If $S \times T$ is semiperfect, is S or T perfect?

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Abstract. The product of a perfect and a semiperfect semigroup is semiperfect. Conversely, if S and T are semigroups such that $S \times T$ is semiperfect then S and T must both be semiperfect. We consider the question whether it follows that S or T is perfect. This question can be answered in the affirmative by showing that every non-perfect semiperfect semigroup admits $\mathbf{Z}$ or $\mathbf{N}_0$ as a minor. We show that the study of the latter question can be reduced to the case of subsemigroups of a rational vector space carrying the identical involution.

Key words: semigroup, positive definite, moment function.

1. Introduction

Suppose $(S, +, *)$ is an abelian semigroup with zero and involution. Such a structure will be called a *$*$-semigroup*, abbreviated 'semigroup' when confusion is unlikely. A function $\varphi \colon S \to \mathbf{C}$ is *positive definite* if

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(s_j + s_k^*) \geq 0$$

for every choice of $n \in \mathbf{N}$, $s_1, \ldots, s_n \in S$, and $c_1, \ldots, c_n \in \mathbf{C}$. Denote by $\mathcal{P}(S)$ the set of all positive definite functions on S.

A *character* on S is a function $\sigma \colon S \to \mathbf{C}$ satisfying $\sigma(0) = 1$, $\sigma(s^*) = \overline{\sigma(s)}$, and $\sigma(s + t) = \sigma(s)\sigma(t)$ for all $s, t \in S$. Denote by S^* the set of all characters on S. Denote by $\mathcal{A}(S^*)$ the least σ-field of subsets of S^* rendering the mapping $\sigma \mapsto \sigma(s) \colon S^* \to \mathbf{C}$ measurable for each $s \in S$. Denote by $F_+(S^*)$ the set of all measures defined on $\mathcal{A}(S^*)$ and integrating $\sigma \mapsto \sigma(s)$ for all $s \in S$. For $\mu \in F_+(S^*)$, define $\mathcal{L}\mu \colon S \to \mathbf{C}$ by

$$\mathcal{L}\mu(s) = \int_{S^*} \sigma(s)\, d\mu(\sigma)$$

for $s \in S$. A function $\varphi \colon S \to \mathbf{C}$ is a *moment function* if $\varphi = \mathcal{L}\mu$ for some $\mu \in F_+(S^*)$, and a moment function φ is *determinate* if there is only one

such μ. Denote by $\mathcal{H}(S)$ the set of all moment functions on S, and by $\mathcal{H}_D(S)$ the subset of determinate moment functions. We have $\mathcal{H}_D(S) \subset \mathcal{H}(S) \subset \mathcal{P}(S)$ since if $\mu \in F_+(S^*)$, $s_1, \ldots, s_n \in S$, and $c_1, \ldots, c_n \in \mathbf{C}$ then

$$\sum_{j,k=1}^{n} c_j \overline{c_k} \mathcal{L}\mu(s_j + s_k^*) = \int_{S^*} \left| \sum_{j=1}^{n} c_j \sigma(s_j) \right|^2 d\mu(\sigma) \geq 0.$$

The semigroup S is *semiperfect* if $\mathcal{H}(S) = \mathcal{P}(S)$, and *perfect* if $\mathcal{H}_D(S) = \mathcal{P}(S)$.

For example, the semigroup $\mathbf{N}_0$ is semiperfect by Hamburger's Theorem. Moment functions on $\mathbf{N}_0$, called *moment sequences*, are sequences $(s_n)_{n=0}^{\infty}$ of reals such that

$$s_n = \int_{\mathbf{R}} x^n \, d\mu(x), \qquad n \in \mathbf{N}_0$$

for some measure μ on $\mathbf{R}$ with moments of all orders. See [1], [13], p. 5, or [2], 6.2.2.

The group $\mathbf{Z}$, *considered with the identical involution*, is likewise semiperfect ([10], see [2], 6.4.1). Moment functions on $\mathbf{Z}$, called *two-sided moment sequences*, are two-sided sequences $(s_n)_{n=-\infty}^{\infty}$ of reals such that

$$s_n = \int_{\mathbf{R} \setminus \{0\}} x^n \, d\mu(x), \qquad n \in \mathbf{Z}$$

for some measure μ on $\mathbf{R} \setminus \{0\}$ satisfying $\int |x|^n \, d\mu(x) < \infty$ for all $n \in \mathbf{Z}$.

Neither of these semigroups is perfect since, e.g., the two-sided moment sequence $n \mapsto e^{n^2/2}$ is indeterminate ([2], 6.4.6).

Sakakibara [12] showed that every semiperfect subsemigroup of $\mathbf{Z}^k$, containing 0 and considered with the identical involution, is $\{0\}$ or isomorphic to $\mathbf{N}_0$ or $\mathbf{Z}$. Thus semiperfectness is a rare phenomenon.

