Pseudo vector bundles and quasifibrations

Martin A. GUEST, Michał KWIECIŃSKI and Boon-Wee ONG

Abstract. We prove a topological result concerning the kernel $\ker d$ of a morphism $d : E \to F$ of holomorphic vector bundles over a complex analytic space. As a consequence, we show that the projectivization $\mathbb{P}(\ker d)$ is a quasifibration up to some dimension. We give an application to the Abel-Jacobi map of a Riemann surface, and to the space of rational curves in the symmetric product of a Riemann surface.

Key words: linear space, symmetric algebra, quasifibration, Abel-Jacobi map.

1. Introduction

Let $d : E \to F$ be a morphism of (smooth) vector bundles over a manifold M. If the rank of $d_m : E_m \to F_m$ is independent of m, the space

$$L = \ker d = \bigcup_{m \in M} \ker d_m$$

is a subbundle of E. In general, however, the fibres of the map $\pi : L \to M$ are vector spaces of varying dimensions, and the topological behaviour of π can be very complicated; it will certainly not be locally trivial.

In the complex analytic category, the topological behaviour of π is more predictable. We shall show that π "resembles" a vector bundle from the viewpoint of homotopy theory, and in particular that the projectivization $\mathbb{P}(\pi) : \mathbb{P}(L) \to M$ is a quasifibration up to some dimension (Theorem 2.3).

Our results are valid in the more general situation where $d : E \to F$ is a morphism of holomorphic vector bundles over a complex analytic space X, and we shall always work in this generality from now on, also assuming that X is nonempty. The space L is an analytic subspace of the total space of E, and we shall call the map $\pi : L \to X$ a *pseudo vector bundle*. A classical example of a pseudo vector bundle is the Zariski tangent space (to a singular complex algebraic variety). Pseudo vector bundles

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have been studied under the label "linear spaces" in analytic geometry (see [7], [18], [19]) and as (spectra of) "symmetric algebras" in commutative algebra (see [10], [21]). There are still many open problems concerning these objects — for example finding necessary and sufficient conditions for their irreducibility or equidimensionality (see [12], [21]). It should be stressed that L is conceptually different from the kernel sheaf of the induced map of sheaves of sections of E and F. Indeed, the induced map may be injective in cases when the map of vector bundles is not.

Theorem 1.1 below is our basic result on the topological structure of pseudo vector bundles. Before stating it, let us just remark that the concepts of pullback, restriction and subbundle apply to pseudo vector bundles in the obvious way.

Theorem 1.1   *There exists a (unique) filtration of X by closed analytic subsets:*

$$X = X_k \supset X_{k+1} \supset \cdots \supset X_l = \emptyset,$$

such that $X_k \neq X_{k+1}$ and

  • *for each i, either $X_i \setminus X_{i+1}$ is empty or the restriction $L|_{X_i \setminus X_{i+1}}$ is a vector bundle of rank i.*

Furthermore, for each i there exists

  • *an open neighbourhood U_i of X_{i+1} in X_i,*
  • *a (strong) deformation retraction $r_i : U_i \to X_{i+1}$,*
  • *a (strong) deformation retraction $\rho_i : L|_{U_i} \to L|_{X_{i+1}}$,*

such that

  • $r_i^t \circ \pi = \pi \circ \rho_i^t$ *on $L|_{U_i}$, where r_i^t, ρ_i^t (respectively) are homotopies of r_i, ρ_i to the identity maps of U_i, $L|_{U_i}$.*
  • ρ_i *restricted to each fibre of L is a linear monomorphism.*

The first statement of the theorem is easily proved by considering a local model of d, that is a matrix A, whose entries are holomorphic functions on an open subset of X. Then the sets X_i of the filtration are described as the zero sets of ideals generated by minors (of the appropriate size) of A and hence are automatically analytic. These ideals are known as Fitting ideals and have been a subject of intensive study in commutative algebra (see e.g. [6]). The remaining statements will be proved later in the paper.

Definition 1.2   We call the integer k appearing in Theorem 1.1 the *min-*

imal rank of L. If $X_k \setminus X_{k+1}$ is dense in X, then we call k the *generic rank* of L and say that L has a generic rank.

Of course, if X is irreducible then L always has a generic rank.

