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Korovkin type approximation theorems on the disk algebra

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Abstract. We investigate BKW-operators on the disk algebra for the test functions $\{1, z, z^2\}$ having forms as $T = (C_\varphi + C_\psi)/2$, $|\varphi| = |\psi| = 1$ on the unit circle. Our studies have some relation to extremal problems on Hardy spaces.

Key words: Korovkin type approximation theorem, disk algebra.

1. Introduction

In 1953, Korovkin [9, 10] proved a well known theorem as follows; if $\{T_n\}_n$ is a sequence of positive operators on $C([0,1])$, the Banach space of real valued continuous functions on $[0,1]$, such that $\|T_n x^j - x^j\|_\infty \to 0$ for $j = 0, 1, 2$ as $n \to \infty$, then $\|T_n f - f\|_\infty \to 0$ for every $f \in C([0,1])$ as $n \to \infty$. Since then, there are many researches on this field from various points of view, see the monograph by Altomare and Campiti [2]. In [16], Wulbert showed that in Korovkin's theorem, the condition of positivity of $\{T_n\}_n$ is replaced by the condition that $\|T_n\| \leq 1$ for every n, see also [1].

Let X be a separable complex Banach space and S be a subset of X. In [14], Takahasi introduced a concept of BKW-operators to generalize Korovkin's approximation theorem. A bounded linear operator T on X is called a BKW-operator for the test functions S if $\{T_n\}_n$ is a sequence of bounded linear operators on X satisfying

i) $\|T_n\| \leq \|T\|$ for every n

and

ii) $\|T_n h - Th\| \to 0$ as $n \to \infty$ for each $h \in S$,

then it holds $\|T_n f - Tf\| \to 0$ for every $f \in X$ as $n \to \infty$. And in [15], Takahasi gave a sufficient conditions on an operator on X to be a BKW-operator. To state this, let $\tilde{S}$ be the closed linear span of S in X. We denote by $U_S = U_S(X)$ the set of $\varphi \in X^*$, the dual space of X, which satisfies that $\|\varphi\| = \|\varphi|_{\tilde{S}}\| = 1$ and $\varphi|_{\tilde{S}}$ has a unique Hahn-Banach extension to X. The

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set U_S is called the uniqueness set for the test functions S. Takahasi proved that a bounded linear operator T on X with $\|T\| = 1$ is a BKW-operator if there exists a weak*-compact subset Y of the closed unit ball of X^* such that $\|f\| = \sup\{|\varphi(f)|; \varphi \in Y\}$ for every $f \in X$ and $T^*\varphi \in U_S$ for every $\varphi \in Y$.

Let Ω be a metrizable compact Hausdorff space and $C(\Omega)$ be the Banach space of complex valued continuous functions on Ω with the supremum norm. In [15], Takahasi showed that if $1 \in S \subset C(\Omega)$ and T is a bounded linear operator on $C(\Omega)$ with $\|T\| = 1$, then T is a BKW-operator for S if and only if $T^*\delta_\zeta \in U_S(C(\Omega))$ for every $\zeta \in \Omega$, where δ_ζ is a point evaluation at ζ. A closed subalgebra A of $C(\Omega)$ is called a function algebra if A contains constant functions and separates the points in Ω, see [4]. We denote by ∂A the Shilov boundary of A, the smallest closed subset of Ω on which every function in A attains the maximum modulus. Let $S \subset A$. By the Hahn-Banach extension theorem and the Riesz representation theorem, we may consider that $U_S(A)$ is a set of Borel measures on ∂A with total variation 1. In [6], the second author, Takagi and Watanabe showed the following theorem.

Theorem A *Suppose that $1 \in S \subset A$ and T is a bounded linear operator on A with $\|T\| = 1$. Then T is a BKW-operator for S if and only if $T^*\delta_\zeta \in U_S(A)$ for every $\zeta \in \partial A$.*

A typical example of function algebras is the disk algebra $A(\Gamma)$. Let Γ be the unit circle in the complex plane, and let $A(\Gamma)$ be the space of complex valued continuous functions on Γ which can be extended analytically in the open unit disk $D = \{|z| < 1\}$. Then $A(\Gamma)$ is a closed subalgebra of $C(\Gamma)$. It is known that $\partial A(\Gamma) = \Gamma$. For $\{z_j\}_{j=1}^n \subset D$, let

$$b(z) = \lambda \prod_{j=1}^{n} \frac{-\bar{z}_j}{|z_j|} \frac{z - z_j}{1 - \bar{z}_j z}, \quad z \in \overline{D},$$