The most important example of a perfect semigroup is $\mathbf{Q}_+$ with its unique involution, the identity ([2], 6.5.6). The product of two perfect semigroups is perfect ([8], Theorem 2), so that, for example, also $\mathbf{Q}_+ \times \mathbf{Q}_+$ is perfect. Every $*$-homomorphic image of a perfect semigroup is perfect ([8], Theorem 1). Since $\mathbf{Q}$, considered with the identical involution, is the image of $\mathbf{Q}_+ \times \mathbf{Q}_+$ under the homomorphism $(s,t) \mapsto s - t$, it follows that $\mathbf{Q}$ is perfect, as first shown in [2], 6.5.10. The direct sum of an arbitrary family of perfect semigroups is perfect ([8], Theorem 3), so that, for example, arbitrary rational vector spaces, considered with the identical involution,

are perfect. In fact, a rational vector space U with arbitrary involution is a perfect semigroup. Indeed, U is the direct sum $U_+ \oplus U_-$ of a linear subspace U_+ carrying the identical involution and a linear subspace U_- carrying the inverse involution ($s^* = -s$). The first factor is perfect by the preceding. The second factor, like every abelian group carrying the inverse involution, is perfect by the discrete version of the Bochner-Weil Theorem. Hence U is perfect by the product theorem.

If S is a perfect semigroup and T is a semiperfect semigroup then the product *-semigroup $S \times T$ is semiperfect. To see this, follow the proof of [3], Proposition 1, where the same is shown for moment functions represented by Radon measures under the assumption that T is finitely generated. Where the proof in [3] refers to the vague compactness of $\{ \mu \in E_+(T^*) \mid \mathcal{L}\mu = \varphi \}$ for $\varphi \in \mathcal{P}(T)$, use [4], Corollary 3.2. Where the proof in [3] refers to a Radon bimeasure theorem, use [8], Lemma 1.

In [3], the following question was posed:

(A) If S and T are *-semigroups such that $S \times T$ is semiperfect, does it follow that S or T is perfect?

The purpose of the present note is to indicate a route that may lead to an affirmative answer to (A). A *face* of a *-semigroup S is a *-subsemigroup X of S such that the conditions $x, y \in S$ and $x + y \in X$ imply $x, y \in X$. Every face of a semiperfect semigroup is semiperfect ([12], 2.1). A *minor* of S is a *-homomorphic image of a face of S. Thus every minor of a semiperfect semigroup is semiperfect. Every minor of a minor of S is a minor of S. We shall be interested in the following question where $\mathbf{N}_0$ and $\mathbf{Z}$ are considered with the identical involution:

(B) If S is a non-perfect semiperfect semigroup, does it follow that S admits $\mathbf{N}_0$ or $\mathbf{Z}$ as a minor?

If the answer to (B) is affirmative, so is the answer to (A). Indeed, suppose the answer to (B) is affirmative. Suppose S and T are *-semigroups such that $S \times T$ is semiperfect; we have to show that S or T is perfect. Suppose neither S nor T is perfect. Now S and T, being *-homomorphic images of $S \times T$ under the projections, are semiperfect. (This follows from [8], Proposition 1.) Thus S is a non-perfect semiperfect semigroup. By the hypothetical affirmative answer to (B) it follows that there exist a face X of S and a *-homomorphism f of X onto a *-semigroup U which is $\mathbf{N}_0$ or $\mathbf{Z}$. Similarly, there exist a face Y of T and a *-homomorphism g of Y onto a semigroup V which is $\mathbf{N}_0$ or $\mathbf{Z}$. Now $X \times Y$ is readily seen to be a face of

$S \times T$, and the equation $h(x,y) = (f(x), g(y))$ defines a $*$-homomorphism h of $X \times Y$ onto $U \times V$. Thus $U \times V$ is a minor of the semiperfect semigroup $S \times T$, hence semiperfect. But $U \times V$ is $\mathbf{N}_0 \times \mathbf{N}_0$, $\mathbf{N}_0 \times \mathbf{Z}$, $\mathbf{Z} \times \mathbf{N}_0$, or $\mathbf{Z} \times \mathbf{Z}$. By the result of Sakakibara cited above, neither of these semigroups is semiperfect. Thus we have the desired contradiction.

The purpose of the present note is to show how the problem of answering question (B) in the affirmative can be reduced to doing the same in the case that the semigroup in question is a subsemigroup of a rational vector space carrying the identical involution. We shall also make some remarks on that case.

2.   Reduction of the problem

A $*$-semigroup H, not necessarily having a zero, is $*$-*archimedean* if for all $x, y \in H$ there exist $z \in H$ and $n \in \mathbf{N}$ such that $n(x + x^*) = y + z$. A $*$-*archimedean component* of a $*$-semigroup is a $*$-archimedean $*$-subsemigroup which is maximal for the inclusion ordering. Every $*$-semigroup is the disjoint union of its $*$-archimedean components (see [9], Section 4.3 for the case of the identical involution).

