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W^*-quantum groups arising from matched pairs of groups

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Abstract. Generalizing the operator algebras defined by Masuda-Nakagami, we introduce a notion of a quasi Woronowicz algebra as a von Neumann algebra version of quantum groups. It is shown that every matched pair of locally compact groups gives rise to two quasi Woronowicz algebras dual to each other.

Key words: von Neumann algebras, quasi Woronowicz algebras, matched pairs of groups.

Introduction

In [M] (see [M1] too), Majid studied the notion of a matched pair of locally compact groups, and showed, among other things, that every matched pair gives rise to two Hopf-von Neumann algebras, called the bicrossproduct Hopf-von Neumann algebras. In general, this bicorssproduct construction produces noncommutative and noncocommutative algebras. Thus it furnishes abundant examples of nontrivial "W^*-quantum groups." If a matched pair is modular in the sense of [M], the associated bicrossproduct algebras turn out to be Kac algebras [ES]. But, if not, which is often the case when the groups in question are continuous, no one ever examined, to the best of author's knowledge, what part in the category of coinvolutive Hopf-von Neumann algebras the bicrossproduct algebras occupy. The purpose of this paper is to try to answer this question. To be more precise, we shall show that the bicrossproduct Hopf-von Neumann algebra associated with a matched pair of groups is a quasi Woronowicz algebra, which is closely related to the object investigated in [MN]. According to this result, we find a concrete example of a deformation automorphism on a Woronowicz algebra which is not induced from the q-deformation of the quantum groups.

The plan of the paper is as follows. Section 1 is concerned with the notation which will be used in the sections that follow. We also introduce the notion of a quasi Woronowicz algebra. The relation to Woronowicz algebras are briefly discussed. The definition of a matched pair of groups is

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also reviewed. Section 2 is devoted to establishing preliminary results that will be applied in Section 3. In Section 3, we show the main theorem that the bicrossproduct Hopf-von Neumann algebra arising from a matched pair is always a quasi Woronowicz algebra. Finally, in Appendix, we deduce some formula for a Radon Nikodym derivative with respect to the Haar measure associated with a bicrossproduct quasi Woronowicz algebra.

1. Notation

In this section, we first give the definition of a quasi Woronowicz algebra. Then we state fundamental facts on this algebra, introducing the notation that will be used in our later discussion. Quasi Woronowicz algebras are almost like Woronowicz algebras introduced in [MN]. It is not too much to say that what is true for Woronowicz algebras is equally true for quasi Woronowicz algebras. Thus, for the general theory of quasi Woronowicz algebras, we may refer readers to [MN]. Our notation will be mainly adopted from this literature. We also recall the definition of a matched pair of groups. For the details of this notion, we refer readers to [M], [M1]. See [LW] also.

Given a von Neumann algebra $\mathcal{M}$ and a faithful normal semifinite weight ψ on $\mathcal{M}$, we introduce subsets $\mathfrak{n}_\psi$, $\mathfrak{m}_\psi$ and $\mathfrak{m}_\psi^+$ of $\mathcal{M}$ by

$$\mathfrak{n}_\psi = \{x \in \mathcal{M} : \psi(x^*x) < \infty\}, \quad \mathfrak{m}_\psi = \mathfrak{n}_\psi^* \mathfrak{n}_\psi, \quad \mathfrak{m}_\psi^+ = \mathfrak{m}_\psi \cap \mathcal{M}_+.$$

We denote by π_ψ the standard (GNS) representation associated with ψ. Its representation space is denoted by $\mathfrak{H}_\psi$. We use the symbol Λ_ψ for the canonical embedding of $\mathfrak{n}_\psi$ into $\mathfrak{H}_\psi$. Let $\mathfrak{a}_\psi = \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*$ and set $\mathfrak{A}_\psi = \Lambda_\psi(\mathfrak{a}_\psi)$, which is the full left Hilbert algebra associated with ψ. For a left bounded vector $\xi \in \mathfrak{H}$ with respect to the left Hilbert algebra $\mathfrak{A}_\psi$, we write $\pi_\ell(\xi)$ for the left multiplication operator corresponding to ξ. For a right bounded vector η, we use $\pi_r(\eta)$ for the corresponding right multiplication operator. The modular automorphism group of ψ is denoted by σ^ψ.

