

## Cyclides

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**Abstract.** We give a classification of cyclides as surfaces which contain many circles and their geometric properties.

*Key words:* cyclides, circles, spheres.

### 0. Introduction

A sphere in  $E^3$  is characterized as a closed surface which contains an infinite number of circles through each point. But we do not know a surface other than a sphere or a plane, which contains infinitely many circles through each point of it.

In 1980, Richard Blum [B] found a closed  $C^\infty$  surface of genus one which contains six circles through each point, and he gave a conjecture:

**Conjecture 1** (R. Blum) A closed  $C^\infty$  surface in  $E^3$  which contains seven circles through each point is a sphere.

In 1984, Koichi Ogiue and Ryoichi Takagi [OT] have given the following:

**Theorem** (K. Ogiue and R. Takagi) *A  $C^\infty$  surface in  $E^3$  is (a part of) a plane or a sphere, if it contains two circles through each point, which are tangent to each other.*

Moreover considering the fact that an ellipsoid contains two circles through each point except only at four points, they gave a conjecture in [OT] such as

**Conjecture 2** (K. Ogiue and R. Takagi) A simply connected complete  $C^\infty$  surface in  $E^3$  is a plane or a sphere, if it contains two circles through each point.

We had the following partial affirmative results toward conjectures 1 and 2:

**Theorem** ([N1]) *A simply connected complete  $C^\infty$  surface in  $E^3$  is a plane or a sphere, if it contains three circles through each point.*

**Theorem** ([N2]) *A closed  $C^\infty$  surface of genus one in  $E^3$  cannot contain seven circles through each point.*

Moreover, we got following theorems.

**Theorem** ([N1]) *A  $C^\infty$  surface in  $E^3$  is (a part of) a plane or a sphere, if it contains three circles through each point, any two of which are tangent to each other or have two points in common.*

**Theorem** ([MN]) *A simply connected complete  $C^1$  surface in  $E^3$  is (a part of) a plane or a sphere, if it contains two circles through each point, which are transversal to each other.*

**Theorem** ([ON2]) *A smooth ovaloid in  $E^3$  is a sphere, if the surface contains a circle of an arbitrary but fixed radius through each point.*

Here, we pay attention to the fact that cyclides contain many extrinsic circles. A cyclide is a surface in a 3-dimensional Euclidean space  $E^3$  defined by a quartic equation of the form

$$(x_1^2 + x_2^2 + x_3^2)^2 + 2(x_1^2 + x_2^2 + x_3^2) \sum_{i=1}^3 b_i x_i + \sum_{i,j=1}^3 a_{ij} x_i x_j + 2 \sum_{i=1}^3 a_i x_i + a = 0.$$

It is known that such a surface corresponds to a complete intersection of two quadrics in a 4-dimensional real projective space via pentaspherical representation. An ordinary torus gives a typical example and quadratic surfaces are considered as singular examples. A closed  $C^\infty$  surface of genus one which contains six circles through each point found by R. Blum is also one of cyclides.

In this paper, we study geometric properties of cyclides and give a classification of cyclides from conformal point of view because we are interested in circles contained in a surface, so it is natural to apply conformal geometry. Main results are summerized as

**Theorem** *A non-singular cyclide is conformally equivalent to a cyclide*

of the form

$$(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_2x_2^2 - 2a_3x_3^2 + a = 0 \quad (a \neq 0),$$

which is topologically a torus, a sphere or two spheres. A cyclide with singularities is conformally equivalent to a quadratic surface.

A cyclide contains  $n$  circles through each non-umbilic point and  $n - 1$  circles through each isolated umbilic point unless it is a sphere or a pair of two spheres, where  $n = 1, 2, 3, 4, 5$  or  $6$ .

According to this theorem, a cyclide is covered by at least one family of circles. Therefore it is natural to regard a cyclide as a surface enveloped by a family of some spheres (which are called Meusnier spheres) determined by a family of circles. G. Darboux ([D]) took an interest in such a property of cyclides and gave concrete expressions for families of such spheres that envelope the given cyclide. We review such a property of cyclides and get the following.

**Proposition** *A cyclide which is topologically a torus is a surface enveloped by three distinct families of Meusnier spheres determined by circles on the cyclide. Each sphere contains one or two circles on the cyclide and it is tangent to the cyclide along the circle or at two points, respectively.*

After all, to the author's knowledge, circular tubes and cyclides are the only surfaces that contain many circles through each point. Therefore it is natural to conjecture the following.

**Conjecture** *A surface in  $E^3$  is a cyclide if it contains two circles through almost every point.*

## 1. Pentaspherical representation

Let  $x_1, x_2, x_3$  be natural coordinates in  $E^3$  and let  $u_1, u_2, u_3, u_4$  be natural coordinates in  $E^4$ . Suppose that  $E^3$  is imbedded in  $E^4$  in such a way that  $u_1 = x_1, u_2 = x_2, u_3 = x_3, u_4 = 0$ . Let  $S^3$  be a unit sphere in  $E^4$  defined by  $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$ . Then the stereographic projection of  $S^3 - \{(0, 0, 0, 1)\}$  onto  $E^3$  is given by

$$x_i = \frac{u_i}{1 - u_4}, \quad 1 \leq i \leq 3$$

or equivalently

$$\left\{ \begin{array}{l} u_1 = \frac{2x_1}{x_1^2 + x_2^2 + x_3^2 + 1} \\ u_2 = \frac{2x_2}{x_1^2 + x_2^2 + x_3^2 + 1} \\ u_3 = \frac{2x_3}{x_1^2 + x_2^2 + x_3^2 + 1} \\ u_4 = \frac{x_1^2 + x_2^2 + x_3^2 - 1}{x_1^2 + x_2^2 + x_3^2 + 1}. \end{array} \right.$$

Let  $P_4(R)$  be a 4-dimensional real projective space with homogeneous coordinates  $v_1, v_2, v_3, v_4, v_\infty$ . Suppose that  $E^4$  is mapped onto  $P_4(R) - \{v_\infty = 0\}$  by

$$u_i = \frac{v_i}{v_\infty}, \quad 1 \leq i \leq 4.$$

Then  $S^3$  can be identified with  $\Sigma = \{(v_1, v_2, v_3, v_4, v_\infty) \in P_4(R) \mid v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0\}$ . This identification combined with the stereographic projection gives a correspondence between geometric objects in  $E^3$  and those in  $\Sigma$ , which is called the *pentaspherical representation*. For example, a sphere in  $E^3$  given by

$$\begin{aligned} \left(x_1 - \frac{a_1}{a_\infty - a_4}\right)^2 + \left(x_2 - \frac{a_2}{a_\infty - a_4}\right)^2 + \left(x_3 - \frac{a_3}{a_\infty - a_4}\right)^2 \\ = \frac{a_1^2 + a_2^2 + a_3^2 + a_4^2 - a_\infty^2}{(a_\infty - a_4)^2} \end{aligned}$$

corresponds to a complete intersection in  $P_4(R)$  given by

$$\left\{ \begin{array}{l} a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 - a_\infty v_\infty = 0 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0, \end{array} \right. \quad (1.1)$$

and a plane in  $E^3$  given by

$$b_1 x_1 + b_2 x_2 + b_3 x_3 = b_4$$

corresponds to a complete intersection in  $P_4(R)$  given by

$$\begin{cases} b_1v_1 + b_2v_2 + b_3v_3 + b_4(v_4 - v_\infty) = 0 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0. \end{cases} \quad (1.2)$$

A circle in  $E^3$  is given as an intersection of a sphere and a plane. In view of (1.1) and (1.2), we see that a circle in  $E^3$  corresponds to a complete intersection in  $P_4(R)$  given by

$$\begin{cases} v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0 \\ a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 - a_\infty v_\infty = 0 \\ b_1v_1 + b_2v_2 + b_3v_3 + b_4(v_4 - v_\infty) = 0. \end{cases} \quad (1.3)$$

A *Möbius transformation* is a projective transformation of  $P_4(R)$  which leaves  $\Sigma$  invariant, and a *conformal transformation* is a transformation of  $E^3$  corresponding to a Möbius transformation.