where λ is a constant with $|\lambda| = 1$. This type of functions $b(z)$ are called finite Blaschke products, and satisfy $|b| = 1$ on Γ. As a special case, a constant function with absolute modulus 1 is also called a finite Blaschke product. If $f \in A(\Gamma)$ and $|f| = 1$ on Γ, then f is a finite Blaschke product (see [5]). For a function φ (may not continuous) on Γ with $|\varphi| = 1$ on Γ, we put $C_\varphi f = f \circ \varphi$ for $f \in A(\Gamma)$. Then C_φ is a bounded linear operator on $A(\Gamma)$ if and only if $\varphi \in A(\Gamma)$. In [6], it is proved that if T is a bounded linear operator on $A(\Gamma)$ with $\|T\| = 1$, then T is a BKW-operator for $\{1, z\}$ if and only if $T = \psi C_\varphi$, where ψ and φ are finite Blaschke products. In [14], Takahasi proved that $a\delta_{\zeta_1} + (1-a)\delta_{\zeta_2} \in U_{\{1,z,z^2\}}(A(\Gamma))$ for $\zeta_1, \zeta_2 \in \Gamma$, $0 \le a \le 1$, and

$$T = aC_{\varphi_1} + (1-a)C_{\varphi_2}, \quad 0 \le a \le 1,$$
$$\varphi_1 \text{ and } \varphi_2 \text{ are finite Blaschke products}$$

is a BKW-operator on $A(\Gamma)$ for $\{1, z, z^2\}$. In [6], it is pointed out that the converse of the above assertion is not true. Let $S_n = \{1, z, z^2, \ldots, z^n\}$. It seems difficult to describe all BKW-operators on $A(\Gamma)$ for the test functions S_n. See [7] for the polydisk and ball algebras.

In this paper, we study BKW-operators on $A(\Gamma)$. In Section 2, we determine measures μ in $U_{S_n}(A(\Gamma))$ such that $\mu \ge 0$. In Section 3, we study BKW-operators T on $A(\Gamma)$ for $\{1, z, z^2\}$ having a special form as follows;

$$T = (C_\varphi + C_\psi)/2, \quad |\varphi| = |\psi| = 1 \text{ on } \Gamma.$$

We give a characterization of a BKW-operator satisfying the above condition.

2. Positive measures in $U_{S_n}(A(\Gamma))$

We denote by $M(\Gamma)$ the set of bounded complex Borel measures on Γ and by $M_{+,1}(\Gamma)$ the set of $\mu \in M(\Gamma)$ with $\mu \ge 0$ and $\|\mu\| = 1$. Let T be a bounded linear operator on $A(\Gamma)$ with $\|T\| = 1$ and $T1 = 1$. Then for each $\zeta \in \Gamma$, we may consider that $T^*\delta_\zeta$ is a bounded Borel measure on Γ and $T^*\delta_\zeta \in M_{+,1}(\Gamma)$. In this section, we study when this operator T is a BKW-operator for the test functions $S_n = \{1, z, \ldots, z^n\}$, see Corollary 2.1. By Theorem A, we need to describe the set $U_{S_n} \cap M_{+,1}(\Gamma)$. In [14], Takahasi proved that

$$\left\{ \sum_{j=1}^n a_j \delta_{\zeta_j};\ \zeta_j \in \Gamma,\ a_j \ge 0,\ \sum_{j=1}^n a_j = 1 \right\} \subset U_{S_n} \cap M_{+,1}(\Gamma).$$

We shall prove that the both sets in the above coincide.

Theorem 2.1 $U_{S_n}(A(\Gamma)) \cap M_{+,1}(\Gamma) = \{\sum_{j=1}^n a_j \delta_{\zeta_j};\ \zeta_j \in \Gamma,\ a_j \ge 0,$

$\sum_{j=1}^{n} a_j = 1\}$.

Proof. Let

$$\mu = \sum_{j=1}^{n} a_j \delta_{\zeta_j}, \quad \zeta_j \in \Gamma, \quad a_j \geq 0, \quad \sum_{j=1}^{n} a_j = 1,$$
$$\text{and} \quad \zeta_i \neq \zeta_j \quad \text{if} \quad i \neq j. \tag{2.1}$$

In [14], Takahasi proved that $\mu \in U_{S_n}$. Here we give a simple proof. Let $\nu \in M(\Gamma)$ with $\|\nu\| = 1$ such that

$$\int_\Gamma z^k d\nu = \int_\Gamma z^k d\mu \quad \text{for} \quad k = 0, 1, 2, \ldots, n. \tag{2.2}$$

