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Parabolic geometries and canonical Cartan connections

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Abstract. Let G be a (real or complex) semisimple Lie group, whose Lie algebra $\mathfrak{g}$ is endowed with a so called $|k|$-grading, i.e. a grading of the form $\mathfrak{g}=\mathfrak{g}_{-k}\oplus\cdots\oplus\mathfrak{g}_{k}$, such that no simple factor of G is of type A_1. Let P be the subgroup corresponding to the subalgebra $\mathfrak{p}=\mathfrak{g}_0\oplus\cdots\oplus\mathfrak{g}_k$. The aim of this paper is to clarify the geometrical meaning of Cartan connections corresponding to the pair (G, P) and to study basic properties of these geometric structures.

Let G_0 be the (reductive) subgroup of P corresponding to the subalgebra $\mathfrak{g}_0$. A principal P-bundle E over a smooth manifold M endowed with a (suitably normalized) Cartan connection $\omega\in\Omega^1(E,\mathfrak{g})$ automatically gives rise to a filtration of the tangent bundle TM of M and to a reduction to the structure group G_0 of the associated graded vector bundle to the filtered vector bundle TM. We prove that in almost all cases the principal P bundle together with the Cartan connection is already uniquely determined by this underlying structure (which can be easily understood geometrically), while in the remaining cases one has to make an additional choice (which again can be easily interpreted geometrically) to determine the bundle and the Cartan connection.

Key words: parabolic geometry, Cartan connection, partially integrable almost-CR-structure, G-structure, filtered manifold.

1. Introduction

It is an idea that goes back to E. Cartan (see [10]) to view manifolds endowed with certain geometric structures as “curved analogs” of homogeneous spaces. More precisely, given a Lie group G and a closed subgroup $H\leq G$, a generalized space corresponding to the homogeneous space G/H (which is simply called a space by Cartan) is a smooth manifold M of the same dimension as G/H, together with a principal H-bundle $E\to M$ over M, which is endowed with a Cartan connection $\omega\in\Omega^1(E,\mathfrak{g})$, that is a trivialization of the tangent bundle of E which is H-equivariant and reproduces the generators of fundamental vector fields. For example, for Riemannian structures (which are not among the structures considered in this paper), the group G is the group of motions of $\mathbb{R}^n$, H is the orthogonal group $O(n)$

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(so G/H is just $\mathbb{R}^n$, and G is exactly the group of isometries of $\mathbb{R}^n$). Given a Riemannian manifold M of dimension n, the principal bundle E is the orthogonal frame bundle of M, and the Cartan connection is the soldering form on this bundle together with the Levi-Civita connection on M.

Already this example (which is among the simplest possible ones) shows, that identifying a manifold as a generalized space in the above sense should be rather the result of a theorem than a definition. The aim of this paper is to clarify the geometrical meaning of such generalized space structures in a (rather wide) special case. Namely, we consider the case where G is semisimple (real or complex) and its Lie algebra is endowed with a so called $|k|$-grading, i.e. a grading of the form $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ and the subgroup, which will be called P, corresponds to the Lie subalgebra $\mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$. In the complex case, this just means that P is a parabolic subgroup of G. Guided by this fact and following Feffermann-Graham and Bailey (see [1]), we call the corresponding geometric structures parabolic geometries. So what we consider contains curved analogs of all compact complex simple homogeneous spaces and real forms of this situation.

Particularly well known examples of structures of this type are the so-called AHS-structures, which are the structures corresponding to groups with $|1|$-graded Lie algebras in the above sense. Among these, there are the conformal and paraconformal (or almost Grassmannian) structures, as well as the classical projective structures (see [6] and the references therein for a discussion of AHS-structures). A very well studied example of a parabolic geometry corresponding to a $|2|$-graded Lie algebra is given by CR-structures with non-degenerate Levi-form. This example is discussed in some detail in this paper, see 4.14–4.16. An important source for examples of general parabolic geometries is twistor theory, see 4.17 for an outline of these examples. Surprisingly, also the geometry of generic six-dimensional codimension-two CR-manifolds fits into the scheme of parabolic geometries, see [18].

The motivation for this work is mainly to provide a basis for a geometrical study of parabolic geometries and of differential operators intrinsic to them. Such operators have been intensively studied in the case of AHS-structures, and in particular in the case of conformal structures (see [6], the references therein, and [8]). Also, powerful results on the existence of invariant differential operators for general parabolic geometries are already available, see [9].

There is a second way to view the results that we shall prove. Given a manifold with a parabolic geometry, i.e. a principal P-bundle together with a suitably normalized Cartan connection, it is easy to see that one can construct certain underlying structures. Our main results then show that the bundle and the Cartan connection are already fully determined by these underlying structures. These structures are rather easy to interpret geometrically, so one can view our results also as proofs for the existence of a canonical Cartan connection for these underlying structures. The construction of canonical Cartan connections also solves the equivalence problem for such structures. In fact, this is the more traditional point of view, say for AHS-structures and CR-structures.

The problem of constructing canonical Cartan connections has a rather long history. First of all, Cartan's original method of equivalence gives a possibility of constructing canonical Cartan connections in a variety of cases. It seems that in some of the cases we consider this method works even under weaker assumptions than we impose. On the other hand, it seems to be hard in this approach to give a general description of the normalizations, which are necessary to ensure the uniqueness of the Cartan connections in a broader setting (say for arbitrary parabolics). The problem of constructing canonical Cartan connections for AHS-structures has been treated in several different ways by various authors (see e.g. [20], [17] (in the torsion free case), [3] (in an associated-bundle setting) and [7]).

In the case of CR-structures, the construction of a canonical Cartan connection is due to E. Cartan (see [11]) for hypersurfaces in $\mathbb{C}^2$ and to N. Tanaka (see [21]) and S.S. Chern and J. Moser (see [12]) for arbitrary CR-manifolds. As an application of our results, we show the existence of a canoncial Cartan connection for the significantly more general class of partially integrable almost-CR-manifolds.

In [22], N. Tanaka treated problems quite closely related to the problems treated here, but from a different point of view and with different aims. His main motivation was studying the corresponding equivalence problems, rather than geometric properties of the structures themselves. We prove a more general version of his main result in 4.5. We believe that, besides working in bigger generality (Tanaka treated only simple groups with trivial center and the case where the algebra is the prolongation of the nonpositive part) and being technically simpler, our approach shows much clearer the geometrical background of the whole situation. In particular, this refers to the emphasis on filtrations (which occur only implicitly in Tanaka's work) and our starting point for the prolongation procedure, which is different from Tanaka's and can be easily understood geometrically.

In [15], T. Morimoto developed a general theory of geometric strucutres on filtered manifolds, mainly motivated by studying equivalence problems. As an application of this general theory, he obtained a general criterion for the existence of canonical Cartan connections, extendeding the results of Tanaka to semisimple groups and other groups (see [16]). Although Morimoto's procedure is simpler than the one presented here (once all the general machinery is developed) we think that our procedure has an advantage: In our approach all the necessary data in the procedure are constructed directly and have a direct geometric interpretation. This is not really visible in the general setting presented here, but it becomes apparent once one works with a concrete structure, see [5]. In contrast to that, Morimoto's procedure uses non-commutative (semi-holonomic) frame bundles and the canonical forms on these bundles, which seem to us to be much harder to interpret geometrically. There is no doubt, however, that the two procedures are essentially equivalent.

The main results

Since parts of this paper are rather technical, we collect here the main results: Let G be a semisimple Lie group with $|k|$-graded Lie algebra $\mathfrak{g}$ as above, P the corresponding subgroup and $G_0 \leq P$ the subgroup corresponding to the Lie subalgebra $\mathfrak{g}_0$. Let M be a smooth manifold with a filtration of the tangent bundle $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ by smooth subbundles such that the rank of T^iM equals the dimension of $\mathfrak{g}_{-i} \oplus \cdots \oplus \mathfrak{g}_{-1}$. The main technical notion in the paper is the notion of harmonic P-frame bundles of degree ℓ (see 3.6 and 3.10). This notion interpolates between two rather simple concepts. For $\ell = 1$ one gets reductions to the structure group G_0 of the associated graded vector bundle to the filtered vector bundle TM, which satisfy a condition called the structure equations (see 3.4). Roughly, this can be interpreted geometrically as follows: First, one has to require that the Lie-bracket of vector fields is compatible with the filtration, i.e. the bracket of a section of T^iM with a section of T^jM is a section of $T^{i+j}M$. Under this assumption, the Lie bracket gives rise to a pointwise Lie algebra structure on the associated graded vector bundle to the tangent bundle, and a P-frame bundle of degree one over M which satisfies the structure equations means exactly that each fiber of the associated graded bundle (with this algebraic bracket) looks like the graded G_0-module $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$. Since the group G_0 is always reductive, this can be easily described explicitly in each case. The other extremal case ($\ell = 2k+1$) of a P-frame bundle is a principal P-bundle over M endowed with a suitably normalized Cartan connection.

The technical core of the paper is Theorem 3.22, which shows that, assuming that a certain Lie algebra cohomology group (which depends on ℓ) is trivial, one can construct a unique (up to isomorphism) harmonic P-frame bundle of degree $\ell + 1$ out of a harmonic P-frame bundle of degree ℓ. Together with a rather simple way to go in the other direction, this shows that, assuming the vanishing of the cohomology group, there is a bijective correspondence between isomorphism classes of harmonic P-frame bundles of degree ℓ and of degree $\ell + 1$.

The relevant cohomology groups have been computed in [23] using Kostant's version of the Bott-Borel-Weil theorem. The result is that in all cases except two families (one of which is the classical projective structures, the other one a certain $|2|$-grading on symplectic algebras which corresponds to a contact-analog of classical projective structures) all the relevant cohomology groups vanish. Thus, except for these two cases, there is a bijective correspondence between reductions to the structure group G_0 of the associated graded vector bundle to the tangent bundle and isomorphism classes of principal P-bundles endowed with suitably normalized Cartan connections. In the remaining cases (except the one of a simple factor corresponding to one-dimensional projective structures, which is really degenerate), we show that the obstructions occur only in the very first step of the prolongation procedure, and in this step one has to make a choice, which should be simply viewed as an ingredient of the structure (in fact the only ingredient in the projective case and the only additional ingredient to a contact structure in the other case). Thus, in these cases we still get canonical Cartan connections.

In Section 4 we discuss the relation of our approach to the one of N. Tanaka and prove a more general version of the main result of his paper [22]. Moreover, we discuss some geometric properties of manifolds with parabolic geometries. In particular, we study the curvature of the canonical Cartan connection and discuss its relation to cohomology. Finally, we outline several examples, in particular AHS-structures and partially integrable almost-CR-structures.

2. $|k|$-graded Lie algebras

In this section, we collect some basic facts about $|k|$-graded semisimple Lie algebras. Our basic reference for these results is [23]. That paper also contains the computations of the cohomology groups which we will need in the sequel. We give an alternative presentation of the Hodge structure on the standard complexes computing these cohomologies in the real and complex case, which seems more conceptual to us than the one of [22] and prove some basic results on groups with $|k|$-graded semisimple Lie algebras.

Definition 2.1 Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. A $|k|$-graded Lie algebra over $\mathbb{K}$ is a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ together with a decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and such that the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$. In the whole paper, we will only deal with *semisimple* $|k|$-graded Lie algebras.

By $\mathfrak{p}$ we will denote the subalgebra $\mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_k$ of $\mathfrak{g}$, and by $\mathfrak{p}_+$ the subalgebra $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ of $\mathfrak{p}$. As we shall see in 2.2 below, the powers of $\mathfrak{p}_+$ are then just given as $\mathfrak{p}_+^i = \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_k$, for all $i = 1, \dots, k$. Moreover, from 2.2 it also follows that a $|k|$-graded semisimple Lie algebra is a direct sum of $|k_i|$-graded simple Lie algebras, and we will assume throughout the paper that all these k_i are bigger than zero, i.e. that none of the simple ideals is contained in $\mathfrak{g}_0$.

Proposition 2.2 *Let $\mathfrak{g}$ be a semisimple $|k|$-graded Lie algebra. Then the following assertions hold:*

1. *There is a unique element $E \in \mathfrak{g}_0$ such that $[E, X] = \ell X$ for all $X \in \mathfrak{g}_\ell$.*
2. *Let B be the Killing form of $\mathfrak{g}$. Then $B(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ unless $i+j = 0$, and B induces an isomorphism $\mathfrak{g}_i^* \cong \mathfrak{g}_{-i}$ of $\mathfrak{g}_0$-modules for all $i = 1, \dots, k$.*
3. *If $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$, then $\mathfrak{g}'$ is homogeneous, i.e. $\mathfrak{g}' = \bigoplus_{i=-k}^{k} (\mathfrak{g}' \cap \mathfrak{g}_i)$. In particular, $\mathfrak{g}$ is a direct sum of simple $|k_i|$-graded Lie algebras (where all k_i are less or equal to k).*
4. *Let $A \in \mathfrak{g}_i$ with $i > -k$ be an element such that $[A, X] = 0$ for all $X \in \mathfrak{g}_{-1}$. Then $A = 0$.*
5. *For $i > -k$ we have $[\mathfrak{g}_{i+1}, \mathfrak{g}_{-1}] = \mathfrak{g}_i$.*

(The last two statements in the case $i = 0$ need that no simple factor of $\mathfrak{g}$

is contained in $\mathfrak{g}_0$.)

Proof. (1)–(3) are shown in Section 3.1 and Lemma 3.1 of [23]. (4) and (5) are proved in Lemma 3.2 of [23] in the simple case, but under the assumption that no simple factor is contained in $\mathfrak{g}_0$, the results for the simple case obviously imply the analogous statements in the semisimple case. □

2.3. The properties of $|k|$-graded Lie algebras collected in 2.2 are sufficient to completely describe the meaning of a $|k|$-grading on a complex simple Lie algebra. Namely, it can be shown (see Section 3.3 of [23]) that given a complex $|k|$-graded simple Lie algebra, one can find a Certan-subalgebra $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathfrak{g}$, a system Δ^+ of positive roots for $(\mathfrak{g}, \mathfrak{h})$, and a subset $\Sigma \subset \Delta_0$ of the corresponding set of simple roots such that the grading of $\mathfrak{g}$ is given by the Σ-height of roots. This means that for any root α, the root space $\mathfrak{g}_\alpha$ is contained in the homogeneous component $\mathfrak{g}_j$, where j is the sum of all coefficients of elements of Σ in the expansion of α as a linear combination of simple roots. In particular, this implies that $\mathfrak{p} \subset \mathfrak{g}$ is the standard parabolic subalgebra corresponding to $\Sigma \subset \Delta^+$, see [4, 2.2].

Conversely, if $\mathfrak{p} \subset \mathfrak{g}$ is any parabolic subalgebra in a complex simple Lie algebra, then choosing a Cartan subalgebra and positive roots appropriately, $\mathfrak{p}$ is the standard parabolic subalgebra corresponding to a set Σ of simple roots. But then the Σ-height gives a $|k|$-grading on $\mathfrak{g}$, where k is the Σ-height of the maximal root.

In particular, specifying a $|k|$-grading for some k on a complex simple Lie-algebra is equivalent to specifying a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. Moreover, the possible gradings of that type and the lengths of these gradings can be read off the expression of the highest root (i.e. the highest weight of the adjoint representation) as a linear combination of simple roots.

Since a $|k|$-grading is the same thing as a parabolic subalgebra, we can use the Dynkin diagram notation as introduced in [4, 2.2] for complex $|k|$-graded Lie algebras. So we take the Dynkin diagram for the Lie algebra $\mathfrak{g}$ and replace the dots corresponding to the simple roots contained in Σ by crosses. Consider for example the Dynkin diagram $\times\!-\!\times\!-\!\bullet$. The underlying Lie algebra of this is $A_3 = \mathfrak{sl}(4, \mathbb{C})$, and we have to consider the standard parabolic corresponding to $\Sigma = \{\alpha_1, \alpha_2\}$. The highest root for A_3 is just the sum of the three simple roots, so its Σ-height is 2, and we get a $|2|$-grading, which is given by:

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_0 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_0 \end{pmatrix}$$

It can also be shown that two Dynkin diagrams (with crosses) represent isomorphic $|k|$-graded Lie algebras if there is an isomorphism of the two diagrams preserving the sets of crosses, see [23, Theorem 3.12], where also the real case is discussed in terms of Satake diagrams.

Finally, it should be remarked that any complex simple Lie algebra admits a (up to isomorphism) unique contact-gradation, i.e. a $|2|$-grading such that $\dim(\mathfrak{g}_{\pm 2}) = 1$, see [23, Section 4.2].

2.4. Next, we have to discuss the Lie algebra cohomology groups of $\mathfrak{g}_-$ with coefficients in $\mathfrak{g}$, which enter in two ways into the theory of parabolic geometries. On one hand, parts of the first cohomology occur as obstructions in the prolongation procedure, and on the other hand the second cohomology is related to the possible values of the curvature of a normalized Cartan connection.

As usual, the chain groups for these cohomology groups are defined as $C^n(\mathfrak{g}_-, \mathfrak{g}) := L(\Lambda^n \mathfrak{g}_-, \mathfrak{g})$, the space of linear maps from the n-th exterior power of $\mathfrak{g}_-$ to $\mathfrak{g}$. Alternatively, one can also view them as multilinear alternating maps. The differential $\partial : C^n(\mathfrak{g}_-, \mathfrak{g}) \to C^{n+1}(\mathfrak{g}_-, \mathfrak{g})$ is given by

$$\begin{aligned}(\partial\varphi)&(X_0, \dots, X_n) \\ &:= \sum_{i=0}^{n} (-1)^i [X_i, \varphi(X_0, \dots, \hat{X}_i, \dots, X_n)] \\ &\quad + \sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_n),\end{aligned}$$

where the hat denotes omission.

We will denote by $L_\ell(\Lambda^n \mathfrak{g}_-, \mathfrak{g})$ or by $C^n_\ell(\mathfrak{g}_-, \mathfrak{g})$ the space of linear maps which are homogeneous of degree ℓ, i.e. for which $\varphi(X_1, \dots, X_n) \in \mathfrak{g}_{i_1 + \cdots + i_n + \ell}$ if each X_j lies in $\mathfrak{g}_{i_j}$. From the definition of ∂ it is obvious that ∂ maps $L_\ell(\Lambda^n \mathfrak{g}_-, \mathfrak{g})$ to $L_\ell(\Lambda^{n+1} \mathfrak{g}_-, \mathfrak{g})$. Accordingly, also the cohomology groups split as $H^n(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_\ell H^n_\ell(\mathfrak{g}_-, \mathfrak{g})$.

Note that the Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ acts on each component $\mathfrak{g}_i$ via the adjoint action. This implies that it acts on each of the spaces $C^n(\mathfrak{g}_-, \mathfrak{g})$, and the action preserves the homogeneity of maps. Moreover, one immediately verifies that the differential ∂ is a homomorphism of $\mathfrak{g}_0$-modules. Consequently, each of the cohomology groups $H^n_\ell(\mathfrak{g}_-, \mathfrak{g})$ is naturally a $\mathfrak{g}_0$-module.

2.5. By 2.2(2), the Killing form on $\mathfrak{g}$ can be used to identify the Lie subalgebra $\mathfrak{g}_-$ with the dual of the Lie subalgebra $\mathfrak{p}_+$. Consequently, for any $\mathfrak{g}$-module V, we can identify the space $C^n(\mathfrak{g}_-, V) \cong \Lambda^n(\mathfrak{g}_-^*) \otimes V$ with the dual space of $\Lambda^n(\mathfrak{p}_+^*) \otimes V^* \cong C^n(\mathfrak{p}_+, V^*)$. In particular, the negative of the dual map of the Lie algebra differential $\partial : C^{n-1}(\mathfrak{p}_+, V^*) \to C^n(\mathfrak{p}_+, V^*)$ can be viewed as a linear map $\partial^* : C^n(\mathfrak{g}_-, V) \to C^{n-1}(\mathfrak{g}_-, V)$, which is called the *codifferential*. Clearly, the codifferential satisfies $\partial^* \circ \partial^* = 0$.

Since the Killing form identifies $\mathfrak{g}_-$ with the dual of $\mathfrak{p}_+$ even as a $\mathfrak{g}_0$-module, we conclude that the codifferential ∂^* is a homomorphism of $\mathfrak{g}_0$-modules.