## 2. Cyclides and their pentaspherical representation

A *cyclide* is a surface in  $E^3$  corresponding to a complete intersection in  $P_4(R)$  of the form

$$\begin{cases} \sum_{i,j=1}^{\infty} a_{ij}v_iv_j = 0 & (a_{ij} = a_{ji}) \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0. \end{cases} \quad (2.1)$$

We see that (2.1) corresponds to

$$\begin{aligned} & (a_{44} + 2a_{4\infty} + a_{\infty\infty})(x_1^2 + x_2^2 + x_3^2)^2 \\ & + 4(x_1^2 + x_2^2 + x_3^2) \sum_{i=1}^3 (a_{i4} + a_{i\infty})x_i \\ & + 2 \sum_{i=1}^3 (2a_{ii} - a_{44} + a_{\infty\infty})x_i^2 + 4 \sum_{i \neq j}^3 a_{ij}x_ix_j \\ & - 4 \sum_{i=1}^3 (a_{i4} - a_{i\infty})x_i + a_{44} - 2a_{4\infty} + a_{\infty\infty} = 0, \end{aligned} \quad (2.2)$$

which is of the form

$$\begin{aligned} \varepsilon(x_1^2 + x_2^2 + x_3^2)^2 + 2(x_1^2 + x_2^2 + x_3^2) \sum_{i=1}^3 b_i x_i \\ + \sum_{i,j=1}^3 a_{ij} x_i x_j + 2 \sum_{i=1}^3 a_i x_i + a = 0 \quad (a_{ij} = a_{ji}), \end{aligned} \quad (2.3)$$

where  $\varepsilon = 1$  or  $\varepsilon = 0$ .

Conversely, we see that a cyclide given by (2.3) corresponds to a complete intersection in  $P_4(R)$  given by

$$\begin{cases} \sum_{i,j=1}^3 a_{ij} v_i v_j + 2 \sum_{i=1}^3 (b_i - a_i) v_i v_4 + 2 \sum_{i=1}^3 (b_i + a_i) v_i v_\infty \\ \quad + (\varepsilon + a)(v_4^2 + v_\infty^2) + 2(\varepsilon - a)v_4 v_\infty = 0 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0. \end{cases} \quad (2.4)$$

Since a Möbius transformation maps a complete intersection of the form (2.1) to a complete intersection of the form (2.1), a conformal transformation maps a cyclide to a cyclide.

### 3. Cyclides of the form $(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_2x_2^2 - 2a_3x_3^2 - a^2 = 0$

Consider the following “standard” form of a cyclide:

$$\begin{aligned} (x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_2x_2^2 - 2a_3x_3^2 - a^2 = 0 \\ (a_1 \geq a_2 \geq a_3, a > 0). \end{aligned} \quad (3.1)$$

We first prove the following.

**Proposition 3.1** *A cyclide (3.1) is topologically a sphere.*

*Proof.* If  $a_1 = a_2 = a_3$ , then (3.1) is an ordinary sphere. Therefore we consider the opposite case, and we may assume without loss of generality that  $a_1 \neq a_2$ . Then  $x_3$ , considered as a function of  $x_1$  and  $x_2$ , is a Morse function on the surface, which has at most 10 critical points. By investigating the indices at the critical points of  $x_3$ , we see that the Euler number of the surface is 2 so that the surface is topologically a sphere.  $\square$

By applying a conformal transformation given by

$$\begin{pmatrix} 1/\sqrt{a} & 0 & 0 \\ 0 & 1/\sqrt{a} & 0 \\ 0 & 0 & 1/\sqrt{a} \end{pmatrix},$$

we see that a cyclide given by (3.1) is conformally equivalent to a cyclide given by

$$(x_1^2 + x_2^2 + x_3^2)^2 - 2\frac{a_1}{a}x_1^2 - 2\frac{a_2}{a}x_2^2 - 2\frac{a_3}{a}x_3^2 - 1 = 0.$$

Therefore we may assume without loss of generality that  $a = 1$  in (3.1) as far as we are interested in conformal properties of cyclides.

**Proposition 3.2** *A cyclide (3.1) contains*

- (i) *two circles (resp. one circle) through each non-umbilic (resp. umbilic) point if  $a_1, a_2$  and  $a_3$  are distinct.*
- (ii) *one circle through each non-umbilic point if either  $a_1 = a_2$  or  $a_2 = a_3$ .*
- (iii) *infinitely many circles through each point if  $a_1 = a_2 = a_3$ .*

*Proof.* Since our assertion is conformal, we assume that  $a = 1$ . Then we see that a cyclide given by (3.1) corresponds to a complete intersection in  $P_4(R)$  given by

$$\begin{cases} a_1v_1^2 + a_2v_2^2 + a_3v_3^2 - 2v_4v_\infty = 0 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0. \end{cases} \quad (3.2)$$

We may assume without loss of generality that  $a_1 \geq a_2 \geq a_3$ . We see that (3.2) is equivalent to

$$\begin{cases} (a_1 - a_2)v_1^2 - (a_2 - a_3)v_3^2 - a_2v_4^2 - 2v_4v_\infty + a_2v_\infty^2 = 0 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0. \end{cases} \quad (3.3)$$

We first consider the case where  $a_1 > a_2 > a_3$ . Then (3.3) can be written as

$$\begin{cases} (\sqrt{a_1 - a_2}v_1 - \sqrt{a_2 - a_3}v_3)(\sqrt{a_1 - a_2}v_1 + \sqrt{a_2 - a_3}v_3) \\ \quad = a_2 \left( v_4 + \frac{1 - \sqrt{a_2^2 + 1}}{a_2} v_\infty \right) \left( v_4 + \frac{1 + \sqrt{a_2^2 + 1}}{a_2} v_\infty \right) \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0. \end{cases}$$

Therefore we get

$$\left\{ \begin{array}{l} v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0 \\ \sqrt{a_1 - a_2}v_1 - \sqrt{a_2 - a_3}v_3 = \alpha \left( v_4 + \frac{1 - \sqrt{a_2^2 + 1}}{a_2} v_\infty \right) \\ \sqrt{a_1 - a_2}v_1 + \sqrt{a_2 - a_3}v_3 = \frac{a_2}{\alpha} \left( v_4 + \frac{1 + \sqrt{a_2^2 + 1}}{a_2} v_\infty \right) \end{array} \right.$$

for an arbitrary  $\alpha$  or

$$\left\{ \begin{array}{l} v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0 \\ \sqrt{a_1 - a_2}v_1 - \sqrt{a_2 - a_3}v_3 = \beta \left( v_4 + \frac{1 + \sqrt{a_2^2 + 1}}{a_2} v_\infty \right) \\ \sqrt{a_1 - a_2}v_1 + \sqrt{a_2 - a_3}v_3 = \frac{a_2}{\beta} \left( v_4 + \frac{1 - \sqrt{a_2^2 + 1}}{a_2} v_\infty \right) \end{array} \right.$$

for an arbitrary  $\beta$ . In view of (1.3), we see that (3.1) contains two families of circles.

We next consider the case where  $a_2 = a_3$  to prove (ii). We see that (3.3) can be written as

$$\left\{ \begin{array}{l} v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0 \\ \sqrt{a_1 - a_2}v_1 = \gamma \left( v_4 + \frac{1 - \sqrt{a_2^2 + 1}}{a_2} v_\infty \right) \\ \sqrt{a_1 - a_2}v_1 = \frac{a_2}{\gamma} \left( v_4 + \frac{1 + \sqrt{a_2^2 + 1}}{a_2} v_\infty \right) \end{array} \right.$$

for an arbitrary  $\gamma$ . This implies that (3.1) contains a family of circles.

The case (iii) is clear.  $\square$

#### 4. Cyclides of the form $(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_2x_2^2 - 2a_3x_3^2 + a^2 = 0$

Consider the following “standard” form of a cyclide:

$$\begin{aligned} (x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_2x_2^2 - 2a_3x_3^2 + a^2 = 0 \\ (a_1 \geq a_2 \geq a_3, a > 0). \end{aligned} \quad (4.1)$$



By the same reason as in the previous section, we may assume without loss of generality that  $a = 1$  in (4.1) as far as we are interested in conformal properties of cyclides.