It should be noted that a pseudo vector bundle is not necessarily a "direct sum" of vector bundles on analytic subsets of X. Indeed, even in the complex analytic situation, the behaviour of the fibres of $L|_{X_i \setminus X_{i+1}}$ near X_{i+1} may be rather complicated. It is in general impossible to extend the vector bundle $L|_{X_i \setminus X_{i+1}}$ to a vector bundle K on X_i such that K is a subbundle of L. The simplest example of such a situation is the kernel of the morphism from a rank two trivial bundle to a rank one trivial bundle over $\mathbb{C}^2$ given by the matrix (x, y).

It is an immediate consequence of Theorem 1.1 and the Dold-Thom criterion [5] that $\mathbb{P}(\pi)$ is a quasifibration up to dimension $2k-1$; this statement appears later as Theorem 2.3. Quasifibrations have played an important but narrowly focused role in algebraic topology (see for example [5], [14], [15], [16], [20], [1]). Our result gives a simple but very general family of new examples.

We shall give a brief review of the theory of quasifibrations in Section 2, and then in Section 3 we give the proof of Theorem 1.1. In Section 4, we give an application to the Abel-Jacobi map of a Riemann surface, and to spaces of rational curves in symmetric products of Riemann surfaces. Finally, in Section 5, we mention an example related to the theory of harmonic maps, and also a counterexample to the statement of Theorem 1.1 in the category of smooth vector bundles.

2. Quasifibrations

We begin by recalling the standard fact from homotopy theory that "any continuous map is homotopic to a fibration". Let $f : X \to Y$ be a continuous map of topological spaces. The *homotopy fibre* of f over y is defined to be the space

$$H_y = \{(x, \gamma) \mid x \in X, \ \gamma : [0,1] \to Y, \ \gamma(0) = y, \ \gamma(1) = f(x)\},$$

i.e. (continuous) paths in Y from y to $f(x)$. This is the fibre over y of the fibration

$$\tilde{f} : \{(x, \gamma) \mid x \in X, \ \gamma : [0,1] \to Y, \ \gamma(1) = f(x)\} \to Y, \quad (x, \gamma) \mapsto \gamma(0).$$

Since the domain of $\tilde{f}$ is homotopy equivalent to X, it follows that — "up to homotopy" — we may identify f with the fibration $\tilde{f}$.

There is a natural inclusion map $f^{-1}(y) \to H_y$. If this is a homotopy equivalence for all $y \in Y$, we say that f is a *quasifibration*. For reasonable spaces X and Y, any fibration has this property, i.e. any fibration is a quasifibration. But in general the property of being a quasifibration is weaker than the property of being a fibration.

It is well known that there is a long exact sequence of homotopy groups

$$\cdots \to \pi_i f^{-1}(y) \to \pi_i X \to \pi_i Y \to \pi_{i-1} f^{-1}(y) \to \cdots$$

if f is a fibration. This remains true if f is a quasifibration, since we may replace H_y by $f^{-1}(y)$ in the long exact sequence of the fibration $\tilde{f}$.

In this paper we shall need the following weaker concept:

Definition 2.1 A map $f : X \to Y$ is a quasifibration up to dimension p if the inclusion $f^{-1}(y) \to H_y$ is a (weak) homotopy equivalence up to dimension p, i.e. if this inclusion map induces isomorphisms of homotopy groups π_i for $i < p$, and a surjection for $i = p$.

Quasifibrations first appeared in [5]. More generally, the concept of "homology fibration (up to dimension p)" was introduced in [14] (Lemma 4.1) and [16]. In all of these papers the main purpose was to study maps between configuration spaces. However, there are other non-trivial applications, for example an approach via quasifibrations to the Bott Periodicity Theorem was suggested in [15] and carried out in [1], and homology fibrations (up to dimension p) were used to study spaces of rational functions in [20].

To prove that a map is a quasifibration, the "Dold-Thom Criterion" of [5] is often used. This criterion may be modified in an obvious way to prove that a map is a quasifibration up to dimension p. The following weaker version of the result of [5] will be sufficient for our purposes.