Then $\nu \in M_{+,1}(\Gamma)$. To prove $\mu \in U_{S_n}$, it is sufficient to show $\nu = \mu$. Put

$$p(z) = \prod_{j=1}^{n} |z - \zeta_j|^2, \quad z \in \Gamma. \tag{2.3}$$

Then we can write $p(z)$ as

$$p(z) = \left(\sum_{j=0}^{n} \alpha_j z^j\right) + \overline{\left(\sum_{j=0}^{n} \alpha_j z^j\right)}, \quad z \in \Gamma. \tag{2.4}$$

Since μ and ν are real measures, by (2.2) we have

$$\int_\Gamma \overline{z}^j d\nu = \int_\Gamma \overline{z}^j d\mu \quad \text{for} \quad j = 0, 1, \ldots, n.$$

Hence by (2.4), $\int_\Gamma p(z) d\nu = \int_\Gamma p(z) d\mu = 0$. Since ν is a probability measure, by (2.3) ν has a form as

$$\nu = \sum_{j=1}^{n} b_j \delta_{\zeta_j}, \quad b_j \geq 0 \quad \text{and} \quad \sum_{j=1}^{n} b_j = 1.$$

By (2.1) and (2.2),

$$\sum_{j=1}^{n} (a_j - b_j) \zeta_j^k = 0 \quad \text{for} \quad k = 1, 2, \ldots, n.$$

Since points $\{\zeta_j\}_{j=1}^n$ are distinct, we have

$$\begin{vmatrix} \zeta_1 & \zeta_2 & \cdots & \zeta_n \\ \zeta_1^2 & \zeta_2^2 & \cdots & \zeta_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ \zeta_1^n & \zeta_2^n & \cdots & \zeta_n^n \end{vmatrix} \neq 0.$$

Therefore we have $a_j = b_j$ for $j = 1, 2, \ldots, n$. Thus we obtain $\nu = \mu$.

Next, we prove the converse inclusion. We use the same idea of the proof of Theorem 1 in [8]. Let $\mu \in U_{S_n} \cap M_{+,1}(\Gamma)$. We shall prove that μ has a form in (2.1). Put

$$\rho(\sigma) = \left(\int_\Gamma z d\sigma, \int_\Gamma z^2 d\sigma, \ldots, \int_\Gamma z^n d\sigma \right) \quad \text{for } \sigma \in M_{+,1}(\Gamma)$$

and

$$\Omega = \{\rho(\sigma); \sigma \in M_{+,1}(\Gamma)\}.$$

Then ρ is a continuous map from $M_{+,1}(\Gamma)$ with the weak*-topology into $\mathbf{C^n}$. Since $M_{+,1}(\Gamma)$ is a weak*-compact convex set, Ω is a compact convex subset of $\mathbf{C^n}$.

Claim 1. $\operatorname{int}\Omega \neq \emptyset$, where $\operatorname{int}\Omega$ denotes the interior of Ω.

To prove this, suppose not. Then there exist complex numbers $c_j, 1 \leq j \leq n$, such that $(c_1, c_2, \ldots, c_n) \neq (0, 0, \ldots, 0)$ and

$$\operatorname{Re} \sum_{j=1}^n c_j \int_\Gamma z^j d\sigma = 0 \quad \text{for every } \sigma \in M_{+,1}(\Gamma).$$

This implies that

$$\int_\Gamma \left(\operatorname{Re} \sum_{j=1}^n c_j z^j \right) d\lambda = 0 \quad \text{for every } \lambda \in M(\Gamma).$$

Since $\operatorname{Re} \sum_{j=1}^n c_j z^j \neq 0$ on Γ, this is absurd.

Claim 2. $\rho(\mu)$ is a boundary point of Ω.

To prove this, suppose that $\rho(\mu) \in \operatorname{int}\Omega$. We shall prove that

$$\rho(\mu) = \rho(\nu) \quad \text{for some } \nu \in M_{+,1}(\Gamma) \text{ with } \nu \neq \mu. \tag{2.5}$$

We note that $M_{+,1}(\Gamma)$ coincides with the weak*-closed convex hull of $\{\delta_\zeta; \zeta \in \Gamma\}$. Since $\{\rho(\delta_\zeta); \zeta \in \Gamma\}$ is a compact subset of $\mathbf{C^n}$, its convex hull coincides with its closed convex hull. Hence

$$\Omega = \text{the convex hull of } \{\rho(\delta_\zeta); \zeta \in \Gamma\},$$

so that

$$\rho(\mu) = \rho\left(\sum_{j=1}^{k} c_j \delta_{\zeta_j}\right) \quad \text{for some} \quad \zeta_j \in \Gamma,\ c_j \geq 0, \ \text{and} \ \sum_{j=1}^{k} c_j = 1. \tag{2.6}$$