In the sequel, we will need the formula for the codifferential in the special case $\partial^* : C^2(\mathfrak{g}_-, \mathfrak{g}) \to C^1(\mathfrak{g}_-, \mathfrak{g})$. To get the explicit formula, let $\{\xi_\alpha\}$ be a basis for $\mathfrak{g}_-$ and $\{\eta_\alpha\}$ the dual basis (with respect to the Killing form) of $\mathfrak{p}_+$. Using these, and identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ using the Killing form, we can compute the dual pairing of $\varphi \in C^n(\mathfrak{g}_-, \mathfrak{g})$ and $\psi \in C^n(\mathfrak{p}_+, \mathfrak{g})$ as

$$\langle \varphi, \psi \rangle = \frac{1}{n!} \sum_{\alpha_1, \dots, \alpha_n} B\left(\varphi(\xi_{\alpha_1}, \dots, \xi_{\alpha_n}), \psi(\eta_{\alpha_1}, \dots, \eta_{\alpha_n})\right).$$

By definition, $\langle \partial^* \varphi, \psi \rangle = -\langle \varphi, \partial \psi \rangle$, and computing the right hand side for $\varphi \in C^2(\mathfrak{g}_-, \mathfrak{g})$ and $\psi \in C^1(\mathfrak{p}_+, \mathfrak{g})$ we get

$$\frac{1}{2} \sum_{\alpha,\beta} B\left(\varphi(\xi_\alpha, \xi_\beta), -[\eta_\alpha, \psi(\eta_\beta)] + [\eta_\beta, \psi(\eta_\alpha)] + \psi([\eta_\alpha, \eta_\beta])\right),$$

and by bilinearity of B this splits as a sum of three terms. Invariance of the Killing form implies that each of the first two terms gives

$$\frac{1}{2} \sum_\beta B\Big(\sum_\alpha [\eta_\alpha, \varphi(\xi_\alpha, \xi_\beta)] \, , \, \psi(\eta_\beta)\Big).$$

For the last term, we can expand $[\eta_\alpha, \eta_\beta] = \sum_\gamma a^\gamma_{\alpha\beta} \eta_\gamma$, where $a^\gamma_{\alpha\beta} = B(\xi_\gamma, [\eta_\alpha, \eta_\beta])$. But again by invariance of the Killing form, $a^\gamma_{\alpha\beta} = B([\xi_\gamma, \eta_\alpha], \eta_\beta)$ and thus

$$\sum_\beta a^\gamma_{\alpha\beta} \xi_\beta = [\xi_\gamma, \eta_\alpha]_-,$$

the component in $\mathfrak{g}_-$ of the Lie bracket $[\xi_\gamma, \eta_\alpha]$. Using this, we can rewrite the last term as

$$\frac{1}{2}\sum_\gamma B\Big(\sum_\alpha \varphi([\eta_\alpha, \xi_\gamma]_-, \xi_\alpha)\ ,\ \psi(\eta_\gamma)\Big).$$

Thus, we see that for $\varphi \in C^2(\mathfrak{g}_-, \mathfrak{g})$ and $X \in \mathfrak{g}_-$ we get:

$$\partial^*\varphi(X) = \sum_\alpha [\eta_\alpha, \varphi(\xi_\alpha, X)] + \frac{1}{2}\sum_\alpha \varphi([\eta_\alpha, X]_-, \xi_\alpha).$$

2.6. Next, we want to show that the codifferential is the adjoint map of the differential with respect to a certain metric. By [22, Lemma 1.5] for any $|k|$-graded simple Lie algebra $\mathfrak{g}$ there is an involutive automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ which is conjugate linear in the complex case, such that $\sigma(\mathfrak{g}_i) = \mathfrak{g}_{-i}$, and $B(X, \sigma(X)) < 0$ for all $0 \neq X \in \mathfrak{g}$. Consequently, $B^*(X,Y) := -B(X, \sigma(Y))$ defines a positive definite inner product in the real case and a positive definite Hermitian inner product in the complex case, which is symmetric by invariance of the Killing form. Applying this construction to each of the simple ideals, we get the same result in the semisimple case.

Now consider the map $\mathcal{F} : C^n(\mathfrak{g}_-, \mathfrak{g}) \to C^n(\mathfrak{p}_+, \mathfrak{g})$ defined by

$$\mathcal{F}(\varphi)(Z_1, \dots, Z_n) := \sigma\left(\varphi(\sigma(Z_1), \dots, \sigma(Z_n))\right).$$

Note that this maps complex linear maps to complex linear maps, but is only conjugate linear in the complex case. A simple direct computation using the fact that σ is compatible with brackets shows that the map $\mathcal{F}$ is compatible with the Lie algebra differential, i.e. $\partial(\mathcal{F}(\varphi)) = \mathcal{F}(\partial\varphi)$.

The form B^* constructed above induces an inner product (which we also denote by B^*) on each of the spaces $C^n(\mathfrak{g}_-, \mathfrak{g}) \cong \Lambda^n(\mathfrak{g}_-^*) \otimes \mathfrak{g}$. Now we claim:

Proposition *The differential $\partial : C^n(\mathfrak{g}_-, \mathfrak{g}) \to C^{n+1}(\mathfrak{g}_-, \mathfrak{g})$ and the codifferential $\partial^* : C^{n+1}(\mathfrak{g}_-, \mathfrak{g}) \to C^n(\mathfrak{g}_-, \mathfrak{g})$ are adjoint with respect to B^*, i.e. $B^*(\partial\varphi, \psi) = B^*(\varphi, \partial^*\psi)$. In particular, for each n and ℓ the space $C^n_\ell(\mathfrak{g}_-, \mathfrak{g})$ splits as a direct sum of the image of ∂ and the kernel of ∂^*, and each cohomology class contains a unique representative, which is harmonic (i.e. ∂-closed and ∂^*-closed).*

Proof. As above let us denote by $\langle\ ,\ \rangle$ the dual pairing between $C^n(\mathfrak{g}_-, \mathfrak{g})$ and $C^n(\mathfrak{p}_+, \mathfrak{g})$ constructed using the Killing form. If $\{\xi_\alpha\}$ is an orthonormal basis for $\mathfrak{g}_-$ with respect to B^*, then the dual basis of $\mathfrak{p}_+$ with respect to the Killing form is by construction given by $\eta_\alpha = -\sigma(\xi_\alpha)$. Using this, one easily concludes that for $\varphi, \psi \in C^n(\mathfrak{g}_-, \mathfrak{g})$ we get $B^*(\varphi, \psi) = (-1)^{n+1}\langle \varphi, \mathcal{F}(\psi)\rangle$, where $\mathcal{F} : C^*(\mathfrak{g}_-, \mathfrak{g}) \to C^*(\mathfrak{p}_+, \mathfrak{g})$ is the map constructed above. But using this we compute:

$$\begin{aligned} B^*(\varphi, \partial\psi) &= (-1)^{n+1}\langle \varphi, \mathcal{F}(\partial\psi)\rangle = (-1)^{n+1}\langle \varphi, \partial(\mathcal{F}(\psi))\rangle \\ &= (-1)^n\langle \partial^*\varphi, \mathcal{F}(\psi)\rangle = B^*(\partial^*\varphi, \psi). \end{aligned}$$

□

2.7. Using the Hodge theory on the standard complex, the cohomology $H^*(\mathfrak{g}_-, \mathfrak{g})$ can now be computed in the complex simple case using Kostant's version of the Bott-Borel-Weil theorem, see [13]. The result that we will need directly in the prolongation procedure is the computation of the first cohomology groups, which have been carried out in [23]. It should be noted here, that in [23] the notation is slightly different from ours, namely what we denote by $H^k_\ell(\mathfrak{g}_-, \mathfrak{g})$ is denoted by $H^{\ell-k+1,k}(\mathfrak{m}, \mathfrak{g})$ there.

Proposition *Let $\mathfrak{g}$ be a complex simple $|k|$-graded Lie algebra. Then for each $\ell > 0$ the cohomology group $H^1_\ell(\mathfrak{g}_-, \mathfrak{g})$ is trivial, except in the following cases (using the Dynkin diagram notation, see* 2.3*):*

1. $\times$, *i.e.* $\mathfrak{g} = A_1$, *and $\mathfrak{p} \subset \mathfrak{g}$ is the Borel subalgebra. In this case, $H^1_2(\mathfrak{g}_-, \mathfrak{g})$ is the only nonzero component with $\ell > 0$.*
2. $\times\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet \cong \bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\times$, *i.e.* $\mathfrak{g} = A_n$ *for some $n > 1$, and $\mathfrak{p}$ is the maximal parabolic corresponding to either the first or the last root. In this case, $H^1_1(\mathfrak{g}_-, \mathfrak{g})$ is the nonzero component.*
3. $\times\!\!-\!\!\bullet \cdots \bullet\!\Leftarrow\!\bullet$, *i.e.* $\mathfrak{g} = C_n$ *for some $n \geq 2$, and $\mathfrak{p}$ is the maximal parabolic corresponding to the first root. In this case, $H^1_1(\mathfrak{g}_-, \mathfrak{g})$ is the nonzero component.*

Proof. see [23, Proposition 5.1]. □

This also completely solves the problem in the real simple case, since by [23, Lemma 3.5] the first cohomology group of positive homogeneity of of a complexification is the complexification of the corresponding real cohomology group of the same homogeneity. To deal with the semisimple case, we have the following

Proposition 2.8 *Let $\mathfrak{g}'$ be a semisimple $|k'|$-graded Lie algebra such that* *no simple factor is contained in* $\mathfrak{g}'_0$ *and* $\mathfrak{g}''$ *be a semisimple* $|k''|$*-graded Lie algebra such that no simple factor is contained in* $\mathfrak{g}''_0$*, and put* $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$*. Then for each* $\ell > 0$ *we have* $H^1_\ell(\mathfrak{g}_-, \mathfrak{g}) \cong H^1_\ell(\mathfrak{g}'_-, \mathfrak{g}') \oplus H^1_\ell(\mathfrak{g}''_-, \mathfrak{g}'')$*. If* $k', k'' \geq 2$*, then the result also holds for* $\ell = 0$*.*

Proof. Since we have $\mathfrak{g}_- = \mathfrak{g}'_- \oplus \mathfrak{g}''_-$ and $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$, we can write any linear map $\psi : \mathfrak{g}_- \to \mathfrak{g}$ as a block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A : \mathfrak{g}'_- \to \mathfrak{g}'$, $B : \mathfrak{g}''_- \to \mathfrak{g}'$, and so on. Now suppose that ψ is a cocycle. Then for all $X, Y \in \mathfrak{g}_-$ we have

$$0 = [X, \psi(Y)] - [Y, \psi(X)] - \psi([X, Y]).$$

Applying this to $X, Y \in \mathfrak{g}'_-$, we get two equations. The first is exactly the cocycle equation for A, while the second says that $C([X,Y]) = 0$, for all X, Y. This means exactly, that C vanishes on $\mathfrak{g}'_{-k'} \oplus \cdots \oplus \mathfrak{g}'_{-2}$. Similarly, for $X, Y \in \mathfrak{g}''_-$ we get that D is a cocycle, and B can be nonzero only on $\mathfrak{g}''_{-1}$. Finally, taking $X \in \mathfrak{g}'_-$ and $Y \in \mathfrak{g}''_-$, we get $[X, B(Y)] = 0$ and $[Y, C(X)] = 0$. But by 2.2(4) this implies that $B(Y) \in \mathfrak{g}'_{-k'}$ and $C(X) \in \mathfrak{g}''_{-k''}$, so B and C can contain only components homogeneous of degree up to $-k' + 1$ respectively $-k'' + 1$, and the result follows. □

2.9. Let $\mathfrak{g}$ be a semisimple $|k|$-graded Lie algebra, and let G be a Lie group with Lie algebra $\mathfrak{g}$. Since G is semisimple, each element of G is determined by its adjoint action up to elements of the center of G. Let $P \subset G$ be the subgroup of those elements which satisfy $\mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \oplus \mathfrak{g}_{i+1} \oplus \cdots \oplus \mathfrak{g}_k$ for each $i = -k, \ldots, k$. This can be interpreted as follows: The $|k|$-grading of $\mathfrak{g}$ gives rise to an associated filtration $\mathfrak{g} = \mathcal{F}_{-k}(\mathfrak{g}) \supset \mathcal{F}_{-k+1}(\mathfrak{g}) \supset \cdots \supset \mathcal{F}_k(\mathfrak{g}) \supset 0$, defined by $\mathcal{F}_i(\mathfrak{g}) := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$, for all $i = -k, \ldots, k$. Clearly, this filtration is compatible with the Lie bracket, i.e. $[\mathcal{F}_i(\mathfrak{g}), \mathcal{F}_j(\mathfrak{g})] \subset \mathcal{F}_{i+j}(\mathfrak{g})$. (In fact, in many points in the sequel it will be more natural to view this filtration as the main structure on $\mathfrak{g}$ and not the actual grading.) Then $P \subset G$ is the subgroup of all elements whose adjoint action is an automorphism of the filtered Lie algebra $\mathfrak{g}$.

We also define a second subgroup G_0 of G as the set of all those g which satisfy that $\mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i$ for all $i = -k, \ldots, k$. By definition, G_0 is a subgroup of P, and it consists of those elements whose adjoint action preserves even the grading of $\mathfrak{g}$.

Proposition *The subgroup P has Lie algebra $\mathfrak{p}$, and G_0 has Lie algebra $\mathfrak{g}_0$.*

Proof. In view of the equation $\mathrm{Ad}(\exp(X)) = e^{\mathrm{ad}(X)}$ for $X \in \mathfrak{g}$, it suffices to show that $\mathfrak{p}$ and $\mathfrak{g}_0$ are precisely the subspaces of those $X \in \mathfrak{g}$ such that $\mathrm{ad}(X)$ preserves the filtration respectively the grading of $\mathfrak{g}$. So suppose that $\mathrm{ad}(X)$ preserves the filtration of $\mathfrak{g}$. Then we can uniquely write $X = X_{-k} + \cdots + X_k$ with $X_i \in \mathfrak{g}_i$. Since $\mathrm{ad}(X)$ preserves the filtration, we must have $[X, E] \in \mathfrak{p}$, where E is the element from 2.2(1). But by definition of E, we have $[X, E] = kX_{-k} + \cdots + X_{-1} - X_1 - \cdots - kX_k$, and this is in $\mathfrak{p}$ if and only if $X \in \mathfrak{p}$. If $\mathrm{ad}(X)$ even preserves the grading then $[X, E]$ must be in $\mathfrak{g}_0$, so $[X, E]$ must be zero in this case, which implies $X \in \mathfrak{g}_0$. □

The structure of the group P is clarified in the following

Proposition 2.10 *Let $g \in P$ be any element. Then there exist unique elements $g_0 \in G_0$ and $X_i \in \mathfrak{g}_i$ for $i = 1, \dots, k$, such that $g = g_0 \exp(X_1) \cdots \exp(X_k)$.*

Proof. (see [21, Lemma 2.6]) Consider the adjoint action $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$. This is an automorphism of the filtered Lie algebra $\mathfrak{g}$. In particular, $\mathrm{Ad}(g)$ maps each $\mathfrak{g}_i$ to $\bigoplus_{j \geq i} \mathfrak{g}_j$. If we just take the lowest component of this map, we get an automorphism of the *graded* Lie algebra $\mathfrak{g}$, which we denote by φ_0. By construction, $\varphi_0(Y)$ is congruent to $\mathrm{Ad}(g)Y$ modulo $\mathfrak{g}_{i+1} \oplus \cdots \oplus \mathfrak{g}_k$ for all $Y \in \mathfrak{g}_i$.

Thus, for $\varphi_1 := \varphi_0^{-1} \circ \mathrm{Ad}(g)$ we get that $\varphi_1(Y)$ is congruent to Y modulo $\mathfrak{g}_{i+1} \oplus \cdots \oplus \mathfrak{g}_k$ for all $Y \in \mathfrak{g}_i$. In particular, for the element $E \in \mathfrak{g}_0$ from 2.2(1), we have $E - \varphi_1(E) \in \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, and we denote by X_1 the component in $\mathfrak{g}_1$ of this element. This means that $\varphi_1(E)$ is congruent to $E - X_1$ modulo $\mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$. Moreover, $\mathrm{Ad}(\exp(-X_1))(E - X_1) = E$, so for $\varphi_2 = \mathrm{Ad}(\exp(-X_1)) \circ \varphi_1$ we see that $\varphi_2(E)$ is congruent to E modulo $\mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$, while for each $Y \in \mathfrak{g}_i$ the element $\varphi_2(Y)$ clearly is congruent to Y modulo $\mathfrak{g}_{i+1} \oplus \cdots \oplus \mathfrak{g}_k$. Inductively, we find elements $X_j \in \mathfrak{g}_j$ and automorphisms φ_j of $\mathfrak{g}$ of the form $\varphi_j = \mathrm{Ad}(\exp(-X_{j-1})) \circ \varphi_{j-1}$, such that $\varphi_j(E)$ is congruent to E modulo $\mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_k$, and such that $\varphi_j(Y)$ is congruent to Y modulo $\mathfrak{g}_{i+1} \oplus \cdots \oplus \mathfrak{g}_k$ for each $Y \in \mathfrak{g}_i$.

Then consider φ_{k+1}. By construction, we have $\varphi_{k+1}(E) = E$. Then for $Y \in \mathfrak{g}_i$ we see that $[E, \varphi_{k+1}(Y)] = \varphi_{k+1}([E, Y]) = i\varphi_{k+1}(Y)$, so $\varphi_{k+1}(Y) \in \mathfrak{g}_i$. But by construction, $\varphi_{k+1}(Y)$ is congruent to Y modulo $\mathfrak{g}_{i+1} \oplus \cdots \oplus \mathfrak{g}_k$, so $\varphi_{k+1}(Y) = Y$. This means that we can write the identity map as

$$\mathrm{Ad}(\exp(-X_k)) \circ \cdots \circ \mathrm{Ad}(\exp(-X_1)) \circ \varphi_0^{-1} \circ \mathrm{Ad}(g),$$

so φ_0 is the adjoint action of $\tilde{g}_0 := g\exp(-X_k)\cdots\exp(-X_1)$, so this element lies in G_0. Moreover, by construction $\tilde{g}_0\exp(X_1)\cdots\exp(X_k)$ has the same adjoint action as g, so since G is semisimple these two elements differ by an element of the center of G, which by definition is contained in G_0. Putting g_0 the product of this element with $\tilde{g}_0$, we get a representation of g as required.

For the uniqueness, assume that we have

$$g_0\exp(X_1)\cdots\exp(X_k) = \hat{g}_0\exp(\hat{X}_1)\cdots\exp(\hat{X}_k).$$

Then $\hat{g}_0 = \exp(-\hat{X}_1)\cdots\exp(-\hat{X}_k)g_0\exp(X_1)\cdots\exp(X_k)$. Considering the adjoint actions of both these elements on each $\mathfrak{g}_i$ and computing modulo $\mathfrak{g}_{i+1}\oplus\cdots\oplus\mathfrak{g}_k$, we see that $g_0^{-1}\hat{g}_0$ lies in the center of G. Thus, in particular we see that $\mathrm{Ad}(\exp(X_1)\cdots\exp(X_k)) = \mathrm{Ad}(\exp(\hat{X}_1)\cdots\exp(\hat{X}_k))$. Applying this to E and computing modulo $\mathfrak{g}_2\oplus\cdots\oplus\mathfrak{g}_k$, we see that $X_1 = \hat{X}_1$. Inductively, $X_i = \hat{X}_i$ for all $i = 1,\dots,k$, and thus also $g_0 = \hat{g}_0$. □

2.11. Next, we define a subgroup $P_+ \subset P$ as the image of $\mathfrak{p}_+ \subset \mathfrak{p}$ under the exponential map. From Proposition 2.10 it follows that the exponential map $\exp : \mathfrak{p}_+ \to P_+$ is a global diffeomorphism. Moreover, again using 2.10, we see that $P/P_+ \cong G_0$, so P is actually a semidirect product of G_0 and P_+. The powers of the nilpotent Lie group P_+ are exactly the exponential images of the powers of $\mathfrak{p}_+$, so we have $P_+^i = \exp(\mathfrak{p}_+^i)$ for all $i = 1,\dots,k$. In the sequel, we will heavily need the quotients P/P_+^i for $i = 1,\dots,k$. Clearly, they are again semidirect products of G_0 with the groups P_+/P_+^i.

Now let $g \in P$ be an element, and write $g = g_0\exp(X_1)\cdots\exp(X_k)$ as in Proposition 2.10. Take an element $h = \exp(Y_i)\cdots\exp(Y_k) \in P_+^i$. Then by the Baker-Campbell-Hausdorff formula we may write $\exp(X_i)\cdots\exp(X_k)h$ as an expression of the form $\exp(Z_i)\cdots\exp(Z_k)$ for certain elements $Z_j \in \mathfrak{g}_j$. Thus, if we decompose the product gh according to 2.10, we must get

$$gh = g_0\exp(X_1)\cdots\exp(X_{i-1})\exp(Z_i)\cdots\exp(Z_k).$$

In particular, this implies that the mapping $G_0 \times (\mathfrak{g}_1\oplus\cdots\oplus\mathfrak{g}_{i-1}) \to P/P_+^i$ which maps $(g_0, X_1,\dots,X_{i-1})$ to the class of $g_0\exp(X_1)\cdots\exp(X_{i-1})$ is a global diffeomorphism.

This construction can also be used to construct for each $i = 1, \dots, k$ a canonical smooth section $s : P/P_+^i \to P/P_+^{i+1}$ of the natural quotient map $P/P_+^{i+1} \to P/P_+^i$. One simply pushes forward the inclusion $G_0 \times (\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{i-1}) \to G_0 \times (\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_i)$ with the diffeomorphism constructed above. Note, however, that these are not group homomorphisms, unless $i = 1$.

