac{|q_1(s)|^2 \cdots |q_{n-1}(s)|^2 \left(|a'_{nn}||q_n(s)|^2 + \cdots + |a'_{nN}||q_N(s)|^2\right)}{4\pi\delta \sum\limits_{1\leq l_1<\cdots<l_n\leq N} (\det(\mathbf{a}_{l_1}\cdots\mathbf{a}_{l_n}))^2 |q_{l_1}(s)|^2\cdots|q_{l_n}(s)|^2}. \tag{3.5}$$

Note that

$$\det\left(P(\mathbf{a}_1\cdots\mathbf{a}_{n-1}\mathbf{a}_k)\right) = a'_{11}\cdots a'_{n-1n-1}a'_{nk}, \qquad k = n,\ldots,N.$$

Because $\mathbf{a}_1,\ldots,\mathbf{a}_{n-1}$ are linearly independent, the entries $a'_{11},\ldots,a'_{n-1n-1}$ are all non-zero. Therefore $\det(\mathbf{a}_1\cdots\mathbf{a}_{n-1}\mathbf{a}_k) \neq 0$ if and only if $a'_{nk} \neq 0$. In other words, the numerator of (3.5) has the non-trivial term

$$|a'_{nk}||q_1(s)|^2\cdots|q_{n-1}(s)|^2|q_k(s)|^2,$$

if and only if the denominator of (3.5) has the non-trivial term

$$(\det(\mathbf{a}_1\cdots\mathbf{a}_{n-1}\mathbf{a}_k))^2 |q_1(s)|^2\cdots|q_{n-1}(s)|^2|q_k(s)|^2.$$

This means that the numerator of (3.5) can be dominated by the denominator of (3.5). Since each summand of the denominator of (3.5) is always positive, we obtain that

$$H(p(s)) \leq \frac{C}{4\pi\delta} \qquad \text{for all } \ s \in [0,1),$$

where $C$ is some constant.

Finally, we prove that $\delta > 0$. Suppose that $\delta = 0$. Then there exists some $t_0 \in [0,1]$ and the following three equations hold:

$$\begin{aligned} p_{n1}\gamma_I^1(t_0) + \cdots + p_{nn}\gamma_I^n(t_0) &= 0 \\ p_{n1}\gamma_J^1(t_0) + \cdots + p_{nn}\gamma_J^n(t_0) &= 0 \\ p_{n1}\gamma_K^1(t_0) + \cdots + p_{nn}\gamma_K^n(t_0) &= 0, \end{aligned} \tag{3.6}$$

where $\gamma_I(t_0) = (\gamma_I^1(t_0), \ldots, \gamma_I^n(t_0)) \in \mathfrak{g}^*$ and $\gamma_J^i(t_0), \gamma_K^i(t_0)$ are defined in a similar fashion. Because $A$ has the maximal rank, there exists some $k = n, \ldots, N$ such that $\{\iota^* u_j\}_{j=1,\ldots,n-1,k}$ forms a basis of $\mathfrak{g}^*$. Then $\gamma_I(t_0)$ can be written as

$$\gamma_I(t_0) = \sum_{j=1}^{n-1} c_j \iota^* u_j + c\iota^* u_k$$

for some constants $c_1, \ldots, c_{n-1}$ and $c$. From (3.1), (3.3) and (3.6), we have

$$0 = \sum_{i=1}^{n} p_{ni}\gamma_I^i(t_0) = \sum_{i=1}^{n}\sum_{j=1}^{n-1} p_{ni}(c_j a_{ij} + c a_{ik}) = c a'_{nk}.$$

Since $a'_{nk} \neq 0$, we have $c = 0$. Hence we obtain that

$$\gamma_I(t_0) = \sum_{j=1}^{n-1} c_j \iota^* u_j.$$

In the same way, we obtain that both $\gamma_J(t_0)$ and $\gamma_K(t_0)$ can be represented by linear combinations of $\iota^* u_1, \ldots, \iota^* u_{n-1}$. Therefore, from Proposition 3.3, $\gamma(t_0)$ is a critical value of $\mu$. This is a contradiction. So at least one of the equations (3.6) does not hold. Hence we have $\delta > 0$. □

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