Theorem 2.2 *Let $f : Y \to X$ be a map, p a positive integer. Assume that X has a filtration by closed subsets:*

$$X = X_k \supset X_{k+1} \supset \cdots \supset X_l = \emptyset,$$

such that

• *for each i, the restriction of f to $f^{-1}(X_i \setminus X_{i+1})$ is a fibration.*
Assume further that for each i there exists

- *an open neighbourhood U_i of X_{i+1} in X_i,*
- *a (strong) deformation retraction $r_i : U_i \to X_{i+1}$,*
- *a (strong) deformation retraction $\rho_i : f^{-1}(U_i) \to f^{-1}(X_{i+1})$,*

such that

- *$r_i^t \circ f = f \circ \rho_i^t$ on $f^{-1}(U_i)$, where r_i^t, ρ_i^t (respectively) are homotopies of r_i, ρ_i to the identity maps of $U_i, f^{-1}(U_i)$.*
- *ρ_i restricted to each fibre of f is a homotopy equivalence up to dimension p.*

Then f is a quasifibration up to dimension p.

Combining this with Theorem 1.1, we obtain the statement in the introduction concerning a projectivized pseudo vector bundle $\mathbb{P}(\pi)$:

Theorem 2.3 *Let $k = \min\{\dim_{\mathbb{C}} \ker d_x \mid x \in X\}$. Then the map*

$$\mathbb{P}(\pi) : \mathbb{P}(L) = \mathbb{P}(\ker d) \to X$$

is a quasifibration up to dimension $2k - 1$.

Proof.   By Theorem 1.1, the "attaching map" $\mathbb{P}(\rho_i|_{\ker d_x})$ is a linear inclusion of the form $\mathbb{C}P^s \to \mathbb{C}P^t$, with $k - 1 \leq s \leq t$. It is well known that any such map is a homotopy equivalence up to dimension $2s + 1$ (for example, because $\mathbb{C}P^t$ may be constructed topologically from $\mathbb{C}P^s$ by adjoining cells of dimensions $2s + 2$, $2s + 4, \ldots, 2t$). $\qquad\qquad\square$

3. Pseudo vector bundles

This section is devoted to the proof of the remaining parts of Theorem 1.1. For simplicity, we shall write $V = X_k \setminus X_{k+1}$ and $S = X_{k+1}$ (S for "singular set").

It is well known that one can simplify the structure of L by an appropriate blowing-up procedure (see [18] or [19]), known as the Nash modification. For our purposes we shall need the following result.

Lemma 3.1 *Suppose L has a generic rank. Then there exists an analytic space X' and a proper surjective analytic map $\nu : X' \to X$, such that*

- *the inverse image $\nu^{-1}V$ is dense in X',*
- *the restriction $\nu|_{\nu^{-1}V} : \nu^{-1}V \to V$ is a biholomorphism,*
- *there exists a vector bundle L', which is a subbundle of the pseudo*

bundle $\nu^* L$, such that

$$L'|_{\nu^{-1}V} = \nu^* L|_{\nu^{-1}V}.$$

Actually, ν becomes unique if we require an additional minimality condition, which we do not need here. The pair (X', ν) may be constructed as follows. The morphism d defines a section of the Grassmannian bundle:

$$\sigma : V \to \mathrm{Grass}_k(E),$$

$$x \to \ker d_x.$$

This section is defined over V only. We define X' to be the closure of the image of σ and the mapping $\nu : X' \to X$ to be the restriction of the natural projection $\mathrm{Grass}_k(E) \to X$. All conditions of Lemma 3.1 are now satisfied. In particular, the bundle L' may be taken to be the restriction of the tautological bundle on the Grassmannian bundle. For the details, we refer to [18]. We just mention that the analyticity of X' is proved by doing a simple calculation in Plücker coordinates, which shows that X' is the closure of an analytically constructible set.

To find the i-th neighbourhood and retractions in Theorem 1.1, one only cares about what goes on inside X_i. Therefore, one may assume $i = k$ and the theorem will follow from the next statement.