We first consider the case where (4.1) has no singular point.

**Proposition 4.1** *A cyclide (4.1) is*

- (i) *a sphere if  $a_1 = a_2 = a_3 = a$ .*
- (ii) *topologically a torus if  $a_1 \geq a_2 > a > a_3$ .*
- (iii) *topologically concentric two spheres if  $a_1 \geq a_2 \geq a_3 > a$ .*
- (iv) *topologically exclusive two spheres if  $a_1 > a > a_2 \geq a_3$ .*

*Proof.* (i) is clear.

(ii): The surface is connected and  $x_1$  considered as a function of  $x_2$  and  $x_3$  is a Morse function on the surface which has 4 critical points. By investigating the indices at the critical points, we see that the Euler number of the surface is 0 so that the surface is topologically a torus.

(iii) and (iv): The surface has two connected components and  $x_1$  considered as a function of  $x_2$  and  $x_3$  is a Morse function on the surface which has at most 12 critical points. By investigating the indices at the critical points of  $x_1$ , we see that the Euler number of each connected component is 2.  $\square$

R. Blum considers the four cases of torus type

(A)

$$(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_1x_2^2 + 2ax_3^2 + a^2 = 0 \quad (a_1 > a)$$

(B<sub>1</sub>)

$$(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_1x_2^2 - 2a_3x_3^2 + a^2 = 0 \\ (a_1 > a > a_3 \neq -a)$$

(B<sub>2</sub>)

$$(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_2x_2^2 + 2ax_3^2 + a^2 = 0 \quad (a_1 > a_2 > a)$$

(C)

$$(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_2x_2^2 - 2a_3x_3^2 + a^2 = 0 \\ (a_1 > a_2 > a > a_3 \neq -a)$$

and proves the following.

**Proposition 4.2** ([B])

- (i) A cyclide of type (A) is an ordinary torus and it contains four circles through each point.
- (ii) A cyclide of type (B<sub>*i*</sub>),  $i = 1, 2$ , contains five circles through each point.
- (iii) A cyclide of type (C) contains six circles through each point.

*Proof.* Since our assertion is conformal, we assume that  $a = 1$  in (4.1). Then we see that a cyclide given by (4.1) corresponds to a complete intersection in  $P_4(R)$  given by

$$\begin{cases} a_1 v_1^2 + a_2 v_2^2 + a_3 v_3^2 - v_4^2 - v_\infty^2 = 0 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0. \end{cases} \quad (4.2)$$

We prove (iii): We see that (4.2) is equivalent to

$$\begin{cases} (a_1 - a_2)v_1^2 - (a_2 - a_3)v_3^2 = (a_2 + 1)v_4^2 - (a_2 - 1)v_\infty^2 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0 \end{cases} \quad (4.2')$$

or

$$\begin{cases} (a_1 - a_3)v_1^2 - (a_3 + 1)v_4^2 = (1 - a_3)v_\infty^2 - (a_2 - a_3)v_2^2 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0 \quad (-1 < a_3 < 1) \end{cases} \quad (4.2'')$$

or

$$\begin{cases} (a_1 + 1)v_1^2 + (a_3 + 1)v_3^2 = 2v_\infty^2 - (a_2 + 1)v_2^2 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0 \quad (a_3 < -1) \end{cases} \quad (4.2''')$$

or

$$\begin{cases} (a_1 - 1)v_1^2 - (1 - a_3)v_3^2 = 2v_4^2 - (a_2 - 1)v_2^2 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0 \end{cases} \quad (4.2''')$$

It is easily seen that (4.2') can be written as

$$\begin{cases} (\sqrt{a_1 - a_2}v_1 - \sqrt{a_2 - a_3}v_3)(\sqrt{a_1 - a_2}v_1 + \sqrt{a_2 - a_3}v_3) \\ \quad = (\sqrt{a_2 + 1}v_4 - \sqrt{a_2 - 1}v_\infty)(\sqrt{a_2 + 1}v_4 + \sqrt{a_2 - 1}v_\infty) \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0. \end{cases}$$

Therefore we get

$$\begin{cases} v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0 \\ \sqrt{a_1 - a_2}v_1 - \sqrt{a_2 - a_3}v_3 = \alpha(\sqrt{a_2 + 1}v_4 - \sqrt{a_2 - 1}v_\infty) \\ \sqrt{a_1 - a_2}v_1 + \sqrt{a_2 - a_3}v_3 = \frac{1}{\alpha}(\sqrt{a_2 + 1}v_4 + \sqrt{a_2 - 1}v_\infty) \end{cases}$$

for an arbitrary  $\alpha$  or

$$\begin{cases} v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0 \\ \sqrt{a_1 - a_2}v_1 - \sqrt{a_2 - a_3}v_3 = \beta(\sqrt{a_2 + 1}v_4 + \sqrt{a_2 - 1}v_\infty) \\ \sqrt{a_1 - a_2}v_1 + \sqrt{a_2 - a_3}v_3 = \frac{1}{\beta}(\sqrt{a_2 + 1}v_4 - \sqrt{a_2 - 1}v_\infty) \end{cases}$$

for an arbitrary  $\beta$ . In view of (1.3), we see that the cyclide contains two families of circles.

By applying the same argument to (4.2'') or (4.2''') and to (4.2'''), we conclude that a cyclide of type (C) contains six circles through each point.

The assertions (i) and (ii) can be proved similarly.  $\square$

**Remark** Let  $\mathfrak{L}$  and  $\mathfrak{M}$  be homotopy classes of a torus generated by a “latitude” and a “meridian”, respectively, so that the fundamental group is generated by  $\mathfrak{L}$  and  $\mathfrak{M}$ . Then we see that the circles corresponding to (4.2') belong to  $\mathfrak{L}$ , the circles corresponding to (4.2'') or (4.2''') belong to  $\mathfrak{M}$  and the circles corresponding to (4.2''') belong to  $\mathfrak{L} + \mathfrak{M}$  and  $\mathfrak{L} - \mathfrak{M}$ , respectively.

**Proposition 4.3** *A cyclide of type (B<sub>1</sub>) is conformally equivalent to a cyclide of type (B<sub>2</sub>) and vice versa.*

*Proof.* Since our assertion is conformal, we consider a cyclide of type (B<sub>2</sub>) with  $a = 1$ :

$$(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_2x_2^2 + 2x_3^2 + 1 = 0. \quad (4.3)$$

Then we see that a cyclide given by (4.3) corresponds to a complete intersection in  $P_4(R)$  given by

$$\begin{cases} a_1v_1^2 + a_2v_2^2 - v_3^2 - v_4^2 - v_\infty^2 = 0 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0. \end{cases} \quad (4.4)$$

We see that (4.4) is transformed to

$$\begin{cases} -v_1^2 - v_2^2 + a_2 v_3^2 + a_1 v_4^2 - v_\infty^2 = 0 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0 \end{cases} \quad (4.5)$$

by a Möbius transformation given by

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We see that (4.5) corresponds to a cyclide

$$\begin{aligned} (x_1^2 + x_2^2 + x_3^2)^2 - 2\frac{a_1 + 3}{a_1 - 1}x_1^2 - 2\frac{a_1 + 3}{a_1 - 1}x_2^2 \\ - 2\frac{a_1 - 2a_2 + 1}{a_1 - 1}x_3^2 + 1 = 0 \end{aligned} \quad (4.6)$$

which is a cyclide of type (B<sub>1</sub>). The converse can be shown similarly.  $\square$

**Remark** We see that (4.3) is transformed to (4.6) by a conformal transformation (an inversion) given by

$$\begin{cases} x'_1 = \frac{2(x_1 - 1)}{(x_1 - 1)^2 + x_2^2 + x_3^2} + 1 \\ x'_2 = \frac{2x_3}{(x_1 - 1)^2 + x_2^2 + x_3^2} \\ x'_3 = \frac{2x_2}{(x_1 - 1)^2 + x_2^2 + x_3^2}. \end{cases}$$

We next consider the case where (4.1) is topologically two spheres.