Let $L_{+,1} = \{\nu \in M_{+,1}(\Gamma); d\nu \ll d\theta/2\pi\}$. Then $L_{+,1}$ is also a weak*-dense convex subset of $M_{+,1}(\Gamma)$, so that $\{\rho(\nu); \nu \in L_{+,1}\}$ is a dense convex subset of Ω. Since $\Omega \subset \mathbf{C^n}$, we have that

$$\operatorname{int}\Omega \subset \{\rho(\nu); \nu \in L_{+,1}\}.$$

Since $\rho(\mu) \in \operatorname{int}\Omega$, there exists $\nu \in L_{+,1}$ such that $\rho(\mu) = \rho(\nu)$. Hence by (2.6),

$$\rho(\mu) = \rho\left(\sum_{j=1}^{k} c_j \delta_{\zeta_j}\right) = \rho(\nu), \quad \sum_{j=1}^{k} c_j \delta_{\zeta_j} \neq \nu.$$

Thus we get (2.5).

Since μ and ν are distinct probability measures, we have $\int_\Gamma f d\mu \neq \int_\Gamma f d\nu$ for some $f \in A(\Gamma)$. This means that $\mu \notin U_{S_n}$. This is a contradiction, so we get Claim 2.

Claim 3. $\mu = \sum_{j=1}^{n} a_j \delta_{\zeta_j}$ for $\zeta_j \in \Gamma$, $a_j \geq 0$ and $\sum_{j=1}^{n} a_j = 1$.

By Claims 1 and 2, there exist complex numbers $\{d_j\}_{j=0}^{n}$ such that

$$d_j \neq 0 \quad \text{for some} \quad j,\ 1 \leq j \leq n, \tag{2.7}$$

$$\operatorname{Re}\left(d_0 + \sum_{j=1}^{n} d_j \int_\Gamma z^j d\mu\right) = 0, \tag{2.8}$$

and

$$\operatorname{Re}\left(d_0 + \sum_{j=1}^{n} d_j \int_\Gamma z^j d\sigma\right) \geq 0 \quad \text{for every} \quad \sigma \in M_{+,1}(\Gamma). \tag{2.9}$$

By (2.9),

$$\int_\Gamma \mathrm{Re}\Big(d_0 + \sum_{j=1}^{n} d_j z^j\Big) d\sigma \geq 0 \quad \text{for every} \quad \sigma \in M_{+,1}(\Gamma),$$

so that we have

$$\mathrm{Re}\Big(d_0 + \sum_{j=1}^{n} d_j z^j\Big) \geq 0 \quad \text{on} \quad \Gamma. \tag{2.10}$$

Moreover by (2.7),

$$\mathrm{Re}\Big(d_0 + \sum_{j=1}^{n} d_j z^j\Big) \not\equiv 0 \quad \text{on} \quad \Gamma. \tag{2.11}$$

Putting $z = e^{i\theta}$, we can write as

$$\mathrm{Re}\Big(d_0 + \sum_{j=1}^{n} d_j z^j\Big) = \sum\{a_{k,l} \sin^k \theta \cos^l \theta;\ 0 \leq k + l \leq n,\ k, l \geq 0\}$$

for some real numbers $\{a_{k,l}\}_{k,l}$. Put

$$F(\theta) = \sum\{a_{k,l} \sin^k \theta \cos^l \theta; 0 \leq k + l \leq n, k, l \geq 0\}, \quad 0 \leq \theta < 2\pi.$$

Then by (2.10) and (2.11), $F(\theta) \geq 0$ and $F(\theta) \not\equiv 0$, $0 \leq \theta < 2\pi$. By (2.8), $\int_0^{2\pi} F(\theta) d\mu(e^{i\theta}) = 0$. Hence to prove Claim 3, we need to prove that the number of zeros of the function $F(\theta)$, $0 \leq \theta < 2\pi$, is less than or equal to n. Since $F(\theta)$ is a 2π-periodic function, to prove this it is sufficient to show that the number of distinct zeros of $F'(\theta)$, $0 \leq \theta < 2\pi$, is less than or equal to $2n + 1$. Here we can write $F'(\theta)$ as

$$F'(\theta) = \sum_{0<k+l\leq n} b_{k,l} \sin^k \theta \cos^l \theta.$$

Put $t = \tan \frac{\theta}{2}$, $\theta \neq \pi$. Then $\sin\theta = 2t/(1+t^2)$ and $\cos\theta = (1-t^2)/(1+t^2)$. Hence the equation $F'(\theta) = 0$ becomes

$$\sum_{0<k+l\leq n} b_{k,l} \Big(\frac{2t}{1+t^2}\Big)^k \Big(\frac{1-t^2}{1+t^2}\Big)^l = 0.$$

This equation has a number of distinct zeros up to $2n$. Since $\tan \frac{\theta}{2}$ is one to one on $[0, \pi] \cup (\pi, 2\pi)$, the number of distinct zeros of $F'(\theta)$, $0 \leq \theta < 2\pi$, is less than or equal to $2n+1$. This completes the proof. □

As an application of Theorems A and 2.1, we have the following.