Lemma 3.2 *Let $\pi : L \to X$ be a pseudo vector bundle on an analytic space X and let S be an analytic subset of X, $V = X \setminus S$. Assume, that $L|_V$ is a vector bundle of rank k. Then there exists*

- *a neighbourhood U of S in X,*
- *a (strong) deformation retraction $r : U \to S$,*
- *a (strong) deformation retraction $\rho : L|_U \to L|_S$,*

such that

- *$r^t \circ \pi = \pi \circ \rho^t$ on $L|_U$, where r^t, ρ^t (respectively) are homotopies of r, ρ to the identity maps of U, $L|_U$.*
- *ρ restricted to each fibre of L is a linear monomorphism.*

Proof. We shall prove the above lemma in three steps.

Step 1. Lemma 3.2 is true in the case when L itself is a vector bundle.

The existence of U and r is obvious (for example by choosing compatible triangularizations of X and S). It is a standard fact concerning vector

bundles (see [2], Lemma 1.4.3) that we have the following isomorphism (of C^0 vector bundles)

$$L|_U \cong r^*(L|_S).$$

Then ρ can be constructed as the composition of this isomorphism with the natural map $r^*(L|_S) \to L|_S$.

Step 2. Lemma 3.2 is true in the case when L has a generic rank.

The assumption allows us to use Lemma 3.1 to produce X', ν and L'. Now, L' is a true vector bundle. Hence we can apply the result of Step 1 to the space X' with the subset $S' = \nu^{-1}(S)$ and the vector bundle L' to obtain U', r' and ρ'. Next, consider the subbundle inclusion

$$\iota : \; L'|_{S'} \subset \nu^* L|_{S'}$$

and

$$\rho' : L'|_{U'} \to L'|_{S'}.$$

Glue $\iota \circ \rho'$ (defined on $L'|_{U'}$) with the identity on $\nu^* L|_{S'}$ to obtain a retraction

$$\rho'' : \nu^* L|_{U'} \to \nu^* L|_{S'}.$$

The map thus defined is continuous, because we have constructed it as the glueing of maps on two closed sets, which agree on their intersection $(L'|_{S'})$.

We now have suitable retractions on the pullback of L by ν. The remaining problem is to push them forward back again.

First of all, since ν is proper and surjective, by elementary analytic topology one proves that the image of U' contains a neighbourhood of S (we leave this to the reader). Since ν is a homeomorphism on the complement of S', the image $U = \nu(U')$ is actually an open neighbourhood of S. Shrinking U if necessary, we may suppose that $U' = \nu^{-1}(U)$. Then $\nu|_{U'} : U' \to U$ is also proper.

It is now easy to find the retraction r. Notice that $\nu|_{S'} \circ r'$ is constant on each fibre of $\nu|_{U'}$. Since $\nu|_{U'} : U' \to U$ is surjective, this implies the existence of a unique map $r : U \to S$, such that

$$r \circ \nu|_{U'} = \nu|_{S'} \circ r'.$$

The continuity of r easily follows from the fact that $\nu|_{U'} : U' \to U$ is proper and surjective and the other properties are immediately transferred from

those of r'.

The construction of ρ from ρ'' follows along the same lines. Instead of ν, we use the induced map $\tilde{\nu} : \nu^*L \to L$ and remark that it is proper and surjective as the base change (fibred product) of the proper surjective map ν by the projection $\pi : L \to X$. Again, ρ is the unique map which satisfies $\rho \circ \tilde{\nu}|_{\nu^*L|_{U'}} = \tilde{\nu}|_{\nu^*L|_{S'}} \circ \rho''$.

Step 3. The general case.

Let $\tilde{S} = \bar{V} \cap S$. Now $\bar{V}$ (the closure of V) is an analytic set, as the closure of a constructible analytic set. We can now apply the result of the previous step to $\bar{V} \supset \tilde{S}$ and the restriction of L. Taking the union of the neighbourhood with S and extending the retractions by the identity on S and on $L|_S$ one obtains the desired neighbourhood and retractions in the general case.   □

With a little more effort (essentially Lemma 3.2 applied inductively) one can prove a stronger result, which we do not need here, but which is perhaps worth noticing:

Remark 3.3 Lemma 3.2 is true wthout the assumption that $L|_V$ is a vector bundle.