**Proposition 4.4** *A cyclide (4.1) of type  $a_1 \geq a_2 \geq a_3 > a$  is conformally equivalent to a cyclide (4.1) of type  $a_1 > a > a_2 \geq a_3$ .*

*Proof.* It follows from Proposition 4.1 that the former is topologically concentric two spheres and the latter is topologically exclusive two spheres. We see that the latter is mapped to the former by an inversion with respect to a sphere which “separates” two portions of the latter.  $\square$

**Proposition 4.5** *A cyclide (4.1) contains two circles (resp. one circle) through each non-umbilic (resp. umbilic) point if  $a_1 > a_2 > a_3 > a$  or  $a_1 > a > a_2 > a_3$  and it contains one circle through each non-umbilic point if  $a_1 > a_2 = a_3 > a$  or  $a_1 = a_2 > a_3 > a$  or  $a_1 > a > a_2 = a_3$ .*

*Proof.* Since our assertion is conformal, we assume that  $a = 1$  in (4.1). Then we see that a cyclide given by (4.1) corresponds to a complete intersection in  $P_4(\mathbb{R})$  given by (4.2). If  $a_1 \geq a_2 \geq a_3 > 1$  is the case, we see that (4.2) is equivalent to (4.2') and we obtain the assertion by the same method as in the proof of Proposition 4.2. Note that a circle passing through an umbilic point may reduce to a point. The case where  $a_1 > 1 > a_2 \geq a_3$  can be proved similarly.  $\square$

Finally we consider the case where (4.1) has singular points.

**Proposition 4.6** *A cyclide (4.1) is*

- (i) *conformally equivalent to an elliptic cone if  $a_1 \geq a_2 > a_3 = a$ .*
- (ii) *conformally equivalent to a hyperbolic cone if  $a_1 > a_2 = a > a_3$ .*
- (iii) *conformally equivalent to two planes if  $a_1 = a_2 = a > a_3$ .*

*Proof.* (i) It is easily seen that  $(0, 0, \pm\sqrt{a})$  are the singular points. By applying an inversion with center at  $(0, 0, \sqrt{a})$ , we see that (4.1) is conformally equivalent to an elliptic cone.

(ii) It is easily seen that  $(0, \pm\sqrt{a}, 0)$  are the singular points. By applying an inversion with center at  $(0, \sqrt{a}, 0)$ , we see that (4.1) is conformally equivalent to a hyperbolic cone.

(iii) It is easily seen that  $\{(x_1, x_2, 0) \mid x_1^2 + x_2^2 = a\}$  is the set of singular points. By applying an inversion with center at  $(0, \sqrt{a}, 0)$ , we see that (4.1) is conformally equivalent to two planes.  $\square$

**Remark** Since we are interested in conformal properties, we do not distinguish between circles and lines in the following:

- (i) A generic elliptic cone contains three circles through each non-singular point.
- (ii) A circular cone contains two circles through each non-singular point.
- (iii) A hyperbolic cone contains one circle through each non-singular point.

## 5. Cyclides of the form $(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_2x_2^2 - 2a_3x_3^2 = 0$

Consider the following particular form of a cyclide:

$$(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1x_1^2 - 2a_2x_2^2 - 2a_3x_3^2 = 0 \quad (a_1 \geq a_2 \geq a_3). \quad (5.1)$$

We see that a cyclide (5.1) has a singular point at the origin and it is conformally equivalent to a quadratic surface, that is, we prove the following.

**Proposition 5.1** *A cyclide (5.1) with the origin removed is*

- (i) *conformally equivalent to an ellipsoid if  $a_1 \geq a_2 \geq a_3 > 0$ .*
- (ii) *conformally equivalent to a hyperboloid of one sheet if  $a_1 \geq a_2 > 0 > a_3$ .*
- (iii) *conformally equivalent to a hyperboloid of two sheets if  $a_1 > 0 > a_2 \geq a_3$ .*
- (iv) *conformally equivalent to an elliptic cylinder if  $a_1 \geq a_2 > a_3 = 0$ .*
- (v) *conformally equivalent to a hyperbolic cylinder if  $a_1 > a_2 = 0 > a_3$ .*
- (vi) *a pair of siamese-twin spheres which is conformally equivalent to two planes if  $a_1 > a_2 = a_3 = 0$ .*

*Proof.* By applying an inversion with center at the origin, we easily verify the assertions.  $\square$

**Remark** Since we are interested in conformal properties, we do not distinguish between circles and lines in the following:

- (i) A generic ellipsoid contains two circles through each non-umbilic point and one circle through each umbilic point.
- (ii) An ellipsoid of revolution contains one circle through each non-umbilic point, unless it is a sphere.
- (iii) A generic hyperboloid of one sheet contains four circles through each point.
- (iv) A hyperboloid of one sheet of revolution contains three circles through each point.
- (v) A generic hyperboloid of two sheets contains two circles through each non-umbilic point and one circle through each umbilic point.
- (vi) A hyperboloid of two sheets of revolution contains one circle through each non-umbilic point.
- (vii) A generic elliptic cylinder contains three circles through each point.
- (viii) A circular cylinder contains two circles through each point.

(ix) A hyperbolic cylinder contains one circle through each point.

## 6. Conformal classification of cyclides

The purpose of this section is to give a conformal classification of cyclides. First of all, we prove the following.

**Proposition 6.1** *A cyclide (2.3) with  $\varepsilon = 1$  can be transformed to a cyclide (2.3) with  $\varepsilon = 0$  by a conformal transformation and vice versa.*

*Proof.* A straightforward computation shows that a translation followed by an inversion with center at a point on the surface is a desired conformal transformation.  $\square$

Therefore it is sufficient to consider cyclides of the form (2.3) with  $\varepsilon = 1$  as far as we are interested in conformal properties.

**Proposition 6.2** *A cyclide (2.3) with  $\varepsilon = 1$  can be transformed to a cyclide of the form*

$$(x_1^2 + x_2^2 + x_3^2)^2 + \sum_{i=1}^3 a_{ii}x_i^2 + 2 \sum_{i=1}^3 a_ix_i + a = 0 \quad (6.1)$$

*by an isometry.*

*Proof.* A straightforward computation shows that a translation  $x_i \longrightarrow x_i + \frac{b_i}{2}$  followed by a rotation is a desired isometry.  $\square$

**Proposition 6.3** *A cyclide (2.3) with  $\varepsilon = 1$  is conformally equivalent to a cyclide of the form*

$$(x_1^2 + x_2^2 + x_3^2)^2 + \sum_{i=1}^3 a_{ii}x_i^2 + 2a_3x_3 + a = 0. \quad (6.2)$$

*Proof.* By Proposition 6.2, it suffices to show that a cyclide of the form (6.1) is conformally equivalent to a cyclide of the form (6.2).