Corollary 2.1 *Let T be a bounded operator on $A(\Gamma)$ such that $\|T\| = 1$ and $T1 = 1$. Then T is a BKW-operator for the test functions S_n if and only if T has a following form;*

$$(Tf)(\zeta) = \sum_{j=1}^{n} a_j(\zeta)(C_{\varphi_j} f)(\zeta), \quad \text{for } \zeta \in \Gamma \quad \text{and} \quad f \in A(\Gamma),$$

where $|\varphi_j| = 1$ on Γ, $a_j(\zeta) \geq 0$ for every j and $\sum_{j=1}^{n} a_j(\zeta) = 1$ for $\zeta \in \Gamma$.

For given functions $\{\varphi_j(\zeta)\}$ and $\{a_j(\zeta)\}$ on Γ satisfying that $|\varphi_j| = 1$ on Γ, $a_j(\zeta) \geq 0$, and $\sum_{j=1}^{n} a_j(\zeta) = 1$ for $\zeta \in \Gamma$, we can defined T as $(Tf)(\zeta) = \sum_{j=1}^{n} a_j(\zeta)(C_{\varphi_j} f)(\zeta)$ for $\zeta \in \Gamma$ and $f \in A(\Gamma)$. Generally, $Tf \notin A(\Gamma)$ for some $f \in A(\Gamma)$, so that T may not be a bounded linear operator on $A(\Gamma)$. If $Tf \in A(\Gamma)$ for $f \in A(\Gamma)$, then T is a bounded linear operator on $A(\Gamma)$. Hence by Corollary 2.1, T becomes a BKW-operator on $A(\Gamma)$ for S_n. We have a question when $Tf \in A(\Gamma)$ for $f \in A(\Gamma)$. It seems difficult to answer. In the next section, we study on this problem when $n = 2$.

3. BKW-operators for $\{1, z, z^2\}$

Let T be a BKW-operator on $A(\Gamma)$ for the test functions $S_2 = \{1, z, z^2\}$ such that $\|T\| = 1$ and $T1 = 1$. Then by Corollary 2.1, T has a form as

$$\begin{aligned}(Tf)(\zeta) &= a(\zeta)(C_\varphi f)(\zeta) + b(\zeta)(C_\psi f)(\zeta), \\ &\qquad \text{for } \zeta \in \Gamma \quad \text{and} \quad f \in A(\Gamma), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}|\varphi(\zeta)| &= |\psi(\zeta)| = 1, \quad a(\zeta) + b(\zeta) = 1, \quad a(\zeta), b(\zeta) \geq 0 \\ &\qquad \text{for every } \zeta \in \Gamma. \end{aligned} \tag{3.2}$$

We note that a, b, φ, and ψ may not be continuous on Γ, see [6].

Suppose that a, b, φ, and ψ are functions on Γ satisfying (3.2), and define T by (3.1). As mentioned in the end of the last section, we have a question when $Tf \in A(\Gamma)$ for every $f \in A(\Gamma)$.

We have another question. For a function $h \in A(\Gamma)$ with $\|h\|_\infty \leq 1$, when there exists a BKW-operator T on $A(\Gamma)$ for the test functions S_2

such that $\|T\| = 1$, $T1 = 1$, and $Tz = h$.

In this section, we study BKW-operators T on $A(\Gamma)$ for $\{1, z, z^2\}$ having a following form;

$$(\#) \qquad T = (C_\varphi + C_\psi)/2, \quad |\varphi| = 1 \text{ and } |\psi| = 1 \text{ on } \Gamma.$$

In this case, $T1 = 1$ and $\|T\| = 1$.

The following lemma follows the definition of BKW-operators.

Lemma 3.1 *Let $1 \in S \subset A(\Gamma)$. Let T be a BKW-operator on $A(\Gamma)$ for S with $\|T\| = 1$. Let T_1 be a bounded linear operator on $A(\Gamma)$ with $\|T_1\| \leq 1$. If $Th = T_1 h$ for $h \in S$, then $T = T_1$.*

For a function $h \in A(\Gamma)$, $h \neq 0$, we can define $(h/\overline{h})(\zeta) = h(\zeta)/\overline{h(\zeta)}$ for almost every $\zeta \in \Gamma$. When $h/\overline{h}$ can be extended continuously on Γ, we consider that $h/\overline{h}$ is an extended function.