4. The Abel - Jacobi Map

Let M be a compact Riemann surface of genus g, and let $J(M)$ be the Jacobian variety of M (a complex torus of dimension g). Let $\mathrm{Sp}^d(M)$ be the d-th symmetric product of M, i.e. $\mathrm{Sp}^d(M) = M^d/\Sigma_d$ where Σ_d is the symmetric group. Although the action of Σ_d is not free, it is known that $\mathrm{Sp}^d(M)$ has the structure of a complex manifold of dimension d (see [9], §3).

The Abel-Jacobi map $j : \mathrm{Sp}^d(M) \to J(M)$ is a holomorphic map, which may be described as follows: if $\{m_1, \ldots, m_d\}$ is an element of $\mathrm{Sp}^d(M)$, then $j(\{m_1, \ldots, m_d\})$ is the isomorphism class of the holomorphic line bundle $L \otimes L_0$ on M, where L is a line bundle corresponding to the divisor $\sum_{i=1}^d m_i$ and L_0 is a fixed line bundle of degree $-d$.

It is a result of Mattuck that j is a holomorphic fibre bundle, with fibre $\mathbb{C}P^{d-g}$, if $d \geq 2g - 1$. A proof of Mattuck's theorem, together with a more explicit description of j, may be found in [9] (see also [17]). It can be

shown from this description that j is in fact a projectivized pseudo vector bundle, for any $d \geq g$. (If $d < g$, j cannot be surjective.) Moreover, for $g \leq d \leq 2g - 2$, this pseudo vector bundle has generic rank $d - g + 1$. Hence we obtain:

Theorem 4.1 *For $g \leq d \leq 2g - 2$, the Abel-Jacobi map $j : \mathrm{Sp}^d(M) \to J(M)$ is a quasifibration up to dimension $2(d - g) + 1$.*

This result may also be verified by a direct (but unilluminating) calculation since the homotopy types of $\mathrm{Sp}^d(M)$, $J(M)$, and the fibres of j are all well-known (see [17]).

Let $\mathrm{Hol}_k(S^2, \mathrm{Sp}^d(M))$ denote the space of holomorphic maps $f : S^2 \to \mathrm{Sp}^d(M)$ whose homotopy class $[f] \in \pi_2 \mathrm{Sp}^d(M) \cong \mathbb{Z}$ is k (≥ 0). Taking the induced topology from the corresponding space $\mathrm{Map}_k(S^2, \mathrm{Sp}^d(M))$ of continuous maps, we have a continuous inclusion map $i_k : \mathrm{Hol}_k(S^2, \mathrm{Sp}^d(M)) \to \mathrm{Map}_k(S^2, \mathrm{Sp}^d(M))$. There are many examples of compact complex manifolds X for which the analogous inclusion map $\mathrm{Hol}(S^2, X) \to \mathrm{Map}(S^2, X)$ is a homotopy equivalence up to some dimension (where the dimension depends on the homotopy class). A detailed explanation of this type of result and its significance can be found, for example, in [8] and [11]. We shall prove that the space $X = \mathrm{Sp}^d(M)$ is another such example, in fact one which is rather different from those considered so far in the literature. The main tool is the following extension of Theorem 4.1:

Theorem 4.2 *For $g \leq d \leq 2g - 2$, the map $j_k : \mathrm{Hol}_k(S^2, \mathrm{Sp}^d(M)) \to J(M)$, defined by $j_k(f) = j \circ f$, is a quasifibration up to dimension $2(d - g) - 1$.*

Proof. The map is well defined as any holomorphic map $j \circ f : S^2 \to J(M)$ is necessarily constant. Hence, the image of f lies entirely in the fibre of j over the point $j \circ f$. To prove the theorem, exactly the same argument may be used as in the case $k = 0$, except that for $k \geq 1$ we need to know that, if $d - g \leq s \leq t$, the inclusion $\mathrm{Hol}_k(S^2, \mathbb{C}P^s) \to \mathrm{Hol}_k(S^2, \mathbb{C}P^t)$ is a homotopy equivalence up to dimension at least $2(d-g)-1$. This follows (for example) from Segal's theorem ([20]) that the inclusion $\mathrm{Hol}_k(S^2, \mathbb{C}P^N) \to \mathrm{Map}_k(S^2, \mathbb{C}P^N)$ is a homotopy equivalence up to dimension $(2N - 1)k$, because the map $\mathrm{Map}_k(S^2, \mathbb{C}P^s) \to \mathrm{Map}_k(S^2, \mathbb{C}P^t)$ is a homotopy equivalence up to dimension at least $2(d - g) - 1$ (by the argument in the proof of Theorem 2.3). $\square$