2.12. By definition, the subgroup $P \subset G$ acts on each of the filtration components $\mathcal{F}_i(\mathfrak{g}) = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$, and for $j > i$ the component $\mathcal{F}_j(\mathfrak{g}) \subset \mathcal{F}_i(\mathfrak{g})$ is a submodule. Thus, we can pass to the quotient $\mathcal{F}_i(\mathfrak{g})/\mathcal{F}_j(\mathfrak{g})$ which is isomorphic as a vector space to $\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{j-1}$. In particular, this leads to a P-action on $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_-$. We will denote all these actions by Ad (and all the corresponding Lie algebra actions by ad) if there is no risk of confusion.

Using that $\mathrm{Ad}(\exp(X)) = e^{\mathrm{ad}(X)}$ it is clear that an element $g \in P$ is contained in P_+^j if and only if $(\mathrm{Ad}(g) - \mathrm{id})(\mathcal{F}_i(\mathfrak{g})) \subset \mathcal{F}_{i+j}(\mathfrak{g})$ for all i. In particular, this implies that P_+^j acts trivial on $\mathcal{F}_i(\mathfrak{g})/\mathcal{F}_{i+j}(\mathfrak{g})$, so the action of P on this space factors to an action of P/P_+^j.

2.13. Using the action of P on $\mathfrak{g}_-$ constructed in 2.12 above, we get an action of P on $C^n(\mathfrak{g}_-, \mathfrak{g}) = L(\Lambda^n \mathfrak{g}_-, \mathfrak{g})$.

Proposition *The codifferential $\partial^* : C^n(\mathfrak{g}_-, \mathfrak{g}) \to C^{n-1}(\mathfrak{g}_-, \mathfrak{g})$ is P-equivariant.*

Proof. (see also [22, 1.12]) To define the codifferential, we have used the Killing form to identify $\mathfrak{g}_-$ with the dual of $\mathfrak{p}_+$ and $\mathfrak{g}$ with its own dual. Now for $X \in \mathfrak{g}_-$ and $b \in P$ we can write $\mathrm{Ad}(b)(X)$ (the adjoint action in $\mathfrak{g}$) as the sum of the action of b on X constructed in 2.12 above plus a component in $\mathfrak{p}$. Since the Killing form vanishes on $\mathfrak{p} \times \mathfrak{p}_+$, we see that invariance of the Killing form implies that $\mathfrak{g}_-$ is in fact dual to $\mathfrak{p}_+$ as a P-module. The identification of $\mathfrak{g}$ with its own dual is, again by invariance of the Killing form, an isomorphism of G- and thus of P-modules.

Thus, we see that for each n the spaces $C^n(\mathfrak{g}_-, \mathfrak{g})$ and $C^n(\mathfrak{p}_+, \mathfrak{g})$ are actually dual P-modules. But then a simple direct computation shows that the Lie algebra differential $\partial : C^n(\mathfrak{p}_+, \mathfrak{g}) \to C^{n+1}(\mathfrak{p}_+, \mathfrak{g})$ is a P-module homomorphism, and the result follows. □

3. P-frame bundles and the prolongation procedure

Throughout this section we fix a semisimple $|k|$-graded Lie algebra $\mathfrak{g}$ and a Lie group G with Lie algebra $\mathfrak{g}$. We continue to use the notation of Section 2.

The aim of this section is to show how to construct principal P-bundles equipped with Cartan connections from underlying structures.

3.1. The basic ingredient in our study is a manifold M together with a filtration of the tangent bundle $T^{-k}M = TM \supset T^{-k+1}M \supset \cdots \supset T^{-1}M$ by subvector bundles, such that for each $i = -k, \dots, -1$ the rank of $T^i(M)$ equals the dimension of $\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{-1}$.

Next, let $p : E \to M$ be a locally trivial fiber bundle. Then we get an induced filtration of the tangent bundle of E as $T^{-k}E = TE \supset T^{-k+1}E \supset \cdots \supset T^{-1}E \supset T^0E := VE$, where VE denotes the vertical bundle of E. This filtration is simply given by $T^iE := (Tp)^{-1}(T^iM)$, where Tp denotes the tangent map to the projection p. Note that if E is a principal bundle with some structure group H, then the principal action of H on E induces an action of H on the tangent bundle TE, and by construction of the induced filtration, each of the subbundles T^iE is invariant under this action.

3.2. Let $p : E \to M$ be a principal bundle with structure group P or P/P^i_+ for some $i = 1, \dots, k$ over a manifold with a filtration of its tangent bundle as in 3.1. In this case, we can prolong the filtration of the tangent bundle of E by putting T^jE the image of $\mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{i-1}$ under the fundamental vector field mapping for $j = 1, \dots, i-1$.

Definition Let ℓ be an integer which is $\leq i$ if the structure group of E is P/P^i_+ and $\leq 2k+1$ if the structure group is P. We define a *frame form* θ of length ℓ on E as a k-tuple $\theta = (\theta_{-k}, \dots, \theta_{-1})$, where θ_j is a smooth section of the bundle $L(T^jE, \mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{\min\{k,j+\ell-1\}})$ of linear maps such that

1. The kernel of θ_j in each point $u \in E$ is exactly the subbundle $T^{j+\ell}_uE$.
2. The forms are mutually compatible, i.e. the restriction of θ_j to $T^{j+1}E$ has vanishing $\mathfrak{g}_j$-component and its components in $\mathfrak{g}_{j+1} \oplus \cdots \oplus \mathfrak{g}_{\min\{k,j+\ell-1\}}$ coincide with the components of θ_{j+1} in that part.
3. Each θ_j is P/P^i_+-equivariant, i.e. $(r^b)^*\theta_j = \mathrm{Ad}(b^{-1}) \circ \theta_j$, where r^b denotes the principal right action of b, and Ad denotes the action of P/P^i_+ on $\mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{\min\{k,j+\ell-1\}}$ introduced in 2.12.
4. For $A \in \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{i-1}$ let ζ_A be the fundamental vector field cor-

responding to A. Then for $j+\ell \leq 0$ we have $\theta_j(\zeta_A) = 0$, while for $j+\ell > 0$, $\theta_j(\zeta_A)$ gives the components of A in $\mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{\min\{k,j+\ell-1\}}$.

Remarks 3.3

1. If θ is a frame form of length ℓ, then simply by forgetting components it gives rise to frame forms of length $1,\dots,\ell-1$. The only point that is not completely obvious is the equivariancy but this holds by definition of the action Ad, see 2.12.
2. In the case $\ell = 1$, the second and last conditions become vacuous. Thus, a frame form of length one is just a collection $\theta = (\theta_{-k},\dots,\theta_{-1})$ such that each θ_j is a smooth section of $L(T^jE, \mathfrak{g}_j)$ which is equivariant and for each point $u \in E$ induces a linear isomorphism $T^j_uE/T^{j+1}_uE \to \mathfrak{g}_j$.
3. On the other hand, if the length ℓ becomes bigger than $k+2$, then some components of θ contain no information, since they are just restrictions of lower components. In the extremal case, $\ell = 2k+1$, the whole information is contained in the form θ_{-k}, which is by definition a Cartan connection.
4. To avoid compicated subscripts, we will in the sequel follow the convention that we simply view the component θ_j of a frame form as having values in $\mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{j+\ell-1}$, i.e. that we agree that $\mathfrak{g}_i = \{0\}$ for $|i| > k$, and we simply forget the components with values in zero spaces.

3.4. The structure equations

Let $p : E \to M$ be a principal bundle with group P/P^i_+ for some $i = 1,\dots,k$, and let θ be a frame form of length one on E. The structure equations impose a certain restriction to the frame form θ (and through that to M), which can be formally written as $d\theta_{i+j} + [\theta_i, \theta_j] = 0$. Since the individual θ_i are only partially defined, this does not make sense as it stands but needs an appropriate interpretation:

The lowest component θ_{-k} of θ is simply a $\mathfrak{g}_{-k}$ valued one-form on E. Thus, we can differentiate it to obtain a $\mathfrak{g}_{-k}$-valued two form $d\theta_{-k}$ on E. Then for each pair (i,j) of negative integers such that $i+j = -k$, each point $u \in E$, and elements $\xi \in T^i_uE$ and $\eta \in T^j_uE$ we consider

$$d\theta_{-k}(\xi,\eta) + [\theta_i(\xi), \theta_j(\eta)] \in \mathfrak{g}_{-k},$$

where we use the bracket $[\ ,\ ] : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{-k}$. This gives a collection of well defined smooth functions $T^iE \otimes T^jE \to \mathfrak{g}_{-k}$, which we call the *structure function* of degree $-k$.

Now assume that θ has the property that the structure function of degree $-k$ is identically zero, and take a pair of negative integers (i, j) such that $i+j > -k$. Let $\tilde{\xi}$ be a section of T^iE and $\tilde{\eta}$ a section of T^jE. Then we can also view $\tilde{\eta}$ as a section of $T^{-k-i}E$, and since $-k-i < j$, we see that $\theta_{-k-i}(\tilde{\eta})$ is identically zero, so vanishing of the structure function of order k implies that $0 = d\theta_{-k}(\tilde{\xi}, \tilde{\eta})$. Since $\theta_{-k}(\tilde{\xi})$ and $\theta_{-k}(\tilde{\eta})$ are identically zero, this means that $\theta_{-k}([\tilde{\xi}, \tilde{\eta}]) = 0$, so the Lie bracket $[\tilde{\xi}, \tilde{\eta}]$ is actually a section of $T^{-k+1}E$.

But this means that if $i+j \geq -k+1$, then $\tilde{\xi}$, $\tilde{\eta}$ and $[\tilde{\xi}, \tilde{\eta}]$ are all sections of $T^{-k+1}E$. Thus, if we extend θ_{-k+1} to a $\mathfrak{g}_{-k+1}$-valued one form and take the exterior derivative, then the value of the resulting two form on $(\tilde{\xi}, \tilde{\eta})$ is independent of the extension. Hence, for $i + j = -k + 1$, $\xi \in T^i_uE$ and $\eta \in T^j_uE$ we get a well defined element

$$d\theta_{-k+1}(\xi, \eta) + [\theta_i(\xi), \theta_j(\eta)] \in \mathfrak{g}_{-k+1}.$$

As above, this gives rise to a smooth function $T^iE \otimes T^jE \to \mathfrak{g}_{-k+1}$ for $i + j = -k + 1$, and the collection of these functions is called the *structure function* of degree $-k + 1$.

Now this procedure can easily be iterated. If the structure function of degree $-k+1$ vanishes identically, then for $i + j > -k + 1$, the Lie bracket of a section of T^iE and a section of T^jE lies in $T^{-k+2}E$, so $d\theta_{-k+2}$ is well defined on such sections, and we get a well defined structure function of degree $-k + 2$, and so on.

Definition

1. We say that the frame form θ of length one *satisfies the structure equations* iff the structure functions of all orders $-k, \dots, -1$ vanish.
2. We say that a frame form of length ℓ satisfies the structure equations iff the underlying frame form of length one has this property.

Remark 3.5 The existence of a frame form which satisfies the structure equations implies subtle conditions on the degree of non-integrability of the sub-bundles T^iE of TE and thus also of the sub-bundles T^iM of TM. In particular, if such a frame form exists, then the Lie bracket of a section of T^iE with a section of T^jE is always a section of $T^{i+j}E$, but in general not of $T^{i+j+1}E$. We will give a detailed discussion of the geometric meaning of the structure equations in 4.2.

Definition 3.6 Let M be a manifold with a filtration of its tangent bundle as in 3.1. For $\ell = 1, \dots, 2k+1$ we define a *P-frame bundle* of degree ℓ over M as a principal fiber bundle $p : E \to M$ with group P/P_+^ℓ together with a frame form θ of length ℓ on E, which satisfies the structure equations.

3.7. By definition, a P-frame bundle of degree one over a manifold M is a principal bundle $E \to M$ with group P/P_+ together with a sequence $\theta = (\theta_{-k}, \dots, \theta_{-1})$ of smooth sections θ_i of the bundle $L(T^iE, \mathfrak{g}_i)$, such that for each point $u \in E$, each θ_i induces an isomorphism between $T_u^iE/T_u^{i+1}E$ and $\mathfrak{g}_i$. Since moreover each θ_i has to be P/P_+-equivariant, this means exactly that the form $\theta_{-k} \oplus \dots \oplus \theta_{-1}$ identifies the bundle E as a reduction to the structure group $P/P_+ \cong G_0$ of the associated graded vector bundle $(T^{-k}M/T^{-k+1}M) \oplus \dots \oplus (T^{-2}M/T^{-1}M) \oplus T^{-1}M$ to the tangent bundle of M. So P-frame bundles of degree one are just reductions to the structure group G_0 of the associated graded to the tangent bundle, which in addition satisfy the structure equations.

Also, the other extremal case is fairly easy to describe. A P-frame bundle of degree $2k+1$ is by definition a principal bundle with group P, which is equipped with a frame from of length $2k+1$, and we have already remarked in 3.3(3) that this frame form is actually a Cartan connection. So P-frame bundles of degree $2k+1$ are just P-principal bundles equipped with a Cartan connection which satisfies the structure equations. We shall discuss later, how the structure equations are related to the curvature of the Cartan connection.

3.8. Let $(p : E \to M, \theta)$ be a P-frame bundle of degree ℓ. Since θ satisfies the structure equations, we know from 3.5 that the Lie bracket of a section of T^iE with a section of T^jE is a section of $T^{i+j}E$ for all $i, j < 0$. But this means, that if $i, j < 0$ are such that $i + j \geq -k$, then we have a well defined tensorial map $d\theta_{i+j} : T^iE \otimes T^jE \to \mathfrak{g}_{i+j} \oplus \dots \oplus \mathfrak{g}_{i+j+\ell-1}$. In particular, since $T^0E = VE$, the vertical bundle, we can form $d\theta_i(\zeta_A, \xi)$ for elements $\xi \in T^iE$ and $A \in \mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_{\ell-1}$.

Lemma *In this situation, we have $d\theta_i(\zeta_A, \xi) = -\operatorname{ad}(A)(\theta_i(\xi))$, where* ad *is the Lie algebra action corresponding to the group action introduced in* 2.12.

Proof. Equivariancy of θ_i reads as $(r^b)^*\theta_i = \mathrm{Ad}(b^{-1}) \circ \theta_i$ for all $b \in P/P_+^\ell$. In particular, we can apply this to $b = \exp(tA)$. Evaluating this on an element $\xi \in T_u^i E$, we get a smooth curve in $\mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_{i+\ell-1}$, and differentiating at zero we get

$$\begin{aligned} \frac{d}{dt}|_{t=0}((r^{\exp(tA)})^*\theta_i)(\xi) &= \frac{d}{dt}|_{t=0} \mathrm{Ad}(\exp(-tA))(\theta_i(\xi)) \\ &= -\mathrm{ad}(A)(\theta_i(\xi)). \end{aligned}$$

Now we can extend θ_i to a globally defined one form $\tilde{\theta}_i$ with values in $\mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_{i+\ell-1}$. Then this one form still satisfies the above equation for $\xi \in T^i E$, and the left hand side simply reads as $\mathcal{L}_{\zeta_A}\tilde{\theta}_i(\xi)$, the Lie derivative along the fundamental vector field. But $\mathcal{L}_{\zeta_A}\tilde{\theta}_i(\xi)$ equals $d(i_{\zeta_A}\tilde{\theta}_i) + i_{\zeta_A}(d\tilde{\theta}_i)$, where i_{ζ_A} denotes the insertion operator. Since ζ_A is a section of $T^0 E \subset T^i E$, the above equation holds on $T^i E$ with $\tilde{\theta}_i$ replaced by θ_i. But since $i_{\zeta_A}\theta_i$ is constant, only the second term remains, and we get $\frac{d}{dt}|_{t=0}((r^{\exp(tA)})^*\theta_i)(\xi) = d\theta_i(\zeta_A, \xi)$. □

3.9. In the situation of 3.8, let $u \in E$ be a point. We define the *torsion* of (E, θ) in u as a linear map $t_\theta(u) : \mathfrak{g}_- \wedge \mathfrak{g}_- \to \mathfrak{g}$, which has homogeneous components of degrees $0, \dots, \ell - 1$ only, as follows: Take $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$, where $i + j + \ell > -k$, and choose elements $\xi \in T_u^i E$ and $\eta \in T_u^j E$ such that $\theta_i(\xi) = X$ and $\theta_j(\eta) = Y$. If $i + j \le -k$, then we define $t_\theta(u)(X, Y)$ as the components in $\mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{i+j+\ell-1}$ of $d\theta_{-k}(\xi, \eta)$. If, on the other hand, $i + j > -k$, then we put $t_\theta(u)(X, Y) := d\theta_{i+j}(\xi, \eta) \in \mathfrak{g}_{i+j} \oplus \dots \oplus \mathfrak{g}_{i+j+\ell-1}$.

We have to show, that this is well defined. Thus, let us assume that we have two elements ξ_1, ξ_2 such that $\theta_i(\xi_1) = \theta_i(\xi_2) = X$. Then their difference is in the kernel of θ_i, which by definition is $T_u^{i+\ell}E$. Now we have to distinguish two cases:

(1) If $i + \ell < 0$, then both $\xi_2 - \xi_1$ and η lie in $T_u^{i+j+\ell}$, and so does the Lie bracket of any two sections of $T^{i+\ell}E$ and $T^j E$. Thus, if $i + j \ge -k$, then by definition $d\theta_{i+j}(\xi_2 - \xi_1, \eta) = 0$, while for $i + j < -k$ the components of $d\theta_{-k}(\xi_2 - \xi_1, \eta)$ that we consider are zero as well.

(2) If $i + \ell \ge 0$, then by definition there is an element $A \in \mathfrak{g}_{i+\ell}$ such that $\xi_2 - \xi_1 = \zeta_A(u)$, the value of the fundamental vector field. But by Lemma 3.8, this implies that $d\theta_{i+j}(\xi_2 - \xi_1, \eta) = -\mathrm{ad}(A)(\theta_{i+j}(\eta))$ (or the respective equation with $i + j$ replaced by $-k$). But if $\theta_{i+j}(\eta)$ (respectively $\theta_{-k}(\eta)$) is nonzero (which means that ℓ is big enough), then it has values in $\mathfrak{g}_j$, so $-\mathrm{ad}(A)(\theta_{i+j}(\eta))$ is an element of $\mathfrak{g}_{i+j+\ell}$ and hence plays no role.

Definition 3.10 Let (E, θ) be a P-frame bundle over M of degree ℓ. Then the homogeneous components t^j_θ of degree $j = 0, \dots, \ell - 1$ of the torsion t_θ define smooth functions on E with values in the space $L_j(\mathfrak{g}_- \wedge \mathfrak{g}_-, \mathfrak{g})$ of homogeneous maps. In 2.5 we have introduced the codifferential ∂^* : $L_j(\mathfrak{g}_- \wedge \mathfrak{g}_-, \mathfrak{g}) \to L_j(\mathfrak{g}_-, \mathfrak{g})$. We call the P-frame bundle (E, θ) *harmonic* if and only if for all $j = 1, \dots, \ell - 1$ we have $\partial^* \circ t^j_\theta = 0$. (Note that the component t^0_θ is already completely determined by the requirement that θ satisfies the structure equations.)

3.11. Underlying P-frame bundles of lower degree

Let $(p : E \to M, \theta)$ be a P-frame bundle of degree $\ell > 1$. We construct from this a P-frame bundle $(\underline{E}, \underline{\theta})$ of degree $\ell - 1$ over M, which is called the *underlying P-frame bundle*.

If $\ell > k+1$ then this construction is completely trivial, since in this case both E and $\underline{E}$ have to be principal P-bundles, so we keep the same bundle and define the new frame form $\underline{\theta}$ by letting $\underline{\theta}_i$ be the first $\ell - 1$ components of θ_i. Obviously, $(\underline{E}, \underline{\theta})$ is a P-frame bundle of degree $\ell - 1$.

So let us assume that $\ell \leq k + 1$. Then E is a principal bundle with group P/P^ℓ_+, and we have the non trivial subgroup $P^{\ell-1}_+/P^\ell_+$, which acts freely on E. Now we define $\underline{E} := E/(P^{\ell-1}_+/P^\ell_+)$, the space of orbits under the action of this group. Then $\pi : E \to \underline{E}$ is a principal bundle with this group, and clearly $\underline{E} \to M$ is a principal bundle with group $P/P^{\ell-1}_+$. Next, we define a frame form $\underline{\theta}$ of length $\ell - 1$ on $\underline{E}$ as follows: Let ξ be an element of $T^i_u\underline{E}$ for some $i = -k, \dots, -1$. Choose a point $x \in E$ with $\pi(x) = u$ and an element $\tilde{\xi} \in T^i_x E$ with $T\pi \cdot \tilde{\xi} = \xi$, and put $\underline{\theta}_i(\xi)$ the components of $\theta_i(\tilde{\xi})$ in $\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+\ell-2}$.