Note that a cyclide (6.1) corresponds to a complete intersection in  $P_4(R)$  given by





which is equivalent to  $\lambda = a_{11}$  and

$$\begin{aligned} &(\lambda^2 - 4a)(\lambda - a_{11})(\lambda - a_{33}) \\ &\quad - 4(a_1^2 + a_2^2)(\lambda - a_{33}) - 4a_3^2(\lambda - a_{11}) = 0. \end{aligned} \quad (6.5)$$

Let  $g(\lambda)$  stand for the left-hand-side of (6.5). Then we obtain  $g(a_{11}) = -4(a_1^2 + a_2^2)(a_{11} - a_{33}) \leq 0$ . If  $a_1 = a_2 = 0$ , then the assertion is trivial. Otherwise, since  $g(a_{11}) < 0$ , (6.5) has at least one simple real solution ( $\neq a_{11}$ ). Let  $\lambda_1$  ( $= a_{11}$ ) and  $\lambda_2$  be simple real solutions of (6.4) and let  $e_1$  and  $e_2$  be vectors satisfying  $Ae_i = \lambda_i Je_i$  ( $i = 1, 2$ ) and  ${}^t e_1 Je_1 = 1$ ,  ${}^t e_2 Je_2 = 1$  and  ${}^t e_1 Je_2 = 0$ . Then we can see that  $A$  is transformed to a matrix of the form

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} \quad (6.6)$$

by a Möbius transformation given by  $(e_1, e_2, e_3, e_4, e_\infty)$ , where  $e_3, e_4$  and  $e_\infty$  are vectors satisfying  ${}^t e_3 Je_3 = 1$ ,  ${}^t e_4 Je_4 = 1$ ,  ${}^t e_\infty Je_\infty = -1$  and  ${}^t e_i Je_j = 0$  ( $1 \leq i \neq j \leq \infty$ ). This implies that (6.1) can be transformed to a cyclide of the form  $(x_1^2 + x_2^2 + x_3^2)^2 + b_3 x_3 (x_1^2 + x_2^2 + x_3^2) + \sum_{i=1}^3 a_{ii} x_i^2 + 2a_3 x_3 + a = 0$  by a conformal transformation. This can be transformed to a cyclide of the form (6.2) by a translation  $x_3 \rightarrow x_3 + \frac{b_3}{2}$ . The case where  $a_{11} > a_{22} = a_{33}$  can be treated similarly.

We finally consider the case where  $a_{11} > a_{22} > a_{33}$ . Let  $f$  stand for the left-hand-side of (6.4). Note that  $f(a_{ii}) = 0$  if and only if  $a_i = 0$  ( $i = 1, 2, 3$ ).

*Case 1:* Neither of  $a_{11}, a_{22}, a_{33}$  is a solution of (6.4).

We see that  $f(a_{11}) = -4a_1^2(a_{11} - a_{22})(a_{11} - a_{33}) < 0$ ,  $f(a_{22}) = -4a_2^2(a_{22} - a_{11})(a_{22} - a_{33}) > 0$  and  $f(a_{33}) = -4a_3^2(a_{33} - a_{11})(a_{33} - a_{22}) < 0$ . This implies that (6.4) has at least two simple real solutions. Therefore, by the same argument as above, we see that  $A$  is transformed to a matrix of the form (6.6) by a Möbius transformation.

*Case 2:* Some of  $a_{11}, a_{22}, a_{33}$  are solutions of (6.4).

Suppose that  $f(a_{11}) = 0$  so that  $a_1 = 0$ . If, in addition, one of  $a_2$  and  $a_3$  is zero, then the assertion is trivial. Therefore we consider the case where

$a_2a_3 \neq 0$ . We see that (6.4) is reduced to

$$(\lambda - a_{11})\{(\lambda^2 - 4a)(\lambda - a_{22})(\lambda - a_{33}) - 4a_2^2(\lambda - a_{33}) - 4a_3^2(\lambda - a_{22})\} = 0$$

which is equivalent to  $\lambda = a_{11}$  and

$$(\lambda^2 - 4a)(\lambda - a_{22})(\lambda - a_{33}) - 4a_2^2(\lambda - a_{33}) - 4a_3^2(\lambda - a_{22}) = 0. \quad (6.7)$$

If  $a_{11}$  is a simple solution of (6.4), then we see that (6.4) has another simple real solution, say,  $\lambda_2$ . Let  ${}^t e_1 = (1, 0, 0, 0, 0)$  and let  $e_2$  be a vector satisfying  $Ae_2 = \lambda_2 Je_2$ ,  ${}^t e_2 Je_2 = 1$  and  ${}^t e_1 Je_2 = 0$ . Then  $A$  is transformed to a matrix of the form (6.6) with  $\lambda_1 = a_{11}$  by a Möbius transformation given by  $(e_1, e_2, e_3, e_4, e_\infty)$ , where  $e_3, e_4$  and  $e_\infty$  are vectors satisfying  ${}^t e_3 Je_3 = 1$ ,  ${}^t e_4 Je_4 = 1$ ,  ${}^t e_\infty Je_\infty = -1$  and  ${}^t e_i Je_j = 0$  ( $1 \leq i \neq j \leq \infty$ ).

If  $a_{11}$  is a double solution of (6.4), then either (6.4) has at least one simple real solution or it has a triple solution. In the former case, we can apply the same argument as above to see that  $A$  is transformed to a matrix of the form (6.6) with  $\lambda_1 = a_{11}$  by a Möbius transformation.

In the latter case, (6.4) can be written as  $(\lambda - a_{11})^2(\lambda - \beta)^3 = 0$  for some  $\beta$  with  $a_{33} < \beta < a_{22}$  so that we have

$$\begin{cases} a_{11} + 3\beta = a_{22} + a_{33} \\ 3\beta(a_{11} + \beta) = a_{22}a_{33} - 4a \\ \beta^2(3a_{11} + \beta) = 4a_2^2 + 4a_3^2 - 4a(a_{22} + a_{33}) \\ a_{11}\beta^3 = 4a_2^2a_{33} + 4a_3^2a_{22} - 4aa_{22}a_{33}. \end{cases} \quad (6.8)$$

We see that the eigenspace  $\{X \mid AX = a_{11}JX\}$  is spanned by  ${}^t(1, 0, 0, 0, 0)$  and  ${}^t\left(0, \frac{-4a_2}{(a_{11}-a_{22})(a_{11}-2)}, \frac{-4a_3}{(a_{11}-a_{33})(a_{11}-2)}, \frac{a_{11}+2}{a_{11}-2}, 1\right)$  if  $a_{11} \neq 2$ . It follows from (6.8) that the square of the norm of the latter is given by  $\frac{4(a_{11}-\beta)^3}{(a_{11}-2)^2(a_{11}-a_{22})(a_{11}-a_{33})} > 0$ . Therefore  $A$  is transformed to a matrix of the form (6.6) with  $\lambda_1 = \lambda_2 = a_{11}$  by a Möbius transformation.

If  $a_{11} = 2$ , then the eigenspace  $\{X \mid AX = 2JX\}$  is spanned by  ${}^t(1, 0, 0, 0, 0)$  and  ${}^t\left(0, \frac{a_2}{a_{22}-2}, \frac{a_3}{a_{33}-2}, 1, 0\right)$  and the latter is of course not null. Therefore  $A$  is transformed to a matrix of the form (6.6) with  $\lambda_1 = \lambda_2 = 2$  by a Möbius transformation.

If  $a_{11}$  is a triple or quadruple solution of (6.4), then we see that the



by a Möbius transformation given by  $(e_1, e_2, e_3, e_4, e_\infty)$ . This implies that (6.2) can be transformed to a cyclide of the form  $(x_1^2 + x_2^2 + x_3^2)^2 + \sum_{i=1}^3 a_{ii}x_i^2 + a = 0$  by a conformal transformation.  $\square$

**Proposition 6.5** *A cyclide (6.2) with  $27a_3^2 = -2a_{33}^3$  and  $12a = -a_{33}^2$  ( $a_{33} < 0$ ) has singular points and it is*

- (i) *conformally equivalent to an elliptic paraboloid if  $(a_{33} - 3a_{11})(a_{33} - 3a_{22}) > 0$ .*
- (ii) *conformally equivalent to a hyperbolic paraboloid if  $(a_{33} - 3a_{11})(a_{33} - 3a_{22}) < 0$ .*
- (iii) *conformally equivalent to a parabolic cylinder if either  $a_{33} = 3a_{11}$  or  $a_{33} = 3a_{22}$ .*
- (iv) *a sphere if  $a_{11} = a_{22} = \frac{1}{3}a_{33}$ .*

*Proof.* We see that  $(0, 0, \pm\sqrt{-\frac{a_{33}}{6}})$  are singular points. By applying an inversion with center at  $(0, 0, \sqrt{-\frac{a_{33}}{6}})$ , we get the assertions (i), (ii), (iii). Under the assumption of (iv), (6.2) is reduced to

$$\left\{ x_1^2 + x_2^2 + \left( x_3 \pm \sqrt{-\frac{a_{33}}{6}} \right)^2 + \frac{2}{3}a_{33} \right\} \left\{ x_1^2 + x_2^2 + \left( x_3 \mp \sqrt{-\frac{a_{33}}{6}} \right)^2 \right\} = 0.$$

This implies that the surface is a sphere with a singular point.  $\square$

**Remark** Since we are interested in conformal properties, we do not distinguish between circles and lines in the following:

- (i) A generic elliptic paraboloid contains two circles through each non-umbilic point and one circle through each umbilic point.
- (ii) An elliptic paraboloid of revolution contains one circle through each non-umbilic point.
- (iii) A hyperbolic paraboloid contains two circles through each point.
- (iv) A parabolic cylinder contains one circle through each point.