Corollary 4.3 *For $g \leq d \leq 2g - 2$ and $k \geq 1$, the natural inclusion map $i_k : \mathrm{Hol}_k(S^2, \mathrm{Sp}^d(M)) \to \mathrm{Map}_k(S^2, \mathrm{Sp}^d(M))$ is a homotopy equivalence up to dimension $2(d - g) - 1$.*

Proof. From Theorem 4.1, the map $j_k' : \mathrm{Map}_k(S^2, \mathrm{Sp}^d(M)) \to \mathrm{Map}(S^2, J(M))$, defined by $j_k'(f) = j \circ f$, is a quasifibration up to dimension $2(d - g) - 1$. The space $\mathrm{Map}(S^2, J(M))$ is homotopy equivalent to $J(M)$ itself, so we may compare the long exact sequences of homotopy groups of j_k and j_k'; the desired result then follows from the Five Lemma and the result of Segal quoted in the proof of Theorem 4.2.   □

For $d > 2g - 2$ all relevant maps in the proof of the corollary are quasifibrations, and so i_k is a homotopy equivalence up to dimension $[2(d-g) - 1]k$ in this case. Thus, the "approximation" improves as $k \to \infty$, as in all other previously known examples of this type. But in the range $g \leq d \leq 2g - 2$ no such improvement appears to be possible.

Corollary 4.3 may be relevant to the quantum cohomology of $\mathrm{Sp}^d(M)$ — see the recent article [3].

5. Further examples

Many examples of pseudo vector bundles may be constructed explicitly by writing down matrix-valued complex analytic functions. It is less easy to find examples where the total space is a smooth variety — the Abel-Jacobi map being one such case. Another case arises naturally in connection with the theory of harmonic maps in differential geometry. It is well known that all harmonic maps from $\mathbb{C}P^1$ to $\mathbb{C}P^2$ may be constructed from holomorphic maps , and it was proved in [4] that each component of the space of harmonic maps $\mathbb{C}P^1 \to \mathbb{C}P^2$ is a smooth variety. The description of this space makes use of a pseudo vector bundle whose total space is smooth; this is described in detail in Section 3 of [13].

Finally, we note that Theorem 1.1 is *false* in the smooth category. There is no guarantee that a suitable neighbourhood U_i of X_i can found; for example the 1×1 matrix-valued function

$$d(u) = \begin{cases} e^{-\frac{1}{u^2}} \sin \dfrac{1}{u} & \text{if} \quad u \neq 0 \\ 0 & \text{if} \quad u = 0 \end{cases}$$

has rank zero on the subset

$$X = \left\{ u = 0 \ \text{ or } \ \frac{1}{\pi n}, \quad n \in \mathbb{Z} \setminus \{0\} \right\}$$

of $\mathbb{R}$, and this set admits no neighbourhood of which it is a deformation retract. Even when a suitable neighbourhood U_i of X_i exists, the following example shows that the fibres of the pseudo vector bundle can behave badly near X. Consider the smooth map

$$d : M \times \mathbb{R}^2 \to M \times \mathbb{R},$$

$$d(m, (x, y)) = (m, xf(m) \cos(m^{-1}) + yf(m) \sin(m^{-1}))$$

where $M = \mathbb{R}$, and where f is a smooth function which vanishes to all orders at zero. Over $\{0\}$ the fibre of $\ker d$ is $\mathbb{R}^2$, and over $M \setminus \{0\}$ the space $\ker d$ is a locally trivial bundle with fibre $\mathbb{R}$. But clearly the conclusions of Theorem 1.1 do not hold.

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Martin A. Guest
Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji-shi
Tokyo 192-0397, Japan
E-mail: martin@comp.metro-u.ac.jp

Michał Kwieciński
Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji-shi
Tokyo 192-0397, Japan
E-mail: kwiecins@im.uj.edu.pl

Boon-Wee Ong
Department of Mathematics
University of Rochester
Rochester, NY 14627, U.S.A.
E-mail: amos@math.rochester.edu