Theorem 1   *If S is a non-perfect semiperfect semigroup then S has a non-perfect semiperfect minor which is a subsemigroup of a rational vector space carrying the identical involution.*

Proof.   Since S is not perfect, by [5], Corollary 3.1, S has a $*$-archimedean component H such that the greatest torsion-free cancellative identical-involution $*$-homomorphic image T of $H \cup \{0\}$ is not Stieltjes perfect. (See [5] for the definition of 'Stieltjes perfect'.) By 'torsion-free cancellative' is meant that T is cancellative and the group $T - T$ is torsion-free, which is equivalent to saying that T is a subsemigroup of a rational vector space. By 'greatest' is meant that every $*$-homomorphic image of $H \cup \{0\}$ which is a subsemigroup of a rational vector space carrying the identical involution is in a canonical way a homomorphic image of T. (Clifford and Preston [9] use the term 'maximal' where we use 'greatest'.) Denote by $\rho \colon H \cup \{0\} \to T$ the quotient mapping. It follows from [5], Corollary 3.1, that T is not perfect. Let X be the set of those $x \in S$ such that $x + H \subset H$. Then X is a face of S, in fact the least face of S containing H. Moreover, $X + H \subset H$. Extend ρ to a mapping of X into $T - T$, also denoted by ρ, by setting

$\rho(x) = \rho(x+y) - \rho(y)$ (difference in the group $T - T$) for $x \in X$ and $y \in H$. It is easy to verify that the definition of $\rho(x)$ is independent of the choice of $y \in H$ and that the extended mapping ρ is again a $*$-homomorphism. Note that $T \subset \rho(X)$. Now $\rho(X)$ is a minor of the semiperfect semigroup S, hence semiperfect. Since $X + H \subset H$ then $\rho(X) + (T \setminus \{0\}) \subset T$. By this fact, if $\rho(X)$ were perfect, by a result of Nakamura and Sakakibara [11] it would follow that T were perfect, a contradiction. Thus $\rho(X)$ is not perfect. We have shown that $\rho(X)$ is a minor of S with the stated properties. $\square$

By Theorem 1, if the answer to question (B) is affirmative in the case of a subsemigroup of a rational vector space carrying the identical involution then it is affirmative in general. This is because a minor of a minor of S is a minor of S.

3.   On the case of subsemigroups of rational vector spaces

If X is a rational vector space, consider X with the topology defined by the condition that a subset G of X is open if and only if $G \cap Y$ is open in Y, in the canonical topology on a finite-dimensional space, for every finite-dimensional linear subspace Y of X. The following result is a generalization of the result of Sakakibara cited in the Introduction.

Theorem 2 *Suppose S is a discrete subsemigroup of a countable rational vector space X carrying the identical involution. If S is semiperfect then S is $\{0\}$ or isomorphic to $\mathbf{N}_0$ or $\mathbf{Z}$.*

Proof.   For every subset V of S, let $E(V)$ be the set of those $v \in V$ such that the conditions $s, t \in S$, $2s, 2t \in V$, and $s + t = v$ imply $s = t$. For every subset U of S, let $C(U)$ denote the union of all finite subsets V of S such that $E(V) \subset U$. Since S carries the identical involution then characters on S are real-valued. They separate points in S since homomorphisms of X into $(\mathbf{R}, +)$ separate points and since if ξ is such a homomorphism then $e^{\xi}|S$ is a character on S. Thus S is what was called '$\mathbf{R}$-separative' in [6]. To see that S is 'C-finite' in the sense of [6], it remains to be shown that the set $C(U)$ is finite if U is a finite subset of S. By [7], Proposition 1, $C(U) \subset \mathrm{Conv}(U)$ where $\mathrm{Conv}(U)$ is the set of those $u \in S$ such that

$$(n+1)u = u + u_1 + \cdots + u_n \tag{1}$$

for some $n \in \mathbf{N}$ and $u_1, \ldots, u_n \in U$. Since S is cancellative, in (1) we can

subtract u from both sides, and we see that $\text{Conv}(U)$ is just the intersection of S with the convex hull of U in the enveloping real vector space. That convex hull is compact, and S is discrete, so the intersection is finite. Hence so is the subset $C(U)$. This proves that S is C-finite.

Since S is countable, cancellative, C-finite, and semiperfect, it follows from [7], Theorem 3, that S is isomorphic to F, $F \times \mathbf{Z}$, or $(E \times \{0\}) \cup (F \times \mathbf{N})$ for some finite abelian group F of exponent 1 or 2 and some subgroup E of F. Since S is a subsemigroup of a torsion-free group, the group F must be the trivial group. Thus S is $\{0\}$ or isomorphic to $\mathbf{Z}$ or $\mathbf{N}_0$. $\square$

We see from Theorem 2 that in order to answer question (B) in the affirmative in the case of countable S, it would suffice to show that if S is a subsemigroup of a countable rational vector space X carrying the identical involution and if S admits an indeterminate moment function then there is a linear subspace Y of X such that if $\phi\colon X \to X/Y$ is the quotient mapping then $\phi(S)$ is a nonzero discrete subsemigroup of X/Y. We have not been able to prove this.

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