To show that this is well defined, let us first assume that we have two choices $\tilde{\xi}_1$ and $\tilde{\xi}_2$ for one point x. Then there is an element $A \in \mathfrak{g}_{\ell-1}$ such that $\tilde{\xi}_2 - \tilde{\xi}_1 = \zeta_A(x)$, so the difference lies in $T^{\ell-1}E$, which is contained in the kernel of θ_i. So let us assume that we have two choices x_1 and x_2 for the point in E. Then there is an element $b \in P^{\ell-1}_+/P^\ell_+ \subset P/P^\ell_+$ such that $x_2 = x_1 \cdot b$, and if $\tilde{\xi}_1 \in T^i_{x_1}E$ is such that $T\pi \cdot \tilde{\xi}_1 = \xi$, then $Tr^b \cdot \tilde{\xi}_1$ is an appropriate choice for $\tilde{\xi}_2 \in T^i_{x_2}E$. But then $\theta_i(\tilde{\xi}_2) = ((r^b)^*\theta_i)(\tilde{\xi}_1) = \mathrm{Ad}(b^{-1})(\theta_i(\tilde{\xi}_1))$. Now by Proposition 2.10, there is an $A \in \mathfrak{g}_{\ell-1}$ such that $b = \exp(A)$, so the right hand side of this equation becomes $e^{\mathrm{ad}(A)}(\theta_i(\tilde{\xi}_1))$, which differs from $\theta_i(\tilde{\xi}_1)$ only in the component $\mathfrak{g}_{i+\ell-1}$, so it again plays no role.

Thus, we have defined $\underline{\theta}_i$ for all i, and one easily verifies that this actually defines a frame form of length $\ell - 1$ on $\underline{E}$. To verify that this frame form satisfies the structure equations, we proceed as follows: Let σ be a local section of the bundle $\pi : E \to \underline{E}$. Since $T\sigma(T^i\underline{E}) \subset T^iE$ for all i, we can form the pullback $\sigma^*\theta_i$ which is a local section of $L(T^i\underline{E}, \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+\ell-1})$. By construction, $\underline{\theta}_i$ coincides with the first $\ell - 1$ components of this pullback.

Let us denote by Θ and $\underline{\Theta}$ the frame forms of length one underlying θ and $\underline{\theta}$, respectively. Then $\underline{\Theta}_{-k}$ locally equals $\sigma^*\Theta_{-k}$, so $d\underline{\Theta}_{-k}$ locally equals $\sigma^*d\Theta_{-k}$. Thus, for $\xi \in T^i_u\underline{E}$ and $\eta \in T^j_u\underline{E}$ such that $i + j = -k$ and u is in the domain of σ we have

$$\begin{aligned} d\underline{\Theta}_{-k}(\xi, \eta) &= d\Theta_{-k}(T\sigma \cdot \xi, T\sigma \cdot \eta) \\ &= -[\Theta_i(T\sigma \cdot \xi), \Theta_j(T\sigma \cdot \eta)] = -[\underline{\Theta}_i(\xi), \underline{\Theta}_j(\eta)], \end{aligned}$$

so the structure function of degree $-k$ on $\underline{E}$ vanishes identically.

Now if we extend Θ_{-k+1} to a one form and pull back this extension along σ, then we get an extension of $\underline{\Theta}_{-k+1}$, so locally we must have $d\underline{\Theta}_{-k+1} = \sigma^*d\Theta_{-k+1}$ on $T^i\underline{E} \otimes T^j\underline{E}$ with $i + j \geq -k + 1$, so as above we conclude that the structure function of degree $-k + 1$ on $\underline{E}$ vanishes identically. Iterating this argument we see that $\underline{\Theta}$ satisfies the structure equations, so $(\underline{E}, \underline{\theta})$ is really a P-frame bundle of degree $\ell - 1$.

The same argument shows that $d\underline{\theta}_i$ equals the first $\ell - 1$ components of $\sigma^*d\theta_i$ on the domain of σ.

Proposition 3.12 *If (E, θ) is a harmonic P-frame bundle of degree $\ell > 1$, then the underlying P-frame bundle $(\underline{E}, \underline{\theta})$ is harmonic, too.*

Proof. If $\ell > k + 1$ then this is completely obvious, so let us assume $\ell \leq k + 1$. For negative integers i and j take $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$, and let $\xi \in T^i_u\underline{E}$ and $\eta \in T^j_u\underline{E}$ be elements such that $\underline{\theta}_i(\xi) = X$ and $\underline{\theta}_j(\eta) = Y$. Then let σ be a local section of $\pi : E \to \underline{E}$ as in 3.11 above. By the last observation in 3.11, $t_{\underline{\theta}}(u)(X, Y)$ equals the first $\ell - 1$ components of $d\theta_{i+j}(T\sigma \cdot \xi, T\sigma \cdot \eta)$. But now let $\tilde{\xi} \in T^i_{\sigma(u)}E$ and $\eta \in T^j_{\sigma(u)}E$ be elements such that $\theta_i(\tilde{\xi}) = X$ and $\theta_j(\tilde{\eta}) = Y$. By construction, the differences $\tilde{\xi} - T\sigma \cdot \xi$ and $\tilde{\eta} - T\sigma \cdot \eta$ lie in $\mathfrak{g}_{i+\ell-1}$ and $\mathfrak{g}_{j+\ell-1}$, respectively. But then arguments as in 3.9 show that the first $\ell - 1$ components of $d\theta_{i+j}(T\sigma \cdot \xi, T\sigma \cdot \eta)$ coincide with the first $\ell - 1$ components of $d\theta_{i+j}(\tilde{\xi}, \tilde{\eta})$, so the result follows. □

3.13. Iterated application of the process of forming the underlying P-

frame bundle shows that from a P-frame bundle of degree $2k+1$, i.e. a principal P-bundle endowed with a suitably normalized Cartan connection, one can construct a P-frame bundle of degree one, i.e. a reduction to the structure group G_0 of the associated graded vector bundle to the filtered vector bundle TM. It is easy to see, that this process can also be carried out in just one step: Given a principal P-bundle $E \to M$ define $E_0 := E/P_+$, which is then a principal G_0-bundle over M. If ω is a Cartan connection on E then one verifies directly (using similar arguments as in 3.11) that for $i<0$ the component ω_i of ω in $\mathfrak{g}_i$ descends to a smooth section θ_i of the bundle $L(T^iE_0, \mathfrak{g}_i)$, and these define a frame form of length one on E_0.

The rest of this section is devoted to the question whether this process can be inverted. We will show that under a cohomological restriction this inversion is possible, i.e. we will construct from a harmonic P-frame bundle of degree ℓ a unique (up to isomorphism) P-frame bundle of degree $\ell+1$. This process is closely related to the theory of prolongation of G-structures, so we call it the prolongation procedure. An iterated application of this procedure will lead to the construction of principal P-bundles equipped with canonical Cartan connections.

Let (E,θ) be a harmonic P-frame bundle of degree ℓ over a manifold M. We start by defining $\hat{E}$ to be the subset of the bundle $\bigoplus_{i=-k}^{-1} L(T^iE, \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_{i+\ell})$ which is formed by all k-tuples $\varphi = (\varphi_{-k}, \dots, \varphi_{-1})$ such that:

1. The first ℓ components of φ_i coincide with $\theta_i(u)$, where u is the base point of φ.
2. The restriction of φ_i to T^{i+1}_uE coincides with $\theta_{i+1}(u)$ (so in particular has zero $\mathfrak{g}_i$-component).
3. $\varphi_{-1}(\zeta_A) = A$ for all $A \in \mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_{\ell-1}$.

Note that for the components φ_j with $j \neq -1$, a condition on compatibility with fundamental vector fields is implied by condition (2), since θ_{j+1} satisfies condition (4) of 3.2.

By $\pi : \hat{E} \to E$ we denote the obvious projection.

Proposition *$\pi : \hat{E} \to E$ is a locally trivial bundle, and each fiber is an affine space with modeling vector space $L_\ell(\mathfrak{g}_-, \mathfrak{g})$, the space of linear maps from $\mathfrak{g}_-$ to $\mathfrak{g}$ which are homogeneous of degree ℓ.*

Proof. Take two elements φ and $\tilde{\varphi}$ of $\hat{E}$ with $\pi(\varphi) = \pi(\tilde{\varphi}) = u$. For some $i = -k, \dots, -1$ consider the difference $\tilde{\varphi}_i - \varphi_i$. By condition (1) from above, this difference has values in $\mathfrak{g}_{i+\ell}$, and by condition (2) it vanishes on $T^{i+1}_u E$. Thus, we can view $\tilde{\varphi}_i - \varphi_i$ as a linear map $T^i_u E/T^{i+1}_u E \to \mathfrak{g}_{i+\ell}$. Now let Θ be the frame form of length one which underlies θ. Then $\Theta_i(u)$ induces an isomorphism $T^i_u E/T^{i+1}_u E \to \mathfrak{g}_i$. Hence, there is a unique linear map $\psi_i : \mathfrak{g}_i \to \mathfrak{g}_{i+\ell}$ such that $\tilde{\varphi}_i(\xi) - \varphi_i(\xi) = \psi_i(\Theta_i(\xi))$ for all $\xi \in T^i_u E$. Now we just have to collect together the ψ_i to a linear map $\mathfrak{g}_- \to \mathfrak{g}$ which is homogeneous of degree ℓ to get the affine structure of the fibers.

To prove the local triviality, it suffices to construct local smooth sections. Consider a subset $U \subset M$ such that all the bundles $T^i M$ for $i = -k, \dots, -1$ and E are trivial over U. This means that we may assume that $TM|_U = U \times \mathfrak{g}_-$ as a filtered vector bundle and that $E|_U = U \times P/P^\ell_+$. We construct a smooth section of $\pi : \hat{E} \to E$ over $p^{-1}(U)$ as follows: The tangent space to each point in $E|_U$ is $TU \times T(P/P^\ell_+)$, and we can identify TU with $U \times \mathfrak{g}_-$ as a filtered vector space. From this trivialization we get projections onto $T^i E|_U$ for $i = -k, \dots, 0$. Composing θ_{i+1} with the projection onto $T^{i+1}E$ we can view it as being defined on $T^i E$ (if $i = -1$ we take θ_0 to be the inverse of the fundamental vector field mapping). The top component of this together with θ_i defines a smooth section $\varphi_i \in L(T^i E, \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_{i+\ell})$, and one immediately verifies that $(\varphi_{-k}(u), \dots, \varphi_{-1}(u)) \in \hat{E}$ for all u. □

3.14. The next step is to define a right action of the group $P/P^{\ell+1}_+$ on the bundle $\hat{E}$. Let φ be an element of $\hat{E}$, put $u := \pi(\varphi)$, and let b be an element of $P/P^{\ell+1}_+$. By b_0 we denote the class of b in P/P^ℓ_+. For each $i = -k, \dots, -1$ we define a linear map $\varphi_i \cdot b : T^i_{u\cdot b_0} E \to \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_{i+\ell}$ by $(\varphi_i \cdot b)(\xi) := \mathrm{Ad}(b^{-1})(\varphi_i(Tr^{b_0^{-1}} \cdot \xi))$. We claim that $\varphi \cdot b = (\varphi_{-k} \cdot b, \dots, \varphi_{-1} \cdot b)$ is again in $\hat{E}$, so we have to verify conditions (1), (2) and (3) of 3.13.

The first ℓ components of $(\varphi_i \cdot b)$ have to be compared with $\theta_i(u \cdot b_0)$. But by 2.12 the first ℓ components coincide with $\mathrm{Ad}(b_0^{-1})$ acting on the first ℓ components of $\varphi_i(Tr^{b_0^{-1}} \cdot \xi)$. Since φ is in $\hat{E}$, these components equal $\theta_i(u)(Tr^{b_0^{-1}} \cdot \xi)$, which by equivariancy equals $\mathrm{Ad}(b_0)(\theta_i(u \cdot b_0)(\xi))$, so (1) is satisfied.

Second, we have to compare the restriction of $\varphi_i \cdot b$ to $T^{i+1}_{u\cdot b_0} E$ with $\theta_{i+1}(u \cdot b_0)$. But for an element $\xi \in T^{i+1}_{u\cdot b_0} E$ the first component of $\varphi_i(Tr^{b_0^{-1}} \cdot \xi)$ is zero, so again by 2.12 this restriction coincides with $\mathrm{Ad}(b_0^{-1})(\varphi_i(Tr^{b_0^{-1}} \cdot \xi))$. Now as above one concludes that (2) is satisfied as well.

To verify (3) we compute:

$$(\varphi_{-1} \cdot b)(\zeta_A(u \cdot b_0)) = \mathrm{Ad}(b^{-1})(\varphi_{-1}(Tr^{b_0^{-1}} \cdot \zeta_A(u \cdot b_0)))$$

$$= \mathrm{Ad}(b^{-1})(\varphi_{-1}(\zeta_{\mathrm{Ad}(b_0)(A)}(u))),$$

and since the action of $P/P_+^{\ell+1}$ on $\mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{\ell-1}$ factors over P/P_+^ℓ, condition (3) is satisfied, too.

If $\varphi \cdot b = \varphi$ for some $\varphi \in \hat{E}$, we must have $b_0 = e$, since the action of P/P_+^ℓ on E is free. But in this case by Proposition 2.10, there is an $A \in \mathfrak{g}_\ell$ such that $b = \exp(A)$, and thus $(\varphi_i \cdot b)(\xi) = \varphi_i(\xi) - \mathrm{ad}(A)(\varphi_i(\xi))$. Now for each $X \in \mathfrak{g}_{-1}$ we can find a $\xi \in T_u^{-1}E$ such that $\varphi_{-1}(\xi) = X$. But then $\varphi \cdot b = \varphi$ implies $[A, X] = 0$ for all $X \in \mathfrak{g}_{-1}$, which implies $A = 0$ by 2.2(4).

Thus, we have a free right action of $P/P_+^{\ell+1}$ on $\hat{E}$, and by definition the projection $\pi : \hat{E} \to E$ is equivariant over the canonical projection $P/P_+^{\ell+1} \to P/P_+^\ell$.

3.15. Since $\hat{E}$ is a locally trivial bundle over E we have the induced filtration of the tangent bundle $T\hat{E}$. Moreover, there is a natural analog of a frame form on $\hat{E}$ defined as follows: Let φ be a point in $\hat{E}$ and put $u := \pi(\varphi)$. An element $\xi \in T_\varphi \hat{E}$ is in $T_\varphi^i \hat{E}$ if and only if $T\pi \cdot \xi \in T_u^i E$. If this is the case, then we define $\hat{\theta}_i(\xi) := \varphi_i(T\pi \cdot \xi)$. Clearly, each $\hat{\theta}_i$ is a smooth section of the bundle $L(T^i\hat{E}, \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+\ell})$. Moreover, from the construction and the properties of φ it follows immediately that $\hat{\theta} = (\hat{\theta}_{-k}, \dots, \hat{\theta}_{-1})$ satisfies the obvious analogs of conditions (1), (2) and (4) of 3.2.

Since the projection π is equivariant, it follows that the subbundles $T^i\hat{E}$ are stable under the action of $P/P_+^{\ell+1}$, and we claim that the components of $\hat{\theta}$ are equivariant. Thus, let us consider $((r^b)^*\hat{\theta}_i)(\xi)$ for some $\xi \in T^i\hat{E}$. By definition, this equals $(\varphi_i \cdot b)(T\pi \cdot Tr^b \cdot \xi) = \mathrm{Ad}(b^{-1})(\varphi_i(Tr^{b_0^{-1}} T\pi Tr^b \cdot \xi))$. But equivariancy of π implies that this equals $\mathrm{Ad}(b^{-1})(\varphi_i(T\pi \cdot \xi)) = \mathrm{Ad}(b^{-1})(\hat{\theta}_i(\xi))$.

3.16. Let $\varphi \in \hat{E}$ be a point and put $u := \pi(\varphi) \in E$. By Proposition 3.13, we can find a section σ of $\hat{E} \to E$ which is defined locally around u and maps u to φ. For $i = -k, \dots, -1$ we can form $\sigma^*\hat{\theta}_i$ which is a locally defined smooth section of $L(T^iE, \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+\ell})$. By definition of the canonical form $\hat{\theta}$, we see that $(\sigma^*\hat{\theta}_i)(u)(\xi) = \sigma_i(u)(\xi)$. Since $\sigma(u) \in \hat{E}$, the first ℓ components of $\sigma_i(u)(\xi)$ coincide with $\theta_i(u)(\xi)$, which means that the first ℓ components of $\sigma^*\hat{\theta}_i$ coincide with θ_i.

Note that since E is a P-frame bundle the derivative $d\sigma^*\hat{\theta}_{i+j} = \sigma^* d\hat{\theta}_{i+j}$ is well defined on $T^iE \otimes T^jE$ locally around u, for all i, j such that $i+j \geq -k$. In particular, for $A \in \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{\ell-1}$ we have the fundamental vector field

$\zeta_A(u)$ and for $\xi \in T^i_u E$ we can form $d\sigma^*\hat{\theta}_i(\zeta_A(u), \xi)$. Now the following weaker analog of Lemma 3.8 holds:

Lemma *For A as above and $\xi \in T^{i+1}_u$ we have*

$$d\sigma^*\hat{\theta}_i(\zeta_A(u), \xi) = -\operatorname{ad}(A)(\sigma_i(u)(\xi)),$$

where the action ad *is well defined, since $\sigma_i(u)(\xi)$ has trivial $\mathfrak{g}_i$-component for $\xi \in T^{i+1}_u E$, see* 2.12.

Proof. We have $d\sigma^*\hat{\theta}_i(\zeta_A(u), \xi) = d\hat{\theta}_i(T\sigma \cdot \zeta_A(u), T\sigma \cdot \xi)$. If we denote fundamental vector fields on $\hat{E}$ by $\hat{\zeta}$, then by equivariancy of the projection $\pi : \hat{E} \to E$ we have $T\pi \cdot \hat{\zeta}_A = \zeta_A$. Thus, there exists an element $\lambda \in V_\varphi \hat{E}$ such that $T\sigma \cdot \zeta_A(u) = \hat{\zeta}_A(u) + \lambda$. Since $\hat{\theta}_i$ is equivariant, the proof of Lemma 3.8 shows that $d\hat{\theta}_i(\hat{\zeta}_A(u), T\sigma \cdot \xi) = -\operatorname{ad}(A)(\hat{\theta}_i(T\sigma \cdot \xi))$. Since by definition $\hat{\theta}_i(T\sigma \cdot \xi) = \sigma_i(u)(\xi)$, we can conclude the proof by showing that for an element λ of the vertical bundle of $\hat{E} \to E$ we have $d\hat{\theta}_i(\lambda, T\sigma \cdot \xi) = 0$.

In Proposition 3.13 we have seen that the fibers of $\hat{E} \to E$ are affine spaces, so we can canonically identify each vertical tangent space with the modeling vector space $L_\ell(\mathfrak{g}_-, \mathfrak{g})$. So for each element ψ in this space, we can consider the constant vertical vector field $\tilde{\psi}$. Now we can imitate the proof of Lemma 3.8 as follows: The flow of $\tilde{\psi}$ up to time t clearly maps φ to $(\xi \mapsto \varphi_i(\xi) + t\psi(\Theta_i(\xi)))$. Now for an element $\bar{\xi} \in T^i_\varphi \hat{E}$ we have by definition $\hat{\theta}_i(\bar{\xi}) = \varphi_i(T\pi \cdot \bar{\xi})$. Pulling this back with the above flow and differentiating in zero we get $\psi(\Theta_i(T\pi \cdot \bar{\xi}))$, so this is just the Lie derivative $\mathcal{L}_{\tilde{\psi}}\hat{\theta}_i(\varphi)(\bar{\xi})$. We can write $\mathcal{L}_{\tilde{\psi}} = d \circ i_{\tilde{\psi}} + i_{\tilde{\psi}} \circ d$, and by definition $i_{\tilde{\psi}}\hat{\theta}_i = 0$, so we finally get $d\hat{\theta}_i(\tilde{\psi}(\varphi), \bar{\xi}) = \psi(\Theta_i(T\pi \cdot \bar{\xi}))$. Since $\xi \in T^{i+1}_u E$ and thus $\Theta_i(\xi) = 0$, we are done. □

3.17. Now we define the *torsion* of φ as a linear map $t_\varphi : \mathfrak{g}_- \wedge \mathfrak{g}_- \to \mathfrak{g}$, which has homogeneous components of degree $0, \dots, \ell$ only, as follows: For $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$ choose $\xi \in T^i_u E$ and $\eta \in T^j_u E$ such that $\varphi_i(\xi) = X$ and $\varphi_j(\eta) = Y$. If $i + j < -k$, then we define $t_\varphi(X, Y)$ to be the components in $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{i+j+\ell}$ of $d\sigma^*\hat{\theta}_{-k}(u)(\xi, \eta)$ and if $i + j \geq -k$, we put $t_\varphi(X, Y) := d\sigma^*\hat{\theta}_{i+j}(u)(\xi, \eta) \in \mathfrak{g}_{i+j} \oplus \dots \oplus \mathfrak{g}_{i+j+\ell}$.