Summarizing these results, we get the following.

**Theorem 6.6** *A cyclide is conformally equivalent to one of the standard forms (3.1) and (4.1) or a quadratic surface.*

## 7. How many circles does a cyclide contain through each point?

R. Blum ([B]) gives an answer to the above question when the cyclide is topologically a torus (cf. Proposition 4.2). Summarizing the results given

in sections 3, 4, 5 and 6, we get a complete answer to the question given as the title of this section.

**Theorem 7.1** *A cyclide contains  $n$  circles through each non-umbilic point and  $n - 1$  circles through each isolated umbilic point unless it is a sphere or a pair of two spheres, where  $n = 1, 2, 3, 4, 5$  or  $6$ .*

## 8. Cyclides as enveloping surfaces of families of spheres: Consideration à la Darboux

Let  $X$  be a tangent vector at a point  $p$  of a surface. Then we can associate a sphere  $S(p, X)$  with the properties that (1)  $S(p, X)$  is tangent to the surface and (2) the radius of  $S(p, X)$  is equal to the normal curvature of the surface in the direction of  $X$ . Let  $\gamma$  be a circle lying on a cyclide. Then, by the theorem of Meusnier, we see that  $S(p, \dot{\gamma})$  contains  $\gamma$ . We call  $S(p, \dot{\gamma})$  the *Meusnier sphere* determined by  $\gamma$  and  $p$ .

According to Theorem 7.1, a cyclide is covered by at least one family of circles. Therefore it is natural to regard a cyclide as a surface enveloped by a family of Meusnier spheres determined by a family of circles. We review such a property of cyclides following G. Darboux ([D]). He gave concrete and explicit expressions for families of such spheres that envelope the given cyclide.

We consider cyclides which are topologically tori, that is, cyclides of the form (4.1) with  $a_1 \geq a_2 > a > a_3$ .

Consider a sphere

$$x_1^2 + x_2^2 + x_3^2 - 2c_1x_1 - 2c_2x_2 - 2c_3x_3 - 2c = 0. \quad (8.1)$$

Since a circle is a plane curve, the necessary condition for a cyclide (4.1) and a sphere (8.1) to intersect along a circle is that a quadratic surface

$$4(c_1x_1 + c_2x_2 + c_3x_3 + c)^2 - 2a_1x_1^2 - 2a_2x_2^2 - 2a_3x_3^2 + a^2 + \lambda(x_1^2 + x_2^2 + x_3^2 - 2c_1x_1 - 2c_2x_2 - 2c_3x_3 - 2c) = 0 \quad (8.2)$$

represents two planes for a suitable choice of a constant  $\lambda$ . Note that these two planes may coincide. We see that (8.2) reduces to

$$(p_1x_1 + p_2x_2 + p_3x_3 + p)(q_1x_1 + q_2x_2 + q_3x_3 + q) = 0 \quad (8.3)$$

so that it represents two planes only when

$$(\lambda^2 - 4a^2)(\lambda - 2a_1)(\lambda - 2a_2)(\lambda - 2a_3) = 0 \quad (8.4)$$

$$(\lambda - 4c)(\lambda - 2a_1)(\lambda - 2a_2)(\lambda - 2a_3) = 0 \quad (8.5)$$

$$\begin{aligned} &4(\lambda - 2a_2)(\lambda - 2a_3)c_1^2 + 4(\lambda - 2a_1)(\lambda - 2a_3)c_2^2 \\ &+ 4(\lambda - 2a_1)(\lambda - 2a_2)c_3^2 + (\lambda - 2a_1)(\lambda - 2a_2)(\lambda - 2a_3) = 0. \end{aligned} \quad (8.6)$$

hold. Note that one of the planes given by (8.3) contains the circle under consideration and another plane also contains a circle on the cyclide. It follows from (8.4) that  $\lambda$  has five possibilities:  $\pm 2a, 2a_1, 2a_2, 2a_3$ .

*Case I:*  $\lambda = 2a$ .

We see from (8.5) and (8.6) that the necessary condition for (8.2) to represent two planes is that  $c_1, c_2$  and  $c_3$  satisfy

$$\frac{2c_1^2}{a_1 - a} + \frac{2c_2^2}{a_2 - a} - \frac{2c_3^2}{a - a_3} = 1$$

and  $c = \frac{a}{2}$ . We can verify that a family of spheres (8.1) with these conditions actually envelopes the given cyclide.

Moreover we see that  $p_1, p_2, p_3, p; q_1, q_2, q_3, q$  in (8.3) satisfy

$$\left( \frac{p_1^2}{a_1 - a} + \frac{p_2^2}{a_2 - a} - \frac{p_3^2}{a - a_3} \right) \left( \frac{q_1^2}{a_1 - a} + \frac{q_2^2}{a_2 - a} - \frac{q_3^2}{a - a_3} \right) = 0$$

and  $p = q = 0$ .

*Case IIa:*  $\lambda = -2a (> 2a_3)$ .

We see from (8.5) and (8.6) that the necessary condition for (8.2) to represent two planes is that  $c_1, c_2$  and  $c_3$  satisfy

$$\frac{2c_1^2}{a + a_1} + \frac{2c_2^2}{a + a_2} + \frac{2c_3^2}{a + a_3} = 1$$

and  $c = -\frac{a}{2}$ . We can verify that a family of spheres (8.1) with these conditions actually envelopes the cyclide.

Moreover we see that  $p_1, p_2, p_3, p; q_1, q_2, q_3, q$  in (8.3) satisfy

$$\left( \frac{p_1^2}{a+a_1} + \frac{p_2^2}{a+a_2} + \frac{p_3^2}{a+a_3} \right) \left( \frac{q_1^2}{a+a_1} + \frac{q_2^2}{a+a_2} + \frac{q_3^2}{a+a_3} \right) = 0$$

and  $p = q = 0$ .

*Case IIb:*  $\lambda = -2a (< 2a_3)$ .

We see from (8.5) and (8.6) that the necessary condition for (8.2) to represent two planes is that  $c_1, c_2$  and  $c_3$  satisfy

$$\frac{2c_1^2}{a+a_1} + \frac{2c_2^2}{a+a_2} + \frac{2c_3^2}{a+a_3} = 1$$

and  $c = -\frac{a}{2}$ . Under these conditions, the quadratic part of (8.2) is semi-definite so that (8.2) cannot represent two planes. Therefore this case does not occur.

*Case III:*  $\lambda = -2a = 2a_3$ .

We get from (8.6) that  $c_3 = 0$ . Hence (8.2) reduces to

$$\begin{aligned} 4(c_1x_1 + c_2x_2 + c)^2 - 2a_1x_1^2 - 2a_2x_2^2 + a^2 \\ - 2a(x_1^2 + x_2^2 - 2c_1x_1 - 2c_2x_2 - 2c) = 0. \end{aligned} \quad (8.2')$$

Note that if (8.2) represents two planes, then (8.2') gives two lines on the  $x_1x_2$ -plane. We see that (8.2') gives two lines on the  $x_1x_2$ -plane only when  $c = -\frac{a}{2}$ . If this is the case, (8.2') reduces to

$$(2c_1^2 - a_1 - a)x_1^2 + 4c_1c_2x_1x_2 + (2c_2^2 - a_2 - a)x_2^2 = 0,$$

which represents two planes containing  $x_3$ -axis. We see that the sphere containing two circles determined by these two planes is tangent to the cyclide only when these two planes are coincidental. The condition for the two planes to be coincidental is given by

$$\frac{2c_1^2}{a+a_1} + \frac{2c_2^2}{a+a_2} = 1.$$

We can verify that, under these conditions, a family of spheres (8.1) with  $c_3 = 0$  actually envelopes the cyclide.