Theorem 3.1 *Let T be a bounded linear operator on $A(\Gamma)$ with $\|T\| = 1$ and $T1 = 1$. Put $Tz = h$ and $Tz^2 = g$. Then we have the following.*

i) *If $h \neq 0$, then T is a BKW-operator for $\{1, z, z^2\}$ satisfying (#) if and only if $h/\overline{h}$ is a finite Blaschke product and $h/\overline{h} = 2h^2 - g$. In this case, we have*

$$\varphi = h + \sqrt{g - h^2} \quad \textit{and} \quad \psi = h - \sqrt{g - h^2},$$

where $\sqrt{g - h^2}$ is one of root functions of $g - h^2$.

ii) *If $h = 0$, then T is a BKW-operator for $\{1, z, z^2\}$ satisfying (#) if and only if g is a finite Blaschke product. In this case, $\varphi = \sqrt{g}$ and $\psi = -\sqrt{g}$.*

Proof. First, we note that $h, g \in A$, $\|h\|_\infty \leq 1$, and $\|g\|_\infty \leq 1$. Suppose that T has a form (#). Then $\varphi + \psi = 2h$ and $\varphi^2 + \psi^2 = 2g$. Since $(\varphi + \psi)^2 = \varphi^2 + \psi^2 + 2\varphi\psi$,

$$2h^2 - g = \varphi\psi. \tag{3.3}$$

Since $h, g \in A$, $\varphi\psi \in A$. Since $|\varphi\psi| = 1$ on Γ, $\varphi\psi$ is a finite Blaschke product. When $h \neq 0$, $h/\overline{h} = (\varphi + \psi)/(\overline{\varphi} + \overline{\psi}) = \varphi\psi = 2h^2 - g$ by (3.3). When $h = 0$, $g = -\varphi\psi$ and g is a finite Blaschke product.

Next, we prove the converse. Suppose that $h \neq 0$. Put

$$b = h/\overline{h} = 2h^2 - g. \tag{3.4}$$

Then by our assumption, b is a finite Blaschke product. Since

$$b = h^2/|h|^2, \tag{3.5}$$

by (3.4) we have

$$g - h^2 = h^2 - b = (-b)(1 - |h|^2). \tag{3.6}$$

Take one root function $\sqrt{g-h^2}$, and we put

$$\varphi = h + \sqrt{g-h^2} \quad \text{and} \quad \psi = h - \sqrt{g-h^2}. \tag{3.7}$$

Then

$$(\varphi + \psi)/2 = h \in A(\Gamma) \quad \text{and} \quad \varphi\psi = 2h^2 - g \in A(\Gamma). \tag{3.8}$$

By (3.6),

$$|h|^2 + \left|\sqrt{g-h^2}\right|^2 = 1. \tag{3.9}$$

Let $\zeta \in \Gamma$. If $h(\zeta) = 0$, then by (3.9) $|(\sqrt{g-h^2})(\zeta)| = 1$, so that $|\varphi(\zeta)| = |\psi(\zeta)| = 1$. If $h(\zeta) \neq 0$, then by (3.5) and (3.6)

$$(\sqrt{g-h^2})(\zeta) = ih(\zeta)\sqrt{1-|h(\zeta)|^2}/|h(\zeta)|$$
$$\text{or} \quad (\sqrt{g-h^2})(\zeta) = -ih(\zeta)\sqrt{1-|h(\zeta)|^2}/|h(\zeta)|.$$

Therefore by (3.7), we have

$$\begin{aligned} |\varphi(\zeta)| = \left|h(\zeta) + \left(\sqrt{g-h^2}\right)(\zeta)\right| &= \left|h(\zeta) \pm \frac{ih(\zeta)}{|h(\zeta)|}\sqrt{1-|h(\zeta)|^2}\right| \\ &= \left||h(\zeta)| \pm i\sqrt{1-|h(\zeta)|^2}\right| = 1 \end{aligned}$$

and similarly $|\psi(\zeta)| = 1$ for every $\zeta \in \Gamma$. Hence

$$|\varphi| = |\psi| = 1 \quad \text{on} \quad \Gamma. \tag{3.10}$$

Put

$$T_0 f = \frac{1}{2}(C_\varphi + C_\psi)f \quad \text{for} \quad f \in A(\Gamma). \tag{3.11}$$