To show that this is well defined, let us first assume that we have two elements $\xi_1, \xi_2 \in T^i_u E$ such that $\varphi_i(\xi_1) = \varphi_i(\xi_2) = X$. Then the difference $\xi_2 - \xi_1$ lies in the kernel of φ_i, which by condition (2) of 3.13 and condition (1) of 3.2 equals $T^{i+\ell+1}_u E$. In particular, φ_{-1} is bijective, so there is no choice in ξ in this case. Now if $i+\ell+1<0$, then one concludes exactly as in case (1) of 3.9 that passing from ξ_1 to ξ_2 does not change the torsion. On the other hand, if $i+\ell+1\geq 0$, then there is an element $A\in\mathfrak{g}_{i+\ell+1}$ such that $\xi_2-\xi_1=\zeta_A(u)$. From above, we know that $i<-1$, so $i+\ell+1<\ell$. Thus for $j>-k$ we can immediately apply Lemma 3.16 to see that $d\sigma^*\hat{\theta}_{i+j}(\xi_2-\xi_1,\eta)=-\operatorname{ad}(A)(Y)$ (or the respective equation with $i+j$ replaced by $-k$), and $[A,Y]$ is an element of $\mathfrak{g}_{i+j+\ell+1}$, so it plays no role. If $j=-k$, then the proof of Lemma 3.16 shows the difference between $d\sigma^*\hat{\theta}_{-k}(\xi_2-\xi_1,\eta)$ and $-\operatorname{ad}(A)(Y)$ lies in $\mathfrak{g}_{-k+\ell}$, so it can play no role either.

To prove independence of the choice of the section σ, we compute the effect of a general change of σ on $d\sigma^*\hat{\theta}_i$. If we have two local sections σ and $\bar{\sigma}$, then from the proof of Proposition 3.13 we see that there is a smooth function ψ with values in $L_\ell(\mathfrak{g}_-,\mathfrak{g})$, such that $\bar{\sigma}_i(u)(\xi)=\sigma_i(u)(\xi)+\psi(u)(\Theta_i(u)(\xi))$. But this means that $\bar{\sigma}^*\hat{\theta}_i=\sigma^*\hat{\theta}_i+\psi\circ\Theta_i$. Differentiating this, we get for elements $\xi\in T^i_uE$ and $\eta\in T^j_uE$:

$$\begin{aligned} d\bar{\sigma}^*\hat{\theta}_{i+j}(\xi,\eta) &= d\sigma^*\hat{\theta}_{i+j}(\xi,\eta)+d\psi(\xi)(\Theta_{i+j}(\eta)) \\ &\quad - d\psi(\eta)(\Theta_{i+j}(\xi))+\psi(d\Theta_{i+j}(\xi,\eta)), \end{aligned}$$

and since $\Theta_{i+j}(\xi)$ and $\Theta_{i+j}(\eta)$ are zero and Θ satisfies the structure equation, this reduces to

$$d\bar{\sigma}^*\hat{\theta}_{i+j}(\xi,\eta)=d\sigma^*\hat{\theta}_{i+j}(\xi,\eta)-\psi([\Theta_i(\xi),\Theta_j(\eta)]).$$

In particular, this implies that the value of $d\sigma^*\hat{\theta}_i$ in u depends only on $\sigma(u)$, so the torsion of φ is really well defined.

We have noted already in 3.16 that the first ℓ components of $\sigma^*\hat{\theta}_i$ coincide with θ_i, which implies that the homogeneous components of degrees less than ℓ of the torsion t_φ coincide with the torsion $t_\theta(u)$, so only the homogeneous component of degree ℓ is really relevant.

3.18. Using the last computation in 3.17, we next compute how the torsion depends on φ. Let us change φ to $\tilde{\varphi}$, $\tilde{\varphi}_i(\xi):=\varphi_i(\xi)+\psi(\Theta_i(\xi))$, and take elements $X\in\mathfrak{g}_i$ and $Y\in\mathfrak{g}_j$. If $\xi\in T^i_uE$ is such that $\varphi_i(\xi)=X$, then by definition $\Theta_i(\xi)=X$, so $\tilde{\varphi}_i(X)=X+\psi(X)$, and $\psi(X)\in\mathfrak{g}_{i+\ell}$. If $i+\ell\geq 0$, we put $\xi':=\zeta_{\psi(X)}(u)$ while for $i+\ell<0$ we choose an element $\xi'\in T^{i+\ell}_uE$ such that $\varphi_{i+\ell}(\xi')=\psi(X)$. In both cases, we then have $\varphi_i(\xi')=\psi(X)$ and $\Theta_i(\xi')=0$, so $\tilde{\varphi}_i(\xi-\xi')=X$. Similarly we define η'.

Let $\tilde{\sigma}$ be a section with $\tilde{\sigma}(u) = \tilde{\varphi}$. We have to compute

$$t_{\tilde{\varphi}}(X,Y) = d\tilde{\sigma}^*\hat{\theta}_{i+j}(\xi - \xi', \eta - \eta'),$$

or the corresponding expression with $i+j$ replaced by $-k$ if $i+j < -k$. In any case, by the last computation in 3.17 this equals $d\sigma^*\hat{\theta}_{i+j}(\xi - \xi', \eta - \eta') - \psi([X,Y])$, since for $i+j < -k$ the bracket $[X,Y]$ is zero. The first of these two terms splits into the sum of $t_\varphi(X,Y)$ with three additional terms. Among these, $d\sigma^*\hat{\theta}_{i+j}(\xi', \eta') = 0$, since by construction $\sigma^*\hat{\theta}_{i+j}$ vanishes on both ξ' and η', and the Lie bracket of two vector fields through these vectors lies in $\mathfrak{g}_{i+j+2\ell}$, so it cannot contribute either. So we have to analyze the term $d\sigma^*\hat{\theta}_{i+j}(\xi', \eta)$ (or the respective term with $i+j$ replaced by k). If $i+\ell \geq 0$, then $\xi' = \zeta_{\psi(X)}(u)$, and by Lemma 3.16 this term gives $-\operatorname{ad}(\psi(X))(Y)$. On the other hand, let us assume that $i+\ell < 0$. Since $\xi' \in T_u^{i+\ell}E$ we have $\sigma^*\hat{\theta}_{i+j}(\xi') = 0$ (respectively, those terms of $\sigma^*\hat{\theta}_{-k}(\xi)$ that we consider are zero). Also, $i+j+\ell < j$, so $\sigma^*\hat{\theta}_{i+j}(\eta) = 0$ as well. Consequently, we have

$$d\sigma^*\hat{\theta}_{i+j}(\xi', \eta) = -\sigma^*\hat{\theta}_{i+j}([\tilde{\xi}', \tilde{\eta}](u)) = -\varphi_{i+j}([\tilde{\xi}', \tilde{\eta}](u)),$$

where $\tilde{\xi}'$ and $\tilde{\eta}$ are vector fields through ξ' and η, respectively. (If $i+j < -k$ a similar equation holds for the appropriate components with $i+j$ replaced by $-k$.) But by construction, $[\tilde{\xi}', \tilde{\eta}](u)$ is an element of $T_u^{i+j+\ell}E$, so applying condition (2) of 3.13 and then several times condition (2) of 3.2, we see that $-\varphi_{i+j}([\tilde{\xi}', \tilde{\eta}](u)) = -\Theta_{i+j+\ell}([\tilde{\xi}', \tilde{\eta}](u))$. Since $\Theta_{i+j+\ell}$ vanishes both on ξ' and η, the latter term equals $d\Theta_{i+j+\ell}(\xi', \eta)$ which by the structure equation equals $-[\psi(X), Y]$.

Together these computations show that the torsion of $\tilde{\varphi} = \varphi + \psi \circ \Theta(u)$ is given by

$$t_{\tilde{\varphi}}(X,Y) = t_\varphi(X,Y) + [\psi(X), Y] + [X, \psi(Y)] - \psi([X,Y]),$$

so $t_{\tilde{\varphi}} = t_\varphi + \partial\psi$, where ∂ denotes the Lie algebra differential introduced in 2.4.

3.19. In 2.6 we have seen that the codifferential $\partial^* : L_\ell(\mathfrak{g}_-, \mathfrak{g}) \to L_\ell(\mathfrak{g}_- \wedge \mathfrak{g}_-, \mathfrak{g})$ is the adjoint with respect to a certain metric of the differential $\partial : L_\ell(\mathfrak{g}_- \wedge \mathfrak{g}_-, \mathfrak{g}) \to L_\ell(\mathfrak{g}_-, \mathfrak{g})$ used above. In particular, this implies that the kernel of ∂^* and the image of ∂ are complementary subspaces of $L_\ell(\mathfrak{g}_- \wedge \mathfrak{g}_-, \mathfrak{g})$. This means, that for each $\varphi \in \hat{E}$, we can find a $\psi \in L_\ell(\mathfrak{g}_-, \mathfrak{g})$ such that for $\tilde{\varphi} = \varphi + \psi \circ \Theta(u)$ we have $\partial^*(t^\ell_{\tilde{\varphi}}) = 0$, where $t^\ell_{\tilde{\varphi}}$ denotes the homogeneous component of degree ℓ of the torsion of $\tilde{\varphi}$. Moreover, from the last formula in 3.18 it is clear that the space of all φ over a point $u \in E$ such that $\partial^*(t^\ell_\varphi) = 0$ is an affine space with modeling vector space $\operatorname{Ker}(\partial) \subset L_\ell(\mathfrak{g}_-, \mathfrak{g})$.

Proposition *Let $\varphi \in \hat{E}$ be a point such that $\partial^*(t^\ell_\varphi) = 0$, and let $b \in P/P^{\ell+1}_+$ be any element. Then also $\partial^*(t^\ell_{\varphi\cdot b}) = 0$.*

Proof. To compute the torsion $t_{\varphi\cdot b}$, we first need a section. Starting from a section σ defined locally around $u = \pi(\varphi)$, we define $\bar{\sigma} := r^b \circ \sigma \circ r^{b_0{}^{-1}}$, where as before b_0 is the class of b in P/P^ℓ_+, and we denote by r the right actions on $\hat{E}$ and E, to get a section defined locally around $u\cdot b_0$ with $\bar{\sigma}(u\cdot b_0) = \varphi\cdot b$. For each $i = -k, \dots, -1$, we then have $\bar{\sigma}^*\hat{\theta}_i = (r^{b_0{}^{-1}})^*\sigma^*(r^b)^*\hat{\theta}_i$. Equivariancy of $\hat{\theta}$ reads as $(r^b)^*\hat{\theta}_i = \operatorname{Ad}(b^{-1}) \circ \theta_i$. Differentiating this, we see that for $\xi \in T^i_{u\cdot b_0}$ and $\eta \in T^j_{u\cdot b_0}$ we have

$$d\bar{\sigma}^*\hat{\theta}_{i+j}(\xi,\eta) = \operatorname{Ad}(b^{-1})(d\sigma^*\hat{\theta}_{i+j}(Tr^{b_0{}^{-1}}\cdot\xi, Tr^{b_0{}^{-1}}\cdot\eta)).$$

To get $t_{\varphi\cdot b}(X,Y)$ for $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$, we have to compute this (or the respective expression with $i+j$ replaced by $-k$ if $i+j < -k$) for ξ such that $(\varphi_i \cdot b)(\xi) = X$ and η such that $(\varphi_j \cdot b)(\eta) = Y$. Since $(\varphi_i \cdot b)(\xi) = X$, we get $\varphi_i(Tr^{b_0{}^{-1}}\cdot\xi) = \operatorname{Ad}(b)(X)$, so we may write $Tr^{b_0{}^{-1}}\cdot\xi = \xi' + \xi''$, where $\varphi_i(\xi') = \operatorname{Ad}_-(b)(X)$, the components of $\operatorname{Ad}(b)(X)$ in $\mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_{-1}$, and $\varphi_i(\xi'') = \operatorname{Ad}_+(b)(X) = \operatorname{Ad}(b)(X) - \operatorname{Ad}_-(b)(X)$. Similarly, we split $Tr^{b_0{}^{-1}}\cdot\eta = \eta' + \eta''$. Now we write

$$\begin{aligned} d\sigma^*\hat{\theta}_{i+j}&(Tr^{b_0{}^{-1}}\cdot\xi, Tr^{b_0{}^{-1}}\cdot\eta) \\ &= d\sigma^*\hat{\theta}_{i+j}(\xi',\eta') + d\sigma^*\hat{\theta}_{i+j}(\xi'', Tr^{b_0{}^{-1}}\cdot\eta) \\ &\quad + d\sigma^*\hat{\theta}_{i+j}(Tr^{b_0{}^{-1}}\cdot\xi, \eta'') - d\sigma^*\hat{\theta}_{i+j}(\xi'',\eta''). \end{aligned}$$

Since $\operatorname{Ad}(b^{-1})$ never moves down in the grading, we may compute this modulo $\mathfrak{g}_{i+j+\ell+1}\oplus\dots\oplus\mathfrak{g}_k$. But modulo this, the term $d\sigma^*\hat{\theta}_{i+j}(\xi'', Tr^{b_0{}^{-1}}\cdot\eta)$ is by Lemma 3.16 congruent to $-[\operatorname{Ad}_+(b)(X), \operatorname{Ad}(b)(Y)]$, and the next two terms are congruent to $-[\operatorname{Ad}(b)(X), \operatorname{Ad}_+(b)(Y)]$ and $[\operatorname{Ad}_+(b)(X), \operatorname{Ad}_+(b)(Y)]$, respectively.

Finally, we claim that the remaining term $d\sigma^*\hat{\theta}_{i+j}(\xi',\eta')$ is congruent to the torsion $t_\varphi(\operatorname{Ad}_-(b)(X), \operatorname{Ad}_-(b)(Y))$. To see this, we have to split ξ' and η' into sums of elements which are mapped by φ_i to one homogeneous component of $\operatorname{Ad}_-(b)(X)$ or $\operatorname{Ad}_-(b)(Y)$, and we only have to consider elements corresponding to components in $\mathfrak{g}_{i'}$ and $\mathfrak{g}_{j'}$ if $i'+j' \leq i+j+\ell$. But in this case, the components in $\mathfrak{g}_{i'+j'} \oplus \cdots \oplus \mathfrak{g}_{i+j+\ell}$ of $d\sigma^*\hat{\theta}_{i+j}$ of these elements coincide with the components of $d\sigma^*\hat{\theta}_{i'+j'}$ of these elements by condition (2) of 3.13.

By the structure equation, the homogeneous component of degree zero of the torsion $t_\varphi(\mathrm{Ad}_-(b)(X), \mathrm{Ad}_-(b)(Y))$ equals $-[\mathrm{Ad}_-(b)(X), \mathrm{Ad}_-(b)(Y)]$, which adds up with the terms from above to $-[\mathrm{Ad}(b)(X), \mathrm{Ad}(b)(Y)]$. Together, we see that

$$d\bar{\sigma}^*\hat{\theta}_{i+j}(\xi,\eta) = -[X,Y] + \mathrm{Ad}(b^{-1})(t_\varphi^{\geq 1}(\mathrm{Ad}_-(b)(X), \mathrm{Ad}_-(b)(Y)))$$

modulo $\mathfrak{g}_{i+j+\ell+1} \oplus \cdots \oplus \mathfrak{g}_k$, where $t_\varphi^{\geq 1}$ denotes the sum of homogeneous components of degree ≥ 1 of the torsion t_φ. By equivariancy of ∂^* (see Proposition 2.13) the proposition follows. □

3.20. Assume now that $H^1_\ell(\mathfrak{g}_-, \mathfrak{g}) = 0$. If $\ell > k$, then this implies that over each point we find a unique φ with $\partial^* t_\varphi = 0$. Clearly, mapping each point to this element defines a smooth section of $\hat{E} \to E$, and by Proposition 3.19 above this section is P-equivariant. Thus, we can simply pull back $\hat{\theta}$ along this section to get a frame form of length $\ell+1$ on E, and with that pullback E clearly is a harmonic P-frame bundle of degree $\ell+1$.

If $\ell \leq k$, then denote by $\tilde{E}$ the subset of all φ such that $\partial^*(t^\ell_\varphi) = 0$, and denote by $\tilde{p} : \tilde{E} \to M$ the projection and by $\tilde{\theta}$ the restriction of $\hat{\theta}$ to $\tilde{E}$. By Proposition 3.19, we have a free right action of $P/P_+^{\ell+1}$ on $\tilde{E}$, which preserves the fibers of $\tilde{p}$. We claim that the action is transitive on each fiber. To see this, assume that φ and $\bar{\varphi}$ are points of $\tilde{E}$ which are in the same fiber of $\tilde{p}$. Then $\pi(\varphi)$ and $\pi(\bar{\varphi})$ are in the same fiber of $p : E \to M$, so there is an element $b_0 \in P/P_+^\ell$ such that $\pi(\bar{\varphi}) = \pi(\varphi) \cdot b_0$. Now let $s : P/P_+^\ell \to P/P_+^{\ell+1}$ be the canonical section introduced in 2.11. Then $\pi(\varphi \cdot s(b_0)) = \pi(\varphi) \cdot b_0 = \pi(\bar{\varphi})$, so there is a map $\psi \in \mathrm{Ker}(\partial) \subset L_\ell(\mathfrak{g}_-, \mathfrak{g})$ such that $\bar{\varphi}_i(\xi) = (\varphi_i \cdot s(b_0))(\xi) + \psi(\Theta_i(\xi))$. Since $H^1_\ell(\mathfrak{g}_-, \mathfrak{g}) = 0$ we must have $\psi = \mathrm{ad}(A)$ for some $A \in \mathfrak{g}_\ell$. But then for $b_1 := \exp(A) \in P/P_+^{\ell+1}$ we clearly have $\varphi \cdot s(b_0) b_1 = \bar{\varphi}$.

Since $\hat{E} \to E$ and $E \to M$ are locally trivial bundles, the projection $\tilde{p} : \tilde{E} \to M$ admits local smooth sections, and since it has a free right action which is transitive on each fiber, it is actually a smooth principal bundle. Moreover, from 3.15 it is clear that $\tilde{\theta}$ is a frame form of length $\ell+1$ on $\tilde{E}$, and by construction it satisfies the structure equations, so $\tilde{p} : \tilde{E} \to M$ is a

P-frame bundle of degree $\ell+1$. Also, the underlying P-frame bundle to $\tilde{E}$ of length ℓ clearly is just E.

To see that the P-frame bundle $(\tilde{E},\tilde{\theta})$ is harmonic, we compute $t_{\tilde{\theta}}(\varphi)(X,Y)$, for $X\in\mathfrak{g}_i$ and $Y\in\mathfrak{g}_j$. To do this, we have to choose $\xi\in T^i_\varphi\tilde{E}$ and $\eta\in T^j_\varphi\tilde{E}$ such that $\tilde{\theta}_i(\xi)=\varphi_i(T\pi\cdot\xi)=X$ and $\tilde{\theta}_j(\eta)=Y$. We do this by choosing $\xi'\in T^i_{\pi(\varphi)}E$ and $\eta'\in T^j_{\pi(\varphi)}E$ such that $\varphi_i(\xi')=X$ and $\varphi_j(\eta')=Y$ and a local smooth section σ with $\sigma(\pi(\varphi))=\varphi$ and putting $\xi=T\sigma\cdot\xi'$ and $\eta=T\sigma\cdot\eta'$. But then $d\tilde{\theta}_{i+j}(\xi,\eta)=d\sigma^*\hat{\theta}_{i+j}(\xi',\eta')=t_\varphi(X,Y)$ (or the same equation with $i+j$ replaced by $-k$ is $i+j<-k$), so $t_{\tilde{\theta}}(\varphi)=t_\varphi$, and thus $(\tilde{E},\tilde{\theta})$ is really harmonic.

3.21. To discuss the question of uniqueness, let us assume that $(\tilde{E},\tilde{\theta})$ is any P-frame bundle of degree $\ell+1$, such that the underlying P-frame bundle of degree ℓ is (E,θ). In particular, this means that we have a smooth mapping $\tilde{p}:\tilde{E}\to E$ which is equivariant over the canonical projection $P/P_+^{\ell+1}\to P/P_+^{\ell}$. For a point $\tilde{u}\in\tilde{E}$ with $\tilde{p}(\tilde{u})=u$ we define $f(\tilde{u})=(f(\tilde{u})_{-k},\dots,f(\tilde{u})_{-1})\in\hat{E}$ as follows: For $\xi\in T^i_uE$ choose an element $\tilde{\xi}\in T^i_{\tilde{u}}\tilde{E}$ such that $T\tilde{p}\cdot\tilde{\xi}=\xi$ and define $f(\tilde{u})_i(\xi):=\tilde{\theta}_i(\tilde{u})(\tilde{\xi})$. One immediately verifies that this is well defined, and since (E,θ) is the underlying P-frame bundle to $(\tilde{E},\tilde{\theta})$, it is an element of $\hat{E}$. Clearly, $f:\tilde{E}\to\hat{E}$ is a smooth fiber bundle homomorphism.