*Case IV:*  $\lambda = 2a_1 (\neq 2a_2)$ .

We get from (8.6) that  $c_1 = 0$ . Hence (8.2) reduces to

$$\begin{aligned} 4(c_2x_2 + c_3x_3 + c)^2 - 2a_2x_2^2 - 2a_3x_3^2 + a^2 \\ + 2a_1(x_2^2 + x_3^2 - 2c_2x_2 - 2c_3x_3 - 2c) = 0. \end{aligned} \quad (8.2'')$$

Note that if (8.2) represent two planes, then (8.2'') gives two lines on the  $x_2x_3$ -plane. But it is impossible, because the quadratic part of (8.2'') is positive definite. Therefore this case does not occur.

*Case V:*  $\lambda = 2a_1 = 2a_2$ .

We see that (8.2) reduces to

$$\begin{aligned} 4(c_1x_1 + c_2x_2)^2 + 8c_3(c_1x_1 + c_2x_2)x_3 + 2(2c_3^2 + a_1 - a_3)x_3^2 \\ - 4(a_1 - 2c)(c_1x_1 + c_2x_2) - 4c_3(a_1 - 2c)x_3 + a^2 \\ - 4a_1c + 4c^2 = 0, \end{aligned}$$

which represents two real planes (possibly coincidental) only if  $c_1 = c_2 = 0$ . If this is the case, we get

$$2(2c_3^2 + a_1 - a_3)x_3^2 - 4c_3(a_1 - 2c)x_3 + a^2 - 4a_1c + 4c^2 = 0.$$

This represents coincidental two planes if and only if

$$2c_3^2(a_1^2 - a^2) - (a_1 - a_3)(a^2 - 4a_1c + 4c^2) = 0.$$

We can verify that, under this condition, a family of spheres (8.1) with  $c_1 = c_2 = 0$  actually envelopes the cyclide.

*Case VI:*  $\lambda = 2a_2$  ( $\neq 2a_1$ ).

We get from (8.6) that  $c_2 = 0$ . Hence (8.2) reduces to

$$\begin{aligned} 4(c_1x_1 + c_3x_3 + c)^2 - 2a_1x_1^2 - 2a_3x_3^2 + a^2 \\ + 2a_2(x_1^2 + x_3^2 - 2c_1x_1 - 2c_3x_3 - 2c) = 0. \end{aligned} \quad (8.2''')$$

Note that if (8.2) represents two planes, then (8.2''') gives two lines on the  $x_1x_3$ -plane. We see that the necessary condition for (8.2''') to give two lines on the  $x_1x_3$ -plane is that  $c_1$ ,  $c_3$  and  $c$  satisfy

$$\frac{2c_1^2}{a_1 - a_2} - \frac{2c_3^2}{a_2 - a_3} = 1 - \frac{(a_2 - 2c)^2}{a_2^2 - a^2}.$$



We can verify that, under this condition, a family of spheres (8.1) with  $c_2 = 0$  actually envelopes the cyclide.

Moreover we see that  $p_1, p_2, p_3, p; q_1, q_2, q_3, q$  in (8.3) satisfy

$$\left( \frac{p_1^2}{a_1 - a_2} - \frac{p_3^2}{a_2 - a_3} \right) \left( \frac{q_1^2}{a_1 - a_2} - \frac{q_3^2}{a_2 - a_3} \right) = 4 \left( \frac{pq}{a_2^2 - a^2} \right)^2$$

and  $p_2 = q_2 = 0$ .

*Case VIIa:*  $\lambda = 2a_3 (< -2a)$ .

We get from (8.6) that  $c_3 = 0$ . Hence (8.2) reduces to

$$4(c_1x_1 + c_2x_2 + c)^2 - 2a_1x_1^2 - 2a_2x_2^2 + a^2 + 2a_3(x_1^2 + x_2^2 - 2c_1x_1 - 2c_2x_2 - 2c) = 0. \quad (8.2''''')$$

Note that if (8.2) represents two planes, then (8.2''''') gives two lines on the  $x_1x_2$ -plane. But it is impossible, because the quadratic part of (8.2''''') is positive definite. Therefore this case does not occur.

*Case VIIb:*  $\lambda = 2a_3 (> -2a)$ .

We see that the necessary condition for (8.2''''') to give two lines on the  $x_1x_2$ -plane is that  $c_1, c_2$  and  $c$  satisfy

$$\frac{2c_1^2}{a_1 - a_3} + \frac{2c_2^2}{a_2 - a_3} = 1 - \frac{(a_3 - 2c)^2}{a_3^2 - a^2}.$$

We can verify that, under this condition, a family of spheres (8.1) with  $c_3 = 0$  actually envelopes the cyclide.

Moreover we see that  $p_1, p_2, p_3, p; q_1, q_2, q_3, q$  in (8.3) satisfy

$$\left( \frac{p_1^2}{a_1 - a_3} + \frac{p_2^2}{a_2 - a_3} \right) \left( \frac{q_1^2}{a_1 - a_3} + \frac{q_2^2}{a_2 - a_3} \right) = 4 \left( \frac{pq}{a_3^2 - a^2} \right)^2$$

and  $p_3 = q_3 = 0$ .

Respective cases occur as follows:

- (A) If  $a_1 = a_2 > a > a_3 = -a$ , then cases I, III and V occur.
- (B<sub>1a</sub>) If  $a_1 = a_2 > a > -a > a_3$ , then cases I, IIa and V occur.
- (B<sub>1b</sub>) If  $a_1 = a_2 > a > a_3 > -a$ , then cases I, V and VIIb occur.
- (B<sub>2</sub>) If  $a_1 > a_2 > a > a_3 = -a$ , then cases I, III and VI occur.
- (Ca) If  $a_1 > a_2 > a > -a > a_3$ , then cases I, IIa and VI occur.
- (Cb) If  $a_1 > a_2 > a > a_3 > -a$ , then cases I, VI and VIIb occur.

Summarizing the above consideration, we get the following.

**Proposition 8.1** *A cyclide which is topologically a torus is a surface enveloped by three distinct families of Meusnier spheres determined by circles on the cyclide. Each sphere contains one or two circles on the cyclide and it is tangent to the cyclide along the circle or at two points, respectively.*

**Remark** Let  $\mathfrak{L}$  and  $\mathfrak{M}$  be homotopy classes of a torus generated by a “latitude” and a “meridian”, respectively, so that the fundamental group is generated by  $\mathfrak{L}$  and  $\mathfrak{M}$ . Then we see that the circles corresponding to case V or VI belong to  $\mathfrak{L}$ , the circles corresponding to case IIa, III or VIIIb belong to  $\mathfrak{M}$  and the circles corresponding to case I belong to  $\mathfrak{L} + \mathfrak{M}$  and  $\mathfrak{L} - \mathfrak{M}$ , respectively.