Then by (3.7),

$$T_0 1 = 1, \quad T_0 z = h, \quad \text{and} \quad T_0 z^2 = g. \tag{3.12}$$

Since

$$\varphi^n + \psi^n = (\varphi^{n-1} + \psi^{n-1})(\varphi + \psi) - \varphi\psi(\varphi^{n-2} + \psi^{n-2}),$$

by (3.8) and by induction we have

$$T_0 z^n \in A(\Gamma) \text{ for every non-negative integer } n. \tag{3.13}$$

Let $f \in A$. Then there exists a sequence of analytic polynomials $\{p_k\}_k$ such that $\|f - p_k\|_\infty \to 0$ as $k \to \infty$. Then by (3.10) and (3.11), $\|T_0 f - T_0 p_k\|_\infty \to 0$ as $k \to \infty$. By (3.13), $T_0 p_k \in A(\Gamma)$, so that we have $T_0 f \in A(\Gamma)$ for every $f \in A(\Gamma)$. As a consequence, T_0 is a bounded linear operator on $A(\Gamma)$ with $\|T_0\| = 1$ and $T_0 1 = 1$. By Corollary 2.1, T_0 is a BKW-operator on $A(\Gamma)$ for $\{1, z, z^2\}$. By (3.12), $T_0 z^j = T z^j$ for $j = 0, 1, 2$. Hence by Lemma 3.1, we have $T = T_0$.

Suppose that $h = 0$ and g is a finite Blaschke product. Put

$$T_1 f = \frac{1}{2}(C_{\sqrt{g}} + C_{-\sqrt{g}})f \quad \text{for } f \in A.$$

Then $T_1 1 = 1$, $T_1 z^{2n-1} = 0$, and $T_1 z^{2n} = g^n$ for $n \geq 1$. In the same way as above, we can prove that $T_1 = T$ and T is a BKW-operator for $\{1, z, z^2\}$. □

Remark 3.1. Let $h \in A$ such that $h/\overline{h}$ is a finite Blaschke product. Then h is a rational function and all such h are described in [3]. For finite Blaschke products b_1 and b_2, put $h = b_1 + b_2$. Then $h/\overline{h} = b_1 b_2$ is a finite Blaschke product. Sum of inner functions are studied in [11, 12, 13]. These functions h are deeply concerned with extremal problems.

The converse of the proof of Theorem 3.1 proves the following actually.

Corollary 3.1 *Suppose that $h, g \in A(\Gamma)$ satisfy the following conditions;*
i) $\|h\|_\infty \leq 1$ *and* $\|g\|_\infty \leq 1$,
ii) $h/\overline{h}$ *is a finite Blaschke product (or* $h = 0$),
iii) $h/\overline{h} = 2h^2 - g$ *(when* $h = 0$, g *is a finite Blaschke product).*
Then there exists a unique bounded linear operator T on $A(\Gamma)$ such that $\|T\| = 1$, $T1 = 1$, $Tz = h$, and $Tz^2 = g$. Moreover T is a BKW-operator on $A(\Gamma)$ for $\{1, z, z^2\}$ having a form (#).

Corollary 3.2 *Let $h \in A$ such that $0 < \|h\|_\infty \leq 1$ and $h/\overline{h}$ is a finite Blaschke product. Then there exists a BKW-operator T on $A(\Gamma)$ for*

$\{1, z, z^2\}$ such that $\|T\| = 1$, $T1 = 1$ and $Tz = h$. In such BKW-operators T, there is a unique operator satisfying (#).

Proof. Put $g = 2h^2 - (h/\overline{h})$. Then $g \in A(\Gamma)$. Since $2h^2 - (h/\overline{h}) = h^2(2|h|^2 - 1)/|h|^2$, $\|g\|_\infty \leq 1$. By Corollary 3.1, we have the first part of our assertion. Let T_1 and T_2 be BKW-operators for $\{1, z, z^2\}$ satisfying (#) such that $\|T_i\| = 1$, $T_i1 = 1$, and $T_iz = h$ for $j = 1, 2$. Then by Theorem 3.1, $T_1z^2 = 2h^2 - h/\overline{h} = T_2z^2$. Hence by Lemma 3.1, $T_1 = T_2$. □

In Corollary 3.2, a BKW-operator T with $\|T\| = 1, T1 = 1$ and $Tz = h$ is not unique generally.