We claim that f is $P/P_+^{\ell+1}$-equivariant. So we have to compute $f(\tilde{u}\cdot b)$ for $b\in P/P_+^{\ell+1}$. If we take $\xi\in T^i_uE$ and $\tilde{\xi}\in T^i_{\tilde{u}}\tilde{E}$ as before, then $Tr^b\cdot\tilde{\xi}$ is a lift of $Tr^{b_0}\cdot\xi$, so we have by equivariancy of $\tilde{\theta}$:

$$\begin{aligned} f(\tilde{u}\cdot b)_i(Tr^{b_0}\cdot\xi) &= \tilde{\theta}_i(\tilde{u}\cdot b)(Tr^b\cdot\tilde{\xi})=\mathrm{Ad}(b^{-1})(\tilde{\theta}_i(u)(\tilde{\xi}))\\ &= (f(\tilde{u})\cdot b)(Tr^{b_0}\cdot\xi), \end{aligned}$$

so f is really equivariant.

Also, it follows immediately from the construction, that $\tilde{\theta}_i=f^*\hat{\theta}_i$ for all i, and finally a computation similar to the one in the end of 3.20 shows that the torsion of $\tilde{\theta}$ in a point $\tilde{u}$ equals the torsion of $f(\tilde{u})$ in the sense of 3.17. In particular, in the situation of 3.20 it follows that f actually is an isomorphism of P-frame bundles. Thus we have completed the proof of the following theorem:

Theorem 3.22 *Let E be a harmonic P-frame bundle of degree ℓ, and suppose that the cohomology group $H^1_\ell(\mathfrak{g}_-,\mathfrak{g})$ vanishes. Then there is an* *(up to isomorphism) unique harmonic P-frame bundle $(\tilde{E}, \tilde{\theta})$ of degree $\ell+1$ whose underlying P-frame bundle of degree ℓ is isomorphic to (E, θ).*

Iterated application of this theorem immediately leads to

Corollary 3.23 *Suppose that G is a semisimple Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a $|k|$-grading, such that all cohomology groups $H^1_\ell(\mathfrak{g}_-, \mathfrak{g})$ with $\ell > 0$ are trivial. (In particular this is satisfied if none of the simple factors of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$ and none of the simple factors is of one of the three types listed in Proposition 2.7). Let M be a smooth manifold with a filtration of its tangent bundle as in 3.1. Then there is a bijective correspondence between isomorphism classes of reductions to the structure group G_0 of the associated graded vector bundle to the tangent bundle, which satisfy the structure equations, and isomorphism classes of principal P-bundles over M endowed with Cartan connections with ∂^*-closed curvature and satisfying the structure equations.*

3.24. The case of nontrivial cohomology

Using Proposition 2.7 together with Proposition 2.8 and the basic results on complexifications noted in 2.7, we see that, except in the case of A_1 (and the case of a simple factor contained in $\mathfrak{g}_0$, which is rather bizarre), the only nontrivial cohomology which can occur is $H^1_1(\mathfrak{g}_-, \mathfrak{g})$. This means that the problems caused by this cohomology group occur actually in the very first prolongation step, that is in the step where we try to construct a P/P^2_+-bundle from a P/P_+-bundle. So in this case we have a principal bundle $E \to M$ with group P/P_+ and a frame form θ on E of length one. Note that in 3.13–3.20 we have not made any assumptions on the cohomology, so all the results from there remain valid in this case. In particular, we can construct the bundle $\hat{E} \to E$, the action of P/P^2_+ on $\hat{E}$, the canonical forms $\hat{\theta}$ on $\hat{E}$, define the torsion of elements of $\hat{E}$ and the elements with co-closed torsion are stable under the action of P/P^2_+.

The group P/P^2_+ is the semidirect product of $G_0 \cong P/P_+$ and P_+/P^2_+. More explicitly, the canonical smooth section $s : G_0 \to P/P^2_+$ introduced in 2.11 is a group homomorphism in this case. Using this, one easily shows that one can find local G_0-equivariant sections from E to the space of elements in $\hat{E}$ with co-closed torsion. Moreover, using the fact that the exponential map induces a diffeomorphism $\mathfrak{g}_1 \cong P_+/P^2_+$ one can glue such local sections using a partition of unity to a global section, which is still G_0-equivariant (compare with the proof of Lemma 3.6 of [6]). Now choosing such a global G_0-equivariant section from E to the space of elements in $\hat{E}$ with co-closed torsion, we can then take the orbit of the image of this section under the group P/P_+^2. By Proposition 3.19, this is still contained in the subspace of all elements having co-closed torsion, and thus we can restrict the frame form to the orbit to get a harmonic P-frame bundle of length two over M.

After making the choice of a section in the first step, we can then finish the prolongation procedure as described before. Geometrically, one has to view the choice of the equivariant section simply as a part of the structure. This is particularly transparent in the case of projective structures (of dimension > 1), in which the P-frame bundle of length one contains no information at all (it is simply the full first order frame bundle), and the whole structure is contained in the choice of the equivariant section (which corresponds to choosing a class of connections in this case).

The second exceptional case is of quite similar nature: In that case, the first order frame bundle is equivalent to specifying a contact structure, and the whole rest of the structure is contained in the additional choice of a section, which can again be interpreted equivalently as choosing a class of partial connections compatible with the contact structure.

Thus, the only case we cannot deal with is the case of simple factors which are either contained in $\mathfrak{g}_0$ or correspond to one-dimensional projective structures, and both these cases are quite degenerate.

4. Parabolic geometries

4.1. By Corollary 3.23 and 3.24, a harmonic P-frame bundle of degree $2\ell+1$ is either already determined by the underlying P-frame bundle of degree one, or by this bundle plus a section of an additional bundle. Thus, in order to understand the geometrical meaning of parabolic geometries (or to understand the structures for which we are able to construct canonical Cartan connections), the main step is to understand the geometrical meaning of a P-frame bundle of degree 1.

So let $\mathfrak{g}$ be a semisimple $|k|$-graded Lie algebra, G a group with Lie algebra $\mathfrak{g}$, and denote the various subgroups and subalgebras as before. Let $p : E \to M$ be a smooth principal bundle with structure group $P/P_+ \cong G_0$ over a smooth manifold M which has the same dimension as $\mathfrak{g}_-$, and let θ be a frame form of length one on E. As we have noted in 3.3(2), for each point $u \in E$ the component θ_i of the frame form θ induces a linear isomorphism $T^i_u E/T^{i+1}_u E \cong \mathfrak{g}_i$ for each $i = -k, \dots, -1$.

Now let $\xi \in T^i_x M$ and $\eta \in T^j_x M$ be tangent vectors, choose a point $u \in E$ with $p(u) = x$ and tangent vectors $\tilde{\xi}$ and $\tilde{\eta}$ over ξ and η. By definition of the induced filtration on TE (see 3.1), we have $\tilde{\xi} \in T^i_u E$ and $\tilde{\eta} \in T^j_u E$, so we can form $[\theta_i(\tilde{\xi}), \theta_j(\tilde{\eta})] \in \mathfrak{g}_{i+j}$. Since $\tilde{\xi}$ and $\tilde{\eta}$ are unique up to vertical vectors and θ_i and θ_j vanish on vertical vectors this element is independent of the choice of $\tilde{\xi}$ and $\tilde{\eta}$. There is an element $\lambda \in T^{i+j}_u E$ (unique up to elements from $T^{i+j+1}_u E$) such that $\theta_{i+j}(\lambda) = [\theta_i(\tilde{\xi}), \theta_j(\tilde{\eta})]$, and we denote by $\{\xi, \eta\} \in T^{i+j}_x M/T^{i+j+1}_x M$ the class of $Tp \cdot \lambda$ (which is independent of the choice of λ).

We claim that $\{\xi, \eta\}$ is also independent of the choice of $u \in E$. If u_1 and u_2 are two points with $p(u_1) = p(u_2) = x$, then there is an element $b \in P/P_+ \cong G_0$ such that $u_2 = u_1 \cdot b$. If $\tilde{\xi} \in T^i_{u_1} E$ and $\tilde{\eta} \in T^j_{u_1} E$ are tangent vectors over ξ and η, then $Tr^b \cdot \tilde{\xi}$ and $Tr^b \cdot \tilde{\eta}$ are tangent vectors over ξ and η with footpoint u_2. But by equivariancy of the frame form, $\theta_i(u_2)(Tr^b \cdot \tilde{\xi}) = ((r^b)^* \theta_i)(u_1)(\tilde{\xi}) = \mathrm{Ad}(b^{-1})(\theta_i(u_1)(\tilde{\xi}))$ and similarly for $\tilde{\eta}$. Now let λ in $T^{i+j}_{u_1} E$ be such that $\theta_{i+j}(\lambda) = [\theta_i(\tilde{\xi}), \theta_j(\tilde{\eta})]$, and consider $Tr^b \cdot \lambda \in T^{i+j}_{u_2} E$. Again by equivariancy we get

$$\begin{aligned}
\theta_{i+j}(u_2)(Tr^b \cdot \lambda) &= \mathrm{Ad}(b^{-1})(\theta_{i+j}(u_1)(\lambda)) \\
&= \mathrm{Ad}(b^{-1})([\theta_i(u_1)(\tilde{\xi}), \theta_j(u_1)(\tilde{\eta})]) \\
&= [\mathrm{Ad}(b^{-1})(\theta_i(u_1)(\tilde{\xi})), \mathrm{Ad}(b^{-1})(\theta_j(u_1)(\tilde{\eta}))] \\
&= [\theta_i(u_2)(Tr^b \cdot \tilde{\xi}), \theta_j(u_2)(Tr^b \cdot \tilde{\eta})],
\end{aligned}$$

and since $Tp \cdot Tr^b \cdot \lambda = Tp \cdot \lambda$ the independence follows.

Thus, from the bundle E together with the frame form θ, we get vector bundle homomorphisms $T^i M \otimes T^j M \to T^{i+j} M/T^{i+j+1} M$ which are skew symmetric if $i = j$. In fact, these homomorphisms define on the associated graded to each tangent space the structure of a graded Lie algebra, which is isomorphic to $\mathfrak{g}_-$.

A very similar structure is however already intrinsic to the filtration of the tangent bundle of M. Let $\xi \in T^i_x M$ and $\eta \in T^j_x M$ be as above, extend them to local vector fields $\tilde{\xi}$ and $\tilde{\eta}$ which have values in $T^i M$ and $T^j M$, respectively, and denote by $L(\xi, \eta)$ the class of the Lie bracket $[\tilde{\xi}, \tilde{\eta}](x)$ in $T_x M/T^{i+j+1}_x M$. If f is a smooth function on M, then $[\tilde{\xi}, f\tilde{\eta}] = f[\tilde{\xi}, \tilde{\eta}] + (\tilde{\xi} \cdot f)\tilde{\eta}$. Since $i + 1 \leq 0$, we see that $\tilde{\eta}$ is a section of $T^{i+j+1} M$, so passing to the class modulo this subbundle, we get something which is linear over smooth functions in the second (and similarly in the first) variable. Thus, L is a well defined tensorial map $T^iM \otimes T^jM \to TM/T^{i+j+1}M$, which is called the (generalized) *Levi-form* corresponding to the filtration of TM. Actually, even the class of the Lie bracket in $TM/T^{\min\{i,j\}}M$ would be well defined, but we do not need this here.

Proposition 4.2 *The frame form θ of length one satisfies the structure equations if and only if the map $\{\ ,\ \} : T^iM \otimes T^jM \to T^{i+j}M/T^{i+j+1}M$ coincides with the generalized Levi form L. In particular, the Lie bracket of vector fields on M has to be compatible with the filtration, i.e. the bracket of a section of T^iM with a section of T^jM has to be a section of $T^{i+j}M$.*

Proof. Let $u \in E$ be a point, $\xi \in T^i_uE$ and $\eta \in T^j_uE$ tangent vectors such that $i + j = -k$. Let us extend ξ and η to smooth sections $\tilde{\xi}$ and $\tilde{\eta}$ of T^iE and T^jE, respectively. Since $i, j > -k$, we have $\theta_{-k}(\tilde{\xi}) = \theta_{-k}(\tilde{\eta}) = 0$, and thus $d\theta_{-k}(\xi, \eta) = -\theta_{-k}([\tilde{\xi}, \tilde{\eta}](u))$. Thus, the structure function of degree $-k$ is identically zero if and only if $\theta_{-k}(u)([\tilde{\xi}, \tilde{\eta}]) = [\theta_i(u)(\xi), \theta_j(u)(\eta)]$. By definition, this is equivalent to the fact that $\{Tp \cdot \xi, Tp \cdot \eta\}$ equals the class in $TM/T^{-k+1}M$ of $Tp \cdot ([\tilde{\xi}, \tilde{\eta}](u))$. But if we choose for $\tilde{\xi}$ and $\tilde{\eta}$ projectable vector fields, then the last expression coincides with the Lie bracket of the projected fields, which by construction extend $Tp \cdot \xi$ and $Tp \cdot \eta$, and thus the class in $TM/T^{-k+1}M$ coincides with $L(Tp \cdot \xi, Tp \cdot \eta)$. Iterating this argument we get the result. □

Remark 4.3 At this place, an interesting relation to the cohomology groups of $\mathfrak{g}_-$ with coefficients in $\mathfrak{g}$ shows up. Namely, suppose that $H^1_0(\mathfrak{g}_-, \mathfrak{g}) = 0$. By definition, this implies that any derivation $\mathfrak{g}_- \to \mathfrak{g}_-$ which is homogeneous of degree zero is given by the bracket with an element of $\mathfrak{g}_0$. But this implies that the group G_0 is a (possibly not connected) covering of the connected component of the group of automorphisms of the *graded* Lie algebra $\mathfrak{g}_-$. Otherwise put, a reduction to the group G_0 of the associated graded to the tangent bundle imposes no further condition on the individual fibers than the structure of a graded Lie algebra isomorphic to $\mathfrak{g}_-$. Hence in these cases, the filtration giving rise to an appropriate Levi form is the only essential ingredient for the corresponding parabolic geometries. Further ingredients (depending on the choice of the group G) can only be of the type of an orientation or an analog of a spin-structure.

On the other hand, if $H^1_0(\mathfrak{g}_-, \mathfrak{g}) \neq 0$, then there must be an additional structure on the individual fibers of the associated graded to the tangent bundle, which is intrinsic to the corresponding parabolic geometry. For example, this can be a complex structure, or a further (local) decomposition into a direct sum of subbundles or a tensor product of vector bundles.

There is a complete list of all complex simple $|k|$-graded Lie algebras which have $H^1_0(\mathfrak{g}_-, \mathfrak{g}) \neq 0$ in [23, Proposition 5.1]. Clearly, this list contains all the $|1|$-graded cases (in which the filtration is trivial), as well as all the contact type structures, i.e. $|2|$-graded algebras with $\dim(\mathfrak{g}_{-2}) = 1$, since in this case the filtration only gives rise to a contact structure, which is well known to be of infinite order. Apart from these obvious cases, there are however only two more series, namely A_ℓ with two crossed roots, one of which is the very first (or last) root, and C_ℓ with the first and last roots crossed. Thus, the case in which the filtration with appropriate Levi form is the only ingredient for the structure is rather typical for the general situation, but most of the structures which have been studied in more detail up to now do not fall into this group.

4.4. $G_0^\#$-structures of type $\mathfrak{m}$

Apart from the structure equations, a P-frame bundle of degree one over a manifold M is just a reduction of the associated graded to the tangent bundle. In particular, it is "less" than a first order G-structure. On the other hand, for a group G corresponding to a $|k|$-graded Lie algebra consider a principal P/P_+^k bundle over M, which is equipped with a frame form θ of length k (so if the frame form satisfies the structure equations, we have a P-frame bundle of degree k). Then the component θ_{-k} of θ is just an equivariant $\mathfrak{g}_-$-valued one-form on the bundle, so it gives a first order P/P_+^k-structure on M. But clearly, at this step more information than this first order structure is already encoded in the frame form θ. So there is no step in the prolongation procedure in which we deal exactly with first-order structures (apart from the structure equations, which are always an additional restriction).

There is a way, however, to formulate the degree one case equivalently in terms of a first order structure. In our setting, this translation seems not to be very natural, but we reproduce it here because of the important role it plays in the papers of N. Tanaka, see [22].

Let us temporarily denote by $GL_{\mathrm{grad}}(\mathfrak{g}_-)$ and $GL_{\mathrm{filt}}(\mathfrak{g}_-)$ the groups of invertible linear maps on $\mathfrak{g}_-$, which preserve the grading or the filtration, respectively. From 2.10 we see that the adjoint action identifies G_0 with a covering of a subgroup of $GL_{\mathrm{grad}}(\mathfrak{g}_-)$ and P with a covering of a subgroup of $GL_{\mathrm{filt}}(\mathfrak{g}_-)$. Obviously, $GL_{\mathrm{grad}}(\mathfrak{g}_-)$ is a subgroup of $GL_{\mathrm{filt}}(\mathfrak{g}_-)$. On the other hand, there is an obvious projection from $\pi : GL_{\mathrm{filt}}(\mathfrak{g}_-) \to GL_{\mathrm{grad}}(\mathfrak{g}_-)$, which corresponds to passing from the filtered vector space $\mathfrak{g}_-$ to the associated graded vector space, which can be canonically identified with $\mathfrak{g}_-$. Let $GL_+(\mathfrak{g}_-)$ be the kernel of this projection. Then it is easy to see, that $GL_{\mathrm{filt}}(\mathfrak{g}_-)$ is the semidirect product of $GL_{\mathrm{grad}}(\mathfrak{g}_-)$ and $GL_+(\mathfrak{g}_-)$.

Now we define $G_0^\# := \{(g,\varphi) \in G_0 \times GL_{\mathrm{filt}}(\mathfrak{g}_-) : \pi(\varphi) = \mathrm{Ad}(g)\}$. This is a Lie subgroup, and we have a canonical inclusion $G_0 \to G_0^\#$, which together with the first projection identifies $G_0^\#$ with the semidirect product of G_0 and $GL_+(\mathfrak{g}_-)$.

Let us consider a manifold M with a filtration of the tangent bundle TM as in 3.1. Let $p : E \to M$ be a principal G_0 bundle, and let θ be a frame form of length one on E. Then the frame form gives a map j from E to the frame bundle of the associated graded bundle to the filtered bundle TM, which can be viewed as the fibered product (over M) of the bundles $\mathrm{Iso}(\mathfrak{g}_i, T^iM/T^{i+1}M)$ of linear isomorphisms for $i = -k, \dots, -1$. Clearly, the latter bundle is a principal bundle with group $GL_{\mathrm{grad}}(\mathfrak{g}_-)$, and this map is a homomorphism of principal bundles over the homomorphism $G_0 \to GL_{\mathrm{grad}}(\mathfrak{g}_-)$ described above. This homomorphism is a reduction of structure group (in the sense of G_0 covering a subgroup of $GL_{\mathrm{grad}}(\mathfrak{g}_-)$).

Similarly, we can consider the bundle $\mathrm{Iso}_{\mathrm{filt}}(\mathfrak{g}_-, TM)$ of filtration preserving linear isomorphism between $\mathfrak{g}_-$ and tangent spaces of M. This is a principal bundle with group $GL_{\mathrm{filt}}(\mathfrak{g}_-)$, and clearly it is a subbundle of the frame bundle of M. Moreover, we have a natural projection Π from $\mathrm{Iso}_{\mathrm{filt}}(\mathfrak{g}_-, TM)$ to the frame bundle of the associated graded bundle to TM, which is a homomorphism of principal bundles over the group homomorphism $GL_{\mathrm{filt}}(\mathfrak{g}_-) \to GL_{\mathrm{grad}}(\mathfrak{g}_-)$ from above.

Now starting from the bundle E from above, we define $E^\# := \{(u,\psi) \in E \times_{G_0} \mathrm{Iso}_{\mathrm{filt}}(\mathfrak{g}_-, TM) : j(u) = \Pi(\psi)\}$. Since j and Π are homomorphisms of principal bundles over compatible group homomorphisms, this is well defined, and one immediately verifies that it is a principal bundle with group $G_0^\#$. The obvious map $E^\# \to \mathrm{Iso}_{\mathrm{filt}}(\mathfrak{g}_-, TM)$ clearly is a reduction of structure group, so $E^\#$ gives rise to a first order $G_0^\#$-structure on M. Conversely, starting from a first order $G_0^\#$-structure on M defined by a bundle $\tilde{E} \to M$ and a one-form $\Theta \in \Omega^1(\tilde{E}, \mathfrak{g}_-)$, we can simply form the principal G_0-bundle $E = \tilde{E}/GL_+(\mathfrak{g}_-)$ over M. Similar arguments as in 3.11 show that for $i = -k, \dots, -1$ the component Θ_i of Θ in $\mathfrak{g}_i$ descends to a smooth section θ_i of the bundle $L(T^iE, \mathfrak{g}_i)$, and these together are a frame form of length one on E.