## 9. Hulahoop surfaces: A construction of cyclides of type (A) and (B<sub>1</sub>)

A *hulahoop surface* is defined in [ON1] to be a smooth surface obtained by revolving a circle around a suitable axis. Let  $\gamma(a, b, r)$ ,  $r > 0$ , be a circle on the  $x_1x_2$ -plane defined by  $(x_1 - a)^2 + (x_2 - b)^2 = r^2$  and let  $\gamma(a, b, r, \alpha)$  be the circle obtained by tilting  $\gamma(a, b, r)$  around the diameter parallel to the  $x_1$ -axis by the angle  $\alpha$ ,  $-\frac{\pi}{2} < \alpha \leq \frac{\pi}{2}$ . Let  $H(a, b, r, \alpha)$  be the surface obtained by rotating  $\gamma(a, b, r, \alpha)$  around the  $x_3$ -axis. Then  $H(a, b, r, \alpha)$  is given by

$$\begin{aligned} & (x_1^2 + x_2^2 + x_3^2)^2 - \frac{4b \cos \alpha}{\sin \alpha} (x_1^2 + x_2^2 + x_3^2)x_3 \\ & - 2(a^2 + b^2 + r^2)(x_1^2 + x_2^2) - 2\left(a^2 + b^2 + r^2 - \frac{2a^2 + 2b^2 \cos^2 \alpha}{\sin^2 \alpha}\right)x_3^2 \\ & + \frac{4b \cos \alpha}{\sin \alpha} (a^2 + b^2 + r^2)x_3 + (a^2 + b^2 + r^2)^2 - 4a^2r^2 = 0. \end{aligned} \tag{9.1}$$

It is easily seen that  $H(a, b, r, \alpha)$  is a smooth surface if and only if  $a = b = 0$  and  $\alpha = \frac{\pi}{2}$  or  $a \neq 0$  and  $(a^2 - r^2) \cos^2 \alpha + b^2 \neq 0$ . We see that  $H(0, 0, r, \frac{\pi}{2})$  is a sphere and otherwise  $H(a, b, r, \alpha)$  is topologically a torus. Note that  $H(a, b, r, \alpha)$  is a cyclide corresponding to a complete intersection in  $P_4(R)$  given by

$$\begin{cases} \sum_{i,j=1}^{\infty} a_{ij}v_i v_j = 0 \\ v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_{\infty}^2 = 0, \end{cases} \quad (9.2)$$

where

$$\begin{cases} a_{11} = a_{22} = -2(a^2 + b^2 + r^2) \\ a_{33} = -2 \left( a^2 + b^2 + r^2 - \frac{2a^2 + 2b^2 \cos^2 \alpha}{\sin^2 \alpha} \right) \\ a_{44} = a_{\infty\infty} = (a^2 + b^2 + r^2)^2 - 4a^2 r^2 + 1 \\ a_{34} = a_{43} = -\frac{2b \cos \alpha}{\sin \alpha} (a^2 + b^2 + r^2 + 1) \\ a_{3\infty} = a_{\infty 3} = \frac{2b \cos \alpha}{\sin \alpha} (a^2 + b^2 + r^2 - 1) \\ a_{4\infty} = a_{\infty 4} = -(a^2 + b^2 + r^2)^2 + 4a^2 r^2 + 1 \\ 0 \text{ otherwise.} \end{cases} \quad (9.3)$$

**Proposition 9.1** *A hula hoop surface is conformally equivalent to a cyclide of type (A) or of type (B<sub>1</sub>).*

*Proof.* Put  $A = (a_{ij})$ , where  $a_{ij}$ 's are given by (9.3), and put

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Consider the equation in  $\lambda$ :  $\det(A - \lambda J) = 0$  or equivalently  $\det(JA - \lambda I) = 0$ , where  $I$  denotes the identity matrix. This is equivalent to  $\lambda = -2(a^2 + b^2 + r^2)$  and

$$\begin{aligned} & \lambda^3 + 2 \left( a^2 + b^2 + r^2 - \frac{2a^2 + 2b^2 \cos^2 \alpha}{\sin^2 \alpha} \right) \lambda^2 \\ & - 4 \left\{ (a^2 + b^2 + r^2)^2 + \frac{4b^2 \cos^2 \alpha}{\sin^2 \alpha} (a^2 + b^2 + r^2) - 4a^2 r^2 \right\} \lambda \end{aligned}$$

$$-8 \left\{ (a^2 + b^2 + r^2)^3 - \frac{2a^2 - 2b^2 \cos^2 \alpha}{\sin^2 \alpha} (a^2 + b^2 + r^2)^2 - 4a^2 r^2 (a^2 + b^2 + r^2) + \frac{8a^4 r^2}{\sin^2 \alpha} \right\} = 0. \quad (9.4)$$

We see that (9.4) has three real solutions, say,  $\lambda_3, \lambda_4, \lambda_\infty$ , which are different from  $-2(a^2 + b^2 + r^2)$ . We first consider the case where  $\lambda_3, \lambda_4$  and  $\lambda_\infty$  are distinct. Then, we can find vectors  $e_3, e_4$  and  $e_\infty$  which satisfy  $Ae_i = \lambda_i J e_i$  ( $i = 3, 4, \infty$ ),  ${}^t e_3 J e_3 = 1, {}^t e_4 J e_4 = 1, {}^t e_\infty J e_\infty = -1, {}^t e_i J e_j = 0$  ( $i \neq j$ ). It is easy to see that the quadratic equation (9.2)<sub>1</sub> is transformed to

$$-2(a^2 + b^2 + r^2)(v_1^2 + v_2^2) + \lambda_3 v_3^2 + \lambda_4 v_4^2 - \lambda_\infty v_\infty^2 = 0 \quad (9.5)$$

by a Möbius transformation given by  $(e_1, e_2, e_3, e_4, e_\infty)$ , where  ${}^t e_1 = (1, 0, 0, 0, 0)$  and  ${}^t e_2 = (0, 1, 0, 0, 0)$ . Note that (9.5), together with  $v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_\infty^2 = 0$ , corresponds to a cyclide of the form  $(x_1^2 + x_2^2 + x_3^2)^2 - 2a_1 x_1^2 - 2a_1 x_2^2 - 2a_3 x_3^2 + 1 = 0$ , which is of type (B<sub>1</sub>).

We next consider the case where (9.4) has multiple solutions. Since  $(a^2 - r^2) \cos^2 \alpha + b^2 \neq 0$ , (9.4) has multiple solutions only when  $b = 0$  and  $\sin \alpha = 1$  or  $b = 0$  and  $\sin \alpha = \frac{a}{r}$ . In this case, it is clear that the surface is nothing but an ordinary torus.  $\square$

## 10. A construction of cyclides of type (B<sub>2</sub>)

Let  $\gamma$  be a curve given as an intersection of a sphere  $x_1^2 + x_2^2 + x_3^2 = r^2$  and an elliptic cylinder  $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$  ( $r > a > b$ ). Then  $\gamma$  is represented as

$$\begin{cases} x_1 = a \cos \theta \\ x_2 = b \sin \theta \\ x_3 = \sqrt{r^2 - a^2 \cos^2 \theta - b^2 \sin^2 \theta}. \end{cases}$$

Consider a family of circles  $c(\theta)$  with the following properties:

- (a)  $c(\theta)$  is vertical
- (b) the center of  $c(\theta)$  is on the  $x_1 x_2$ -plane
- (c)  $c(\theta)$  is tangent to  $O\gamma(\theta)$  at  $\gamma(\theta)$ , where  $O$  denotes the origin.

Then  $c(\theta)$  is a circle with center at  $\left( \frac{ar^2 \cos \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta}, \frac{br^2 \sin \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta}, 0 \right)$  and of radius  $r \sqrt{\frac{r^2 - a^2 \cos^2 \theta - b^2 \sin^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}$ . Therefore the surface  $M(a, b, r)$  generated

by the family of circles is represented as

$$\left\{ \begin{array}{l} x_1 = \frac{ar \cos \theta (r + \sqrt{r^2 - a^2 \cos^2 \theta - b^2 \sin^2 \theta} \cos \varphi)}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \\ x_2 = \frac{br \sin \theta (r + \sqrt{r^2 - a^2 \cos^2 \theta - b^2 \sin^2 \theta} \cos \varphi)}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \\ x_3 = \frac{r \sqrt{r^2 - a^2 \cos^2 \theta - b^2 \sin^2 \theta}}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \sin \varphi. \end{array} \right.$$

By eliminating  $\theta$  and  $\varphi$  from these equations, we obtain

$$\begin{aligned} (x_1^2 + x_2^2 + x_3^2)^2 - 2 \frac{r^2(2r^2 - a^2)}{a^2} x_1^2 \\ - 2 \frac{r^2(2r^2 - b^2)}{b^2} x_2^2 + 2r^2 x_3^2 + r^4 = 0 \end{aligned}$$

which is a cyclide of type  $(B_2)$ .

Note that the above construction is different from an enveloping by a family of spheres given as Case III in §8.

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