Example 3.1. Let $h = z/2$. Then $h/\overline{h} = z^2$. Then by Takahasi's theorem [14],

$$T = (C_\varphi + C_\psi)/2, \quad \varphi = e^{\frac{\pi}{3}i}z, \quad \psi = e^{-\frac{\pi}{3}i}z$$

is a BKW-operator for $\{1, z, z^2\}$ satisfying $T1 = 1$, $Tz = z/2$, and $Tz^2 = -z^2/2$. Also

$$T_0 = (C_{-z} + 3C_z)/4.$$

is a BKW-operator for $\{1, z, z^2\}$ satisfying $T_01 = 1$, $T_0z = z/2$, and $T_0z^2 = z^2$.

By Theorem 3.1 ii), we have the following.

Corollary 3.3 *There are uncountable many BKW-operators T on $A(\Gamma)$ for $\{1, z, z^2\}$ satisfying $\|T\| = 1$, $T1 = 1$, and $Tz = 0$.*

Next, we study that for BKW-operators T satisfying (#), when both φ and ψ are continuous or analytic.

Corollary 3.4 *Let $h \in A(\Gamma)$ such that $0 < \|h\| \leq 1$ and $h/\overline{h}$ is a finite Blaschke product. Let T be a BKW-operators on $A(\Gamma)$ for $\{1, z, z^2\}$ such that $Tz = h$ and $T = (C_\varphi + C_\psi)/2$, $|\varphi| = |\psi| = 1$ on Γ. Then we have the following.*

i) *If number of zeros of $h/\overline{h}$ in D, counting multiplicities, is even or $h^2 - h/\overline{h}$ vanishes at some points in Γ, then φ and ψ are continuous on Γ.*

ii) *If $h^2 - h/\overline{h} = f^2$ for some $f \in A(\Gamma)$, then φ and ψ are finite Blaschke products.*

Proof. Put $Tz^2 = g$. Then by Theorem 3.1, we have

$$2h^2 - g = h/\overline{h} \tag{3.14}$$

and

$$\varphi = h + \sqrt{g - h^2} \quad \text{and} \quad \psi = h - \sqrt{g - h^2}. \tag{3.15}$$

By (3.14),

$$g - h^2 = h^2 - h/\overline{h} = (-h/\overline{h})(1 - |h|^2). \tag{3.16}$$

We note that the usual root function $\sqrt{1 - |h|^2}$ is continuous on Γ.

i) Suppose that number of zeros of $h/\overline{h}$ in D is even. Then we can take such as $\sqrt{-h/\overline{h}}$ is continuous on Γ. Hence by (3.15) and (3.16), φ and ψ are continuous.

Suppose that $h^2 - h/\overline{h}$ vanishes at some points in Γ. Then $1 - |h|^2$ vanishes at these points. Hence we can take $\sqrt{g - h^2}$ such as

$$\sqrt{g - h^2} = \sqrt{-h(1 - |h|^2)/\overline{h}} \text{ is continuous on } \Gamma.$$

ii) Suppose that $h^2 - h/\overline{h} = f^2$ for some $f \in A(\Gamma)$. Then by (3.16), we can take such as $\sqrt{g - h^2} = f$. Hence $\varphi, \psi \in A(\Gamma)$, so that both of these functions are finite Blaschke products. □

Remark 3.2. Let $0 < r < 1$. If T_0 is a bounded linear operator on $A(\Gamma)$ such that

$$(\#_r) \qquad T_0 = rC_\varphi + (1 - r)C_\psi, \quad |\varphi| = |\psi| = 1 \text{ on } \Gamma,$$

then by Takahasi's theorem [14], T_0 is a BKW-operator for $\{1, z, z^2\}$ and $T1 = 1$. For a bounded linear operator T on $A(\Gamma)$ such that $\|T\| = 1$ and $T1 = 1$, put $Tz = h$ and $Tz^2 = g$. We do not known conditions on h and g for which T has a form $(\#_r)$.

Remark 3.3. Let T_1 be a bounded linear operator on $A(\Gamma)$ such that

$$(\#_-) \qquad T_1 = (C_\varphi - C_\psi)/2, \quad |\varphi| = |\psi| = 1 \text{ on } \Gamma.$$

Then T_1 may not be a BKW-operator for $\{1, z, z^2\}$. For, let $T_2 = (C_z - C_{-z})/2$ and

$$Pf = \left(\int_0^{2\pi} f(e^{i\theta})e^{-i\theta}d\theta/2\pi\right) z \quad \text{for } f \in A(\Gamma).$$

Then T_2 and P are bounded linear operators on $A(\Gamma)$ satisfying that $T_2 \neq P$, $\|T_2\| = \|P\| = 1$, $T_2 1 = P1 = 0$, $T_2 z = Pz = z$, and $T_2 z^2 = Pz^2 = 0$. Then by Lemma 3.1, T_2 is not a BKW-operator for $\{1, z, z^2\}$.

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