If E and F are principal G_0-bundles over M equipped with frame forms of length one such that there is an isomorphism $E \to F$ which is compatible with the frame forms, then one easily sees that $E^\#$ and $F^\#$ are equivalent first order $G_0^\#$-structures. On the other hand, it is easy to see that $GL_+(\mathfrak{g}_-)$ is a vector group. Using this together with the fact that $G_0^\#$ is the semidirect product of G_0 and $GL_+(\mathfrak{g}_-)$, one shows similarly as in 3.24 that the bundle $E^\# \to E$ always has a global G_0-equivariant section. Using this, one shows that if $E^\#$ and $F^\#$ are equivalent $G_0^\#$-structures, then there is an isomorphism between E and F which is compatible with the frame forms. Thus, we have established a bijective correspondence between isomorphism classes of first order $G_0^\#$-structures on M and isomorphism classes of principal G_0-bundles equipped with frame forms of length one over M.

So it remains to discuss the structure equations in the $G_0^\#$-picture, and this is fairly easy to do: Let $\tilde{E} \to M$ be a principal $G_0^\#$-bundle together with a one form $\Theta \in \Omega^1(\tilde{E}, \mathfrak{g}_-)$ as above, and form the quotient $E = \tilde{E}/GL_+(\mathfrak{g}_-)$ with the induced frame form θ of length one. Then the component θ_i of θ is induced by restricting the $\mathfrak{g}_i$-component Θ_i of Θ to $T^i\tilde{E}$. From this one easily concludes that θ satisfies the structure equations if and only if for each $i, j = -k, \dots, -1$ and $\xi \in T^i\tilde{E}$ and $\eta \in T^j\tilde{E}$ we have that $d\Theta(\xi, \eta)$ is congruent to $[\Theta(\xi), \Theta(\eta)]$ modulo $\mathfrak{g}_{i+j+1} \oplus \dots \oplus \mathfrak{g}_{-1}$. In this case, Tanaka calls the corresponding $G_0^\#$-structure "of type $\mathfrak{m}$" ($\mathfrak{m}$ is his notation for $\mathfrak{g}_-$). Thus, we recover the main result of [22] (which is proved in the case that G is simple and has trivial center there):

Theorem 4.5 *Suppose that G is a semisimple Lie group with $|k|$-graded Lie algebra $\mathfrak{g}$, such that all cohomology groups $H^1_\ell(\mathfrak{g}_-, \mathfrak{g})$ for $\ell > 0$ are trivial. Then for any manifold M with a filtration of the tangent bundle as in 3.1, there is a bijective correspondence between isomorphism classes of first order $G_0^\#$-structures of type $\mathfrak{m}$ on M and of principal P-bundles over M equipped with a Cartan connection with ∂^*-closed curvature and satisfying the structure equations.*

Remark 4.6 The main difference between the prolongation procedure of Tanaka and the one described here, lies in the first $k-1$ steps. Roughly, it can be described as follows: Tanaka starts with a $G_0^{\#}$-structure of type $\mathfrak{m}$ and then works "down", refining it step by step, until he arrives at a first order P/P_+^k-structure with special properties (which corresponds to a P-frame bundle of degree k in our picture). In contrast to this, our approach is starting with the group G_0 and work "up" step by step over the quotients P/P_+^i, until we arrive at this point. So in these first steps, not only the procedure is different, but also the data that we work with. From the step of a P-frame bundle of degree k on, the prolongation procedures still differ, but the data are the same.

4.7. The curvature of the canonical Cartan connection

Let $(p : E \to M, \theta)$ be a harmonic P-frame bundle of degree $2k+1$. As we have already noted in 3.3(3), the component $\omega := \theta_{-k} \in \Omega^1(E, \mathfrak{g})$ of the frame form θ is a Cartan connection in this case. This means that $\omega(u) : T_uE \to \mathfrak{g}$ is a linear isomorphism for each $u \in E$, ω is equivariant, so $(r^b)^*\omega = \mathrm{Ad}(b^{-1}) \circ \omega$ for all $b \in P$, and it reproduces the generators of fundamental fields, so $\omega(\zeta_A) = A$ for all $A \in \mathfrak{p}$.

In general, the curvature of a Cartan connection is defined to be the $\mathfrak{g}$-valued two-form $K := d\omega + \frac{1}{2}[\omega, \omega]$, i.e. for $\xi, \eta \in T_uE$ we have $K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$. Now suppose that ξ is vertical, so $\xi = \zeta_A(u)$ for some A in $\mathfrak{p}$. Then by Lemma 3.8, we get $d\omega(\zeta_A(u), \eta) = -[A, \omega(\eta)]$, and thus $K(\zeta_A(u), \eta) = 0$, so the curvature is a horizontal form. Moreover, equivariancy of ω immediately implies that $(r^b)^*K = \mathrm{Ad}(b^{-1}) \circ K$, so K is equivariant, too. Hence, we can view K as a two-form on M with values in the vector bundle $E \times_P \mathfrak{g}$ associated to the adjoint representation of P on $\mathfrak{g}$.

There is another very convenient way to view the curvature as follows: Since K is horizontal, its value in $u \in E$ is completely determined by the function $\kappa(u) : \mathfrak{g}_- \wedge \mathfrak{g}_- \to \mathfrak{g}$, which is defined by $\kappa(u)(X, Y) := K(\omega(u)^{-1}(X), \omega(u)^{-1}(Y))$. Thus, K is completely determined by the smooth function $\kappa : E \to C^2(\mathfrak{g}_-, \mathfrak{g})$. By definition, κ coincides with the torsion of θ as introduced in 3.9. In particular, since (E, θ) is harmonic, we know that $\partial^* \circ \kappa = 0$.

There are two natural ways to split the function κ into components. First, we may split $\kappa = \kappa_{-k} + \cdots + \kappa_k$, according to the splitting $\mathfrak{g} =$

$\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$. In traditional terminology, the form $\kappa_- := \kappa_{-k} + \cdots + \kappa_{-1}$ is called the *torsion* and the form $\kappa_{\mathfrak{p}} := \kappa_0 + \cdots + \kappa_k$ is called the *curvature* of ω. If the form κ_- is identically zero, the corresponding P-frame bundle is called *torsion-free*, and if the form κ is zero, the corresponding bundle is called *flat*. Note that the P-frame bundle $(G \to G/P, \omega)$, where ω is the left Maurer-Cartan form, is flat by the Maurer-Cartan equation.

The second natural way is to split κ as $\sum_i \kappa^{(i)}$ according to homogeneous degrees. The importance of this splitting lies in the fact that since κ coincides with the torsion of the P-frame bundle (E, θ) in the sense of 3.9, we see from the proof of Proposition 3.12 that various homogeneous components of κ are already visible on the P-frame bundles of lower degree underlying (E, θ). In particular, the structure equation exactly means that the homogeneous components $\kappa^{(i)}$ are zero for all $i \leq 0$, so the decomposition of κ reads as $\kappa = \sum_{i=1}^{3k} \kappa^{(i)}$.

4.8. To give a geometrical interpretation of torsion and curvature, note that since the Cartan connection gives rise to a trivialization of the tangent bundle of E, it can in particular be viewed as a generalized connection on E, that is a projection onto the vertical bundle. This vertical projection $V : TE \to VE$ is given by mapping $\xi \in T_uE$ to $\xi - \omega(u)^{-1}(\omega_-(\xi)) = \zeta_{\omega_{\mathfrak{p}}(\xi)}$, where we split $\omega = \omega_- + \omega_{\mathfrak{p}}$ according to the splitting $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$. Thus, we also get a horizontal distribution given by $H_u := \omega(u)^{-1}(\mathfrak{g}_-)$.

Now for any $X \in \mathfrak{g}$ we can consider the vector field $\tilde{X} \in \mathcal{X}(E)$ defined by $\tilde{X}(u) := \omega(u)^{-1}(X)$, thus obtaining a map $\mathfrak{g} \to \mathcal{X}(E)$. Let $\mathcal{X}_h(E)$ denote the space of horizontal (with respect to ω) vector fields on E. This space becomes a Lie algebra with the bracket $[\ ,\ ]_h$ given by the horizontal projection of the usual Lie bracket, i.e. $[\xi, \eta]_h := [\xi, \eta] - V([\xi, \eta])$ for $\xi, \eta \in \mathcal{X}_h(E)$. Now we can characterize vanishing of torsion and curvature as follows:

Proposition *Let $E \to M$ be a P-frame bundle of degree $2k+1$, $\omega \in \Omega^1(E, \mathfrak{g})$ the corresponding Cartan connection, κ its curvature and H the horizontal distribution induced by ω. Then*

1. *The curvature component $\kappa_{\mathfrak{p}}$ is identically zero if and only if the horizontal distribution H is integrable, i.e. the Lie bracket of two horizontal fields is horizontal, too.*
2. *The torsion component κ_- is identically zero if and only if the mapping $\mathfrak{g}_- \to \mathcal{X}_h(E)$ given by $X \mapsto \tilde{X}$ is a Lie algebra homomorphism for the*

bracket $[\ ,\ ]_h$.

Proof. Let $X, Y \in \mathfrak{g}_-$ and consider the Lie bracket $[\tilde{X}, \tilde{Y}]$. Then the horizontal part of this, which can be computed as $\omega^{-1}(\omega_-([\tilde{X}, \tilde{Y}]))$ is by definition just $[\tilde{X}, \tilde{Y}]_h$. Now since $\omega(\tilde{X})$ and $\omega(\tilde{Y})$ are constant, we see that by definition of the exterior derivative, we have $d\omega(\tilde{X}, \tilde{Y}) = -\omega([\tilde{X}, \tilde{Y}])$, or equivalently $\kappa(X, Y) = [X, Y] - \omega([\tilde{X}, \tilde{Y}])$. The component of this in $\mathfrak{g}_-$ equals by the above observation $[X, Y] - \omega([\tilde{X}, \tilde{Y}]_h)$, so the second part follows.

Also, if the horizontal distribution is integrable, then we must have $[\tilde{X}, \tilde{Y}]_h = [\tilde{X}, \tilde{Y}]$, and thus $\kappa_{\mathfrak{p}}(X, Y) = 0$, so the necessity in the first part is clear. Finally, we can write each horizontal vector field ξ as $\sum_i \xi_i \tilde{e}_i$, where $\{e_i\}$ is a basis of $\mathfrak{g}_-$ and the ξ_i are smooth functions on E. Now

$$\left[\sum \xi_i \tilde{e}_i, \sum \eta_j \tilde{e}_j\right] = \sum_{i,j} (\xi_i \eta_j [\tilde{e}_i, \tilde{e}_j] + \xi_i (\tilde{e}_i \cdot \eta_j) \tilde{e}_j + \eta_j (\tilde{e}_j \cdot \xi_i) \tilde{e}_i),$$

and this is horizontal if and only if $[\tilde{e}_i, \tilde{e}_j]$ is horizontal for all i and j. □

4.9. In the general case, we can get more information on the curvature using the following result, which is called the *Bianchi identity* (compare with [6, 2.4])

Proposition *The curvature κ satisfies the equation*

$$(\partial \circ \kappa)(X, Y, Z) + \sum_{cycl} (\kappa(\kappa_-(X, Y), Z) + \tilde{X} \cdot \kappa(Y, Z)) = 0$$

for all $X, Y, Z \in \mathfrak{g}_-$, where ∂ is the Lie algebra differential introduced in 2.4, *$\sum_{cycl}$ denotes the sum over all cyclic permutations of (X, Y, Z), and $\tilde{X}$ is the horizontal vector field corresponding to X as in* 4.8.

Proof. The definition of κ, applied to the vector fields $[\tilde{X}, \tilde{Y}]$ and $\tilde{Z}$ reads as

$$\kappa(\omega_-([\tilde{X}, \tilde{Y}]), Z) = d\omega([\tilde{X}, \tilde{Y}], \tilde{Z}) + [\omega([\tilde{X}, \tilde{Y}]), Z].$$

Using the fact that $\omega(\tilde{Z}) = Z$ is constant, we get from the definition of the exterior derivative that

$$d\omega([\tilde{X}, \tilde{Y}], \tilde{Z}) = -\tilde{Z} \cdot \omega([\tilde{X}, \tilde{Y}]) - \omega([[\tilde{X}, \tilde{Y}], \tilde{Z}]).$$

From the proof of 4.8 we know that $\omega([\tilde{X}, \tilde{Y}]) = [X, Y] - \kappa(X, Y)$. Inserting this into the above equation and rearranging terms, we get

$$\begin{aligned}-[\kappa(X,Y),Z]-\kappa([X,Y],Z)+\kappa(\kappa_-(X,Y),Z)+\tilde{Z}\cdot\kappa(X,Y)\\ =\omega([[\tilde{X},\tilde{Y}],\tilde{Z}])-[[X,Y],Z].\end{aligned}$$

Forming the sum over all cyclic permutations of (X,Y,Z), the right hand side vanishes by the Jacobi identity for vector fields and for the Lie bracket in $\mathfrak{g}$, and the first two terms on the left hand side add up to $(\partial\circ\kappa)(X,Y,Z)$. □

Corollary 4.10 *Let $\kappa=\sum_{i=1}^{3k}\kappa^{(i)}$ be the splitting of the curvature into homogeneous components as in 4.8. Then $\partial\circ\kappa^{(1)}$ is identically zero. More generally, if $\kappa^{(j)}$ is identically zero for all $j<i$, the $\partial\circ\kappa^{(i)}$ is identically zero.*

Proof. We have to split the Bianchi identity into homogeneous parts to see this. Evaluate the Bianchi identity on elements X, Y, and Z, which are homogeneous of degree $|X|$, $|Y|$, and $|Z|$, and consider the homogeneous component of degree $|X|+|Y|+|Z|+i$ of the result for some $i>0$. Since we have observed that ∂ preserves homogeneous degrees in 2.4, the first term in the Bianchi identity contributes $\partial\circ\kappa^{(i)}$ in this degree. All contributions of the second term in this degree must be of the form $\kappa^{(j)}(\kappa_-^{(i-j)}(X,Y),Z)$ (or a cyclic permutation of the arguments) for some j with $0<j<i$. Finally, the last term can only contribute summands as $\tilde{Z}\cdot\kappa^{(i+|Z|)}(X,Y)$, and since $|Z|<0$, the result follows. □

4.11. Using the adjointness of the codifferential and the differential that we have proved in 2.6, one can split $C^n(\mathfrak{g}_-,\mathfrak{g})$ as $\mathrm{Im}(\partial)\oplus\mathrm{Im}(\partial^*)\oplus(\mathrm{Ker}(\partial)\cap\mathrm{Ker}(\partial^*))$ for each n. Moreover, $\mathrm{Im}(\partial^*)\oplus(\mathrm{Ker}(\partial)\cap\mathrm{Ker}(\partial^*))=\mathrm{Ker}(\partial^*)$ and $(\mathrm{Ker}(\partial)\cap\mathrm{Ker}(\partial^*))$ (the harmonic part) can be identified with $H^n(\mathfrak{g}_-,\mathfrak{g})$. Since both ∂ and ∂^* preserve the homogeneous degree, this decomposition is compatible with the decomposition into homogeneous degrees.

The curvature κ has values in $C^2(\mathfrak{g}_-,\mathfrak{g})$, so we can also split it according to this decomposition. By construction, κ is co-closed, so its $\mathrm{Im}(\partial)$-component is zero. Now by Corollary 4.10 we also have $\partial\circ\kappa^{(1)}=0$, so $\kappa^{(1)}$ has harmonic values and thus can be viewed as a smooth function with values in the cohomology group $H^2_1(\mathfrak{g}_-,\mathfrak{g})$. Similarly, if we already know that $\kappa^{(j)}$ is identically zero for all $j<i$, then $\kappa^{(i)}$ can be viewed as a smooth function with values in $H^2_i(\mathfrak{g}_-,\mathfrak{g})$. As we have mentioned be-

fore, these cohomology groups can be computed using Kostant's version of the Bott-Borel-Weil theorem, so this gives computable information about certain curvature components being automatically trivial. Moreover, this result provides not only the $\mathfrak{g}_0$-module structure of the cohomology but it contains also an explicit description of harmonic representatives for the individual irreducible components. In several cases, this can be used to restrict the possibilities for the values of κ further.

Finally, the splitting from above allows us to consider the harmonic part of the curvature κ, and as above we see that κ is identically zero if and only if this harmonic part is identically zero.

Proposition 4.12 *Let $(E \to M, \omega)$ be a P-frame bundle of degree $2k+1$. Then the following are equivalent:*

1. *The P-frame bundle E is flat.*
2. *The harmonic part of the curvature κ is identically zero.*
3. *The mapping $\mathfrak{g} \to \mathcal{X}(E)$ given by $X \mapsto \tilde{X}$ as in 4.8 is a homomorphism of Lie algebras.*
4. *M is locally isomorphic to G/P, i.e. for each $x \in M$ there are neighborhoods U of x and V of $0 = eP \in G/P$ such that $E|_U$ is isomorphic to $G|_V$ as a P-frame bundle.*

Proof. The equivalence of (1) and (2) was already observed in 4.11 above, and the equivalence of (1) and (3) follows immediately from part (2) of Proposition 4.8 and equivariancy of ω. The fact that (4) implies (1) is clear, since G/P is flat by the Maurer-Cartan equation. So it remains to prove that (1) implies (4).

Consider the product $E \times G$ and the form $\Omega := pr_1^*\omega - pr_2^*\omega^{MC}$, where ω denotes the Cartan connection on E and ω^{MC} denotes the Maurer-Cartan form on G. The kernel of Ω is a distribution on $E \times G$ of constant rank equal to the dimension of $\mathfrak{g}$. Now using the fact that both ω and ω^{MC} are flat, we can compute the derivative of Ω as

$$\begin{aligned} d\Omega &= -\frac{1}{2}([pr_1^*\omega, pr_1^*\omega] + [pr_2^*\omega^{MC}, pr_2^*\omega^{MC}]) \\ &= -\frac{1}{2}([\Omega, pr_1^*\omega] + [pr_2^*\omega^{MC}, \Omega]). \end{aligned}$$

But this implies that the differential of each component of Ω lies in the ideal generated by the components of Ω, so the kernel of Ω is an integrable distribution by the Frobenius theorem, so one has the corresponding foliation.

Now let $x \in M$ be a point, and let σ be a local section of E defined around x. Let L be the connected component of the leaf of the foliation through $(\sigma(x), e)$ in a small neighborhood. Then by definition of the distribution, the projections $pr_1 : L \to E$ and $pr_2 : L \to G$ are local diffeomorphisms, so they give rise to a local diffeomorphism Φ from a neighborhood of $\sigma(x)$ to a neighborhood of $e \in G$ (whose graph is exactly L). Now this neighborhood contains a neighborhood of the form $\{\sigma(y) \cdot b : y \in U, b \in W\}$, where U is an open neighborhood of x in M and W is an open neighborhood of the identity in P.

We then define $\varphi : U \to G/P$ by mapping y to the class of $\Phi(\sigma(y))$. Using that both pr_1 and pr_2 are equivariant, one concludes that φ is a diffeomorphism locally around x. Moreover, we can obviously extend it equivariantly to a local isomorphism of principal bundles (which coincides with the equivariant extension of Φ), and by construction this map pulls back ω^{MC} to ω. □

4.13. AHS-structures

As a first special case of our general constructions, we discuss the case of *almost Hermitian symmetric* structures or AHS structures. This is the case of $|1|$-graded Lie algebras $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Examples of these structures are conformal structures and almost Grassmannian structures. These structures (particularly the conformal ones) have been studied in detail by many authors, see [6] and the references therein. In this case, things simplify considerably. First of all, the filtration on TM as introduced in 3.1 has length one, so one can simply forget it. Frame forms of length ℓ are simply one forms with values in $\mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{\ell-2}$. Moreover, the structure equations as introduced in 3.4 become vacuous. Thus, in the AHS-case, a P-frame bundle of degree one over a manifold M is just a (first order) G_0-structure on M described by a $\mathfrak{g}_{-1}$-valued one form θ_{-1} on E (compare with [7, 1.2]).

To apply the prolongation procedure, we first have to consider the space $\hat{E}$ introduced in 3.13. By definition, this is the subspace of $L(TE, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$ formed by all $\varphi : T_uE \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ which have $\theta_{-1}(u)$ as $\mathfrak{g}_{-1}$-component and satisfy $\varphi(\zeta_A(u)) = (0, A)$ for all $A \in \mathfrak{g}_0$. So this is precisely the space constructed (pointwise) in [7, 1.2]. The torsion of φ in the sense of 3.17 has only one relevant component, namely the component $\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$. So in this case we only need the component in $\mathfrak{g}_{-1}$ of the form $\hat{\theta}$ on $\hat{E}$ introduced in 3.15. But then from 3.16 we see, that we may compute the torsion of φ

simply as $t_\varphi(X,Y) = d\theta_{-1}(u)(\varphi^{-1}(X), \varphi^{-1}(Y))$ for $X, Y \in \mathfrak{g}_{-1}$. Thus, we see that the first step in our prolongation procedure coincides exactly with the constructions carried out in [7, 1.2–1.6].

Similarly, one can analyze the second step in the prolongation procedure and show that our procedure coincides with the one carried out in [7, Section 2].

4.14. Partially integrable almost CR-structures

Next, we discuss the parabolic geometry containing codimension one CR-structures. These correspond to a $|2|$-graded Lie algebra, and have been is extensively studied in the literature. The construction of the canonical Cartan connection for CR-manifolds is due to E. Cartan (see [11]) for dimension three and to N. Tanaka (see [21]) and S.S. Chern and J. Moser (see [12]) for arbitrary dimensions. As we shall see below, a parabolic geometry for that case is a more general structure, namely a partially integrable almost CR-structure. Hence by our general method we get canonical Cartan connections in this more general situation. Here we only outline how to specialize our procedure to this case, a more detailed discussion will appear in [5]. For simplicity, we will restrict the discussion to the case of positive definite Levi form.

The basic setup in this case is as follows: Put $\mathfrak{g} = \mathfrak{su}(n+1,1)$. Let us number the coordinates on $\mathbb{C}^{n+2}$ as $x_0, \dots, x_{n+1}$, and choose as the Hermitian form $(x,y) \mapsto 2x_0\bar{y}_{n+1} + \langle (x_1,\dots,x_n), (y_1,\dots,y_n)\rangle$, where $\langle\ ,\ \rangle$ denotes the standard positive definite Hermitian form on $\mathbb{C}^n$. Then $\mathfrak{g}$ consists of all matrices of the form

$$\begin{pmatrix} z & Z & ib \\ X & A & -Z^* \\ ia & -X^* & -\bar{z} \end{pmatrix},$$

where the blocks are of sizes 1, n, and 1, $z \in \mathbb{C}$, $X \in \mathbb{C}^n$, $Z \in \mathbb{C}^{n*}$, $A \in \mathfrak{u}(n)$ with $\mathrm{tr}(A) = \bar{z} - z$, and $a, b \in \mathbb{R}$. Now one defines a $|2|$-grading on $\mathfrak{g}$ by giving degree -2 to the entry corresponding to a, -1 to the one corresponding to X, 0 to the ones corresponding to z and A, 1 to those corresponding to Z, and 2 to the one corresponding to b. From the block form it is obvious, that this is actually a $|2|$-grading.

Next, let G be the adjoint group of $\mathfrak{su}(n+1,1)$. We can identify G with the quotient of $SU(n+1,1)$ by its center, which is isomorphic to $\mathbb{Z}_{n+2}$, given by the roots of unity times the identity matrix. Thus, we will compute in

$SU(n+1,1)$ keeping in mind that we work modulo the center. First, one easily verifies that the matrices which are in the subgroup G_0 (see 2.9) must be block diagonal (with blocks of sizes 1, n, and 1), and using this, one verifies that they must be of the form

$$\begin{pmatrix} \varphi & 0 & 0 \\ 0 & \Phi & 0 \\ 0 & 0 & \varphi/|\varphi|^2 \end{pmatrix},$$

for some $\varphi \in \mathbb{C}$ and some $\Phi \in U(n)$ such that $\frac{\varphi^2}{|\varphi|^2} \det \Phi = 1$. We denote this element by (φ, Φ).

We have to discuss the adjoint action of G_0 on $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. In a notation similar as above, let us denote an element of $\mathfrak{g}_-$ as a pair (a, X). A direct computation shows that $\mathrm{Ad}(\varphi, \Phi)(a, X) = (\frac{a}{|\varphi|^2}, \varphi^{-1}\Phi X)$. Now for two elements $X, Y \in \mathfrak{g}_{-1}$ the bracket $[X, Y]$ is just the imaginary part of $\langle X, Y\rangle$. Clearly, the adjoint action preserves this bracket.

Conversely, let us assume that we have a complex linear automorphism f of $\mathfrak{g}_{-1}$ such that $[f(X), f(Y)] = \alpha[X, Y]$ for all $X, Y \in \mathfrak{g}_{-1}$ and some (fixed) real number α. Since $\mathrm{Re}(\langle X, Y\rangle) = -\mathrm{Im}(\langle X, iY\rangle)$, we get $[X, iX] = -i\langle X, X\rangle$, and since f is complex linear, this implies on one hand that α must be positive and on the other hand that $\langle f(X), f(Y)\rangle = \alpha\langle X, Y\rangle$. Thus, $\frac{f}{\sqrt{\alpha}}$ is unitary, so the absolute value of the determinant of f is $\alpha^{n/2}$. Now let $\varphi \in \mathbb{C}$ be a complex number such that $\varphi^{-n-2} = \alpha \det(f_1)$. (The non-uniqueness of φ exactly reflects the fact that we work modulo the center of $SU(n+1,1)$.) Taking the absolute value in that equation, we get $\alpha = 1/|\varphi|^2$, which immediately implies that φf is unitary. Moreover, $\det(\varphi f) = \varphi^n \det(f) = \frac{|\varphi|^2}{\varphi^2}$, so $(\varphi, \varphi f)$ is an element of G_0. Thus, we can identify G_0 with the group of all pairs (α, f) as above. (Clearly, the number α is determined by f.)

4.15. Now suppose that M is a smooth manifold of dimension $2n+1$ with a subbundle $T^{-1}M \subset TM$ of real rank $2n$ and that $p : E \to M$ is a principal G_0-bundle with a frame form $\theta = (\theta_{-2}, \theta_{-1})$ of length one on E. As we have noticed in 3.3, in each point $u \in E$ the form $\theta_{-1}(u)$ gives an isomorphism $T^{-1}_{p(u)}M \cong T^{-1}_u E/V_u E \cong \mathfrak{g}_{-1}$ and thus in particular a complex structure J on $T^{-1}_{p(u)}M$. It is independent of the choice of the point u since the adjoint action of G_0 on $\mathfrak{g}_{-1}$ preserves the complex structure. Similarly, since the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is preserved by the adjoint action, it pulls back to a bilinear skew symmetric bundle map $\{\ ,\ \}: T^{-1}M \times T^{-1}M \to (TM/T^{-1}M)$. This map is totally real in the sense that $\{J(\xi), J(\eta)\} = \{\xi, \eta\}$ for all $\xi, \eta \in T_x^{-1}M$ and $x \in M$.

Conversely, assume that we have a smooth manifold M of dimension $2n+1$, together with a rank n complex subbundle $T^{-1}M$ of TM and a bilinear pairing $\{\ ,\ \} : T^{-1}M \times T^{-1}M \to (TM/T^{-1}M)$ which is non-degenerate at each point and totally real. Then for a point $x \in M$ we can fix an identification of $T_xM/T_x^{-1}M$ with $\mathbb{R}$. Then one easily sees that on each $T_x^{-1}M$ the map $\{\ ,\ \}$ is the imaginary part of a non degenerate Hermitian form. Let us in addition assume that this Hermitian form is positive definite for each x and an appropriate isomorphism $T_xM/T_x^{-1}M \to \mathbb{R}$. Note that this also fixes an orientation of the line-bundle $TM/T^{-1}M$ (which means just deciding between a positive definite or a negative definite form).

Then let E be the set of all pairs (φ_1, φ_2), where $\varphi_1 : \mathfrak{g}_{-1} \to T_x^{-1}M$ is a complex linear isomorphism and $\varphi_2 : \mathfrak{g}_{-2} \to T_xM/T_x^{-1}M$ is a linear isomorphism for some $x \in M$, such that $\{\varphi_1(X), \varphi_1(Y)\} = \varphi_2([X, Y])$, for all $X, Y \in \mathfrak{g}_{-1}$. Let $p : E \to M$ denote the obvious projection. Then one verifies directly that this is a smooth principal G_0-bundle, where G_0 acts by composition with the adjoint action from the right.

Moreover, we define a frame form θ on E as follows: Let $Tp : TE \to TM$ be the tangent map of p. Take a point $\varphi = (\varphi_1, \varphi_2) \in E$ and a tangent vector $\xi \in T_\varphi E$. Then $Tp \cdot \xi$ is an element of $T_{p(\varphi)}M$, so we can form its class $[\xi]$ in $T_{p(\varphi)}M/T^{-1}_{p(\varphi)}M$. But the component φ_2 of the point φ is an isomorphism of the latter space with $\mathfrak{g}_{-2}$, and we define $\theta_{-2}(\varphi)(\xi) := \varphi_2^{-1}([\xi])$. This gives a well defined one form $\theta_{-2} \in \Omega^1(E, \mathfrak{g}_{-2})$. Next, by definition an element $\xi \in T_\varphi E$ lies in the subbundle $T_\varphi^{-1}E$ if and only if $Tp \cdot \xi \in T^{-1}_{p(\varphi)}M$. But in this case we can define $\theta_{-1}(\xi) := \varphi_1^{-1}(Tp \cdot \xi)$. From the definitions, one verifies directly that $\theta = (\theta_{-1}, \theta_{-2})$ is actually a frame form of length one on E.

Thus, we see that giving a principal G_0-bundle with a frame form of length one on it is equivalent to specifying a rank n complex subbundle $T^{-1}M$ in TM and a non-degenerate skew pairing $\{\ ,\ \} : T^{-1}M \times T^{-1}M \to (TM/T^{-1}M)$ which is positive definite and compatible with the complex structure as explained above. Finally, by Proposition 4.2 it is clear that the frame form θ satisfies the structure equations if and only if the pairing is actually given by the Levi-form.

Let us compare this to the usual concept of almost-CR-manifolds:

Clearly, specifying a rank n complex subbundle $T^{-1}M \subset TM$ is equivalent to specifying a rank n complex subbundle V in the complexified tangent bundle $T_{\mathbb{C}}M$ such that $V \cap \bar{V} = 0$, by letting V be the holomorphic part of $T^{-1}M \otimes \mathbb{C}$ and conversely putting $T^{-1}M = (V \oplus \bar{V}) \cap TM$. As above, let J be the complex structure on $T^{-1}M$. The fact that the (real) Levi-form on M is a totally real map is clearly equivalent to the fact that for sections ξ, η of $T^{-1}M$ the difference $[\xi, \eta] - [J(\xi), J(\eta)]$ is also a section of $T^{-1}M$. In the complex picture, this is easily seen to be equivalent to the fact that the bracket of two sections of V is a section of $V \oplus \bar{V}$, which is exactly the definition of a *partially integrable* almost-CR-structure (M, V), see [14]. For general almost-CR manifolds, one defines the Levi-form $V \times V \to T_{\mathbb{C}}M/(V \oplus \bar{V})$ via the class of $-i[\xi, \bar{\eta}]$. Now one easily verifies that the real Levi form introduced above is up to a fixed scalar multiple exactly the imaginary part of this Levi form. Consequently, the structures we consider are exactly partially integrable almost-CR-structures with positive definite Levi-form.

4.16. Integrability and torsion

Next, we want to characterize CR-structures with positive definite Levi-form in our picture. By definition, an almost-CR-manifold M is CR if and only if the subbundle $V \subset T_{\mathbb{C}}M$ from 4.15 above is integrable. This can be reformulated as follows: Let ξ and η be smooth sections of $T^{-1}M$, and consider the sections $[\xi, \eta] - [J(\xi), J(\eta)]$ and $[J(\xi), \eta] + [\xi, J(\eta)]$ of TM. As observed above, these are actually sections of $T^{-1}M$, and the integrability is equivalent to the fact that $[\xi, \eta] - [J(\xi), J(\eta)] = -J([J(\xi), \eta] + [\xi, J(\eta)])$.

This integrability problem can be related to our constructions as follows: Suppose we have finished all the prolongations, so we have a principal P-bundle $p : E \to M$ together with a Cartan connection $\omega \in \Omega^1(E, \mathfrak{g})$. If ξ is a local section of $T^{-1}M$, then we can lift it locally to a smooth section $\tilde{\xi}$ of $T^{-1}E$, which is p-related to ξ, i.e. such that $Tp \cdot \tilde{\xi}(u) = \xi(p(u))$. Similarly, for a second section η we find $\tilde{\eta}$. Since the bracket of p-related vector fields is again p-related we get $[\xi, \eta] = Tp \cdot [\tilde{\xi}, \tilde{\eta}]$.

Now put $X := \omega_{-1}(\tilde{\xi})$, $A := \omega_{\mathfrak{p}}(\tilde{\xi})$. Then $\tilde{\xi}(u) = \omega^{-1}(X(u)) + \zeta_{A(u)}$. Similarly, we write $\tilde{\eta}(u) = \omega^{-1}(Y(u)) + \zeta_{B(u)}$. Moreover, Tp induces an isomorphism of $T^{-1}_{p(u)}M$ with $T^{-1}_u E/V_u E$, which in turn is isomorphic via $\omega(u)$ to $\mathfrak{g}_{-1}$, and these are complex linear isomorphisms. Thus, to understand the T^{-1} component of the Lie bracket $[\xi, \eta]$ we have to compute $\omega_{-1}([\tilde{\xi}, \tilde{\eta}])$.

By definition of the exterior derivative, we get

$$\omega_{-1}([\tilde{\xi},\tilde{\eta}](u)) = \tilde{\xi}(u)\cdot Y(u) - \tilde{\eta}(u)\cdot X(u) - d\omega_{-1}(\tilde{\xi}(u),\tilde{\eta}(u)).$$

Next, by definition of the curvature K of ω (see 4.7), we can compute

$$d\omega_{-1}(\tilde{\xi}(u),\tilde{\eta}(u)) = K_{-1}(\tilde{\xi}(u),\tilde{\eta}(u)) - [A_0(u),Y(u)] - [X(u),B_0(u)],$$

where we split A and B according to the splitting $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Finally, by definition $K_{-1}(\tilde{\xi}(u),\tilde{\eta}(u)) = \kappa_{-1}(X(u),Y(u))$, and since both X and Y have degree -1, this equals $\kappa^{(1)}(X(u),Y(u))$. Collecting the computations together, we get

$$\begin{aligned}\omega_{-1}([\tilde{\xi},\tilde{\eta}](u)) &= \tilde{\xi}(u)\cdot Y(u) - \tilde{\eta}(u)\cdot X(u) - \kappa^{(1)}(X(u),Y(u)) \\ &\quad + [A_0(u),Y(u)] + [X(u),B_0(u)].\end{aligned}$$

Next, observe that if $\omega^{-1}(X(u))+\zeta_{A(u)}$ is p-related to ξ, then $\omega^{-1}(iX(u))+\zeta_{A(u)}$ is p-related to $J(\xi)$. Using this, one directly verifies that in the expression corresponding to $[\xi,\eta]-[J(\xi),J(\eta)]+J([J(\xi),\eta]+[\xi,J(\eta)])$ all terms except those coming from $\kappa^{(1)}$ cancel, so that integrability is equivalent to

$$\begin{aligned}&\kappa^{(1)}(u)(X,Y) - \kappa^{(1)}(u)(iX,iY) \\ &\qquad + i(\kappa^{(1)}(u)(iX,Y) + \kappa^{(1)}(u)(X,iY)) = 0,\end{aligned}$$

for all $u \in E$ and $X,Y \in \mathfrak{g}_{-1}$. Note that since this involves only the homogeneous component of degree one of the curvature it is already visible in the first prolongation step.

We just outline briefly how to proceed further: By the Bianchi identity (see 4.9), we know that for each $u \in E$ the map $\kappa^{(1)}(u) : \Lambda^2\mathfrak{g}_- \to \mathfrak{g}$ is ∂-closed and ∂^*-closed. Now we can extend this map to the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(n+2,\mathbb{C})$, which is $|2|$-graded using the same block form as for $\mathfrak{g}$. In the complex case, the subspace $\mathfrak{g}_{-1}$ splits as a $\mathfrak{g}_0$-module into a direct sum of two irreducible modules, and the condition on $\kappa^{(1)}$ from above is equivalent to the fact that the complexification preserves both these submodules. But now the complexification still is ∂- and ∂^*-closed, so it is the harmonic representative for a cohomology class in $H^2_1(\mathfrak{g}^{\mathbb{C}}_-,\mathfrak{g}^{\mathbb{C}})$. But Kostant's version of the Bott-Borel-Weil theorem (see [13]) also gives an explicit description of representatives of these cohomology classes. Looking at these, one sees that they can never preserve the submodules, so that integrability is actually equivalent to $\kappa^{(1)} = 0$. If this is the case, then one can analyze $\kappa^{(2)}$ in a similar way and see that actually integrability implies that the structure is torsion free, so all components of κ in $\mathfrak{g}_-$ vanish.

4.17. Examples related to twistor theory

To finish, we discuss a family of examples of parabolic geometries which is closely related to twistor theory and Penrose transforms. These examples are not interesting from the point of view of the prolongation procedure, since one gets the canonical Cartan connections for free, but we think that they show the importance of understanding the geometrical properties of structures of this type. The basic idea underlying these examples is easy to explain: Suppose that we have a Lie group G with $|k|$-graded Lie algebra $\mathfrak{g}$ and that $\mathfrak{q} \subset \mathfrak{p}$ is a subalgebra which gives rise to an $|\ell|$-grading of $\mathfrak{g}$.

In the complex case, such subalgebras are particularly easy to find. In this case, we know from 2.3 that the original $|k|$-grading on $\mathfrak{g}$ is determined by a standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, which corresponds to a set $\Sigma \subset \Delta_0$ of simple roots, which is exactly the set of those simple roots whose root spaces are contained in $\mathfrak{p}_+$. Now we simply take a second subset Σ' of Δ_0 such that $\Sigma' \supset \Sigma$. By construction, then the standard parabolic $\mathfrak{q}$ corresponding to Σ' is a subalgebra of $\mathfrak{p}$.

Returning to the general case, we consider the corresponding subgroups $Q \subset P \subset G$. If we have a manifold M with a parabolic geometry corresponding to P, then we have a P-principal bundle $E \to M$ endowed with a Cartan connection $\omega \in \Omega^1(E, \mathfrak{g})$ with ∂^*-closed curvature. Now since P acts freely on E, also the subgroup Q of P acts freely on E, so we can form the orbit space $M' := E/Q$. Then the canonical projection $p : E \to M'$ is a principal Q-bundle, and one immediately verifies that the form ω itself is a Cartan connection on $p : E \to M'$. Finally, the curvature κ of ω is also closed under the operator ∂^* corresponding to the subalgebra $\mathfrak{q}$. To see this, one only has to notice that although the subalgebra $\mathfrak{g}_-$ corresponding to $\mathfrak{q}$ is bigger than the one corresponding to $\mathfrak{p}$, the additional elements correspond to horizontal vectors on $E \to M'$ but to vertical vectors on $E \to M$, so the curvature vanishes on these elements. Together with the formula for ∂^* from 2.5 this implies the result. Thus, we see that the parabolic geometry corresponding to $P \subset G$ on M is almost the same thing as a parabolic geometry corresponding to $Q \subset G$ on M'. Note that the corresponding construction in the flat (homogeneous) case is the basis for applications of Penrose transforms to representation theory as described in [4].

For an explicit example, consider the case $\mathfrak{g} = \mathfrak{sl}(4,\mathbb{R})$ with the $|1|$-grading corresponding to the block form $\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}$, where all blocks are of size 2×2. The complexification of this corresponds to the Dynkin diagram $\bullet\!\!-\!\!\times\!\!-\!\!\bullet$ in the notation of 2.3. The corresponding geometric structure is an almost Grassmannian (or paraconformal) structure of type $(2,2)$. This means that one deals with 4-dimensional manifolds equipped with a volume form and two rank two bundles whose tensor product is isomorphic to the tangent bundle (see [2] for a discussion of almost Grassmannian structures and their twistor theory). Via the well known isomorphism $\mathfrak{sl}(4,\mathbb{C}) \cong \mathfrak{so}(6,\mathbb{C})$, upon complexification this gives also information on 4-dimensional conformal manifolds. (To have this directly in the real setting one has to consider pseudo-Riemannian conformal manifolds in split-signature $(2,2)$.)

The simplest instance of the construction outlined above is now to consider $\Sigma' = \{\alpha_1, \alpha_2\} \supset \Sigma = \{\alpha_2\}$. This gives a $|2|$-grading on $\mathfrak{g}$ which is the obvious real version of the example in 2.3 corresponding to $\times\!\!-\!\!\times\!\!-\!\!\bullet$. In this case, the manifold E/Q can be easily described explicitly as follows: The Lie algebra $\mathfrak{p}$ has two obvious irreducible two dimensional representations corresponding to the two diagonal 2×2-blocks. The associated bundles to these representations are exactly the two rank two bundles whose tensor product gives the tangent bundle. Taking the first of these (which corresponds to the upper block), one can pass to the projectivization (i.e. the space of all lines in this representation), and the subgroup Q is exactly the isotropy subgroup of a suitable point in this projectivization. Using this, one easily proves that the manifold M' is exactly the total space of the associated bundle corresponding to this projectivization, so it is exactly the projectivization of the rank two bundle from above. In the language of twistor theory, this is the correspondence space. Thus, the correspondence space carries canonically a parabolic geometry of the type $\times\!\!-\!\!\times\!\!-\!\!\bullet$.

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