A *coinvolutive Hopf-von Neumann algebra* is a triple $(\mathcal{M}, \delta, R)$ in which:

(1) $\mathcal{M}$ is a von Neumann algebra;

(2) δ is an injective normal $*$-homomorphism, called a *coproduct* (or a *comultiplication*), from $\mathcal{M}$ into $\mathcal{M} \bar{\otimes} \mathcal{M}$ with the coassociativity condition: $(\delta \otimes id_{\mathcal{M}}) \circ \delta = (id_{\mathcal{M}} \otimes \delta) \circ \delta$;

(3) R is a $*$-antiautomorphism of $\mathcal{M}$, called a *coinvolution* or a *unitary antipode*, such that $R^2 = id_{\mathcal{M}}$ and $\sigma \circ (R \otimes R) \circ \delta = \delta \circ R$, where σ is

the usual flip.

A *quasi Woronowicz algebra* is a family $\mathbb{W} = (\mathcal{M}, \delta, R, \tau, h)$ in which:

(1) $(\mathcal{M}, \delta, R)$ is a coinvolutive Hopf-von Neumann algebra;

(2) τ is a continuous one-parameter automorphism group of $\mathcal{M}$, called the *deformation automorphism*, which commutes with the coproduct δ and the antipode R;

(3) h is a τ-invariant faithful normal semifinite weight on $\mathcal{M}$, called the *Haar measure* of $\mathbb{W}$, satisfying the following conditions:

(a) *Quasi left invariance*: For any ϕ in $\mathcal{M}_*^+$, we have $(\phi \otimes h) \circ \delta(x) = h(x)\phi(1)$ for all $x \in \mathfrak{m}_h^+$;

(b) *Strong left invariance*: For any $x, y \in \mathfrak{n}_h$ and $\phi \in \mathcal{M}_*$ which is analytic with respect to the adjoint action of the deformation automorphism τ on $\mathcal{M}_*$, the following equality holds:

$$(\phi \otimes h)((1 \otimes y^*)\delta(x)) = (\phi \circ \tau_{-i/2} \circ R \otimes h)(\delta(y^*)(1 \otimes x)).$$

(c) *Commutativity*: $h \circ \sigma_t^{h \circ R} = h$ for all $t \in \mathbf{R}$ (or, equivalently, $h \circ R \circ \sigma_t^h = h \circ R$).

Remark that only difference between a Woronowicz algebra and a quasi Woronowicz algebra is the requirement that the weight h is left invariant or quasi left invariant. In other words, in the definition of a Woronowicz algebra, one requires that h should satisfy $(\phi \otimes h) \circ \delta(x) = h(x)\phi(1)$ for all $\phi \in \mathcal{M}_*^+$ and *all* $x \in \mathcal{M}_+$. At the present stage, the author does not know whether left invariance and quasi left invariance are distinct notions, although it is clear that left invariance implies quasi left invariance. Let us briefly expalin the reason why we work with quasi Woronowicz algebras rather than with Woronowicz algebras in this note. In the paper [MN], there is a crucial gap at the end of the proof of Proposition 3.8. Because of this gap, we do <u>not</u> yet know that the dual Woronowicz algebra in the sense of [MN] is really a Woronowicz algebra. One can, however, easily see that the dual <u>*is*</u> a quasi Woronowicz algebra. Moreover, most of the argument in [MN] goes through perfectly without any change even if we start with a quasi Woronowicz algebra, not with a Woronowicz algebra. In particular, the duality for quasi Woronowicz algebras holds true. (There are some points in which we really have to be careful, but those points are irrelevant to our discussion that follows). This is why we would like to insist on working with quasi Woronowicz algebras. Besides, as we see in Section 3, every matched pair of (locally compact) groups gives rise to a quasi Woronowicz algebra. Hence there are plenty of examples of quasi Woronowicz algebras. It is worth remarking that, if a quasi Woronowicz algebra $\mathbb{W}$ satisfies $\tau_t = id$ and $\sigma^{h\circ R} = \sigma^h$, not only that $h \circ \sigma^{h\circ R} = h$, then one can show that $\mathbb{W}$ is actually a Kac algebra [ES].

Throughout this section, we fix a quasi Woronowicz algebra $\mathbb{W} = (\mathcal{M}, \delta, R, \tau, h)$. Identifying $\mathcal{M}$ with $\pi_h(\mathcal{M})$, we always think of $\mathcal{M}$ as represented on the Hilbert space $\mathfrak{H} := \mathfrak{H}_h$. We denote by Δ and J the modular operator and the modular conjugation of h, respectively. Since h is τ-invariant, $\Lambda_h(x) \mapsto \Lambda_h(\tau_t(x))$ $(x \in \mathfrak{n}_h,\ t \in \mathbf{R})$ defines a one-parameter unitary group on $\mathfrak{H}$. We write H for the analytic generator of this one-parameter unitary group: $H^{it}\Lambda_h(x) := \Lambda_h(\tau_t(x))$. An element $\phi \in \mathcal{M}_*$ is said to be $L^2(h)$-bounded if

$$\sup\{|\phi(x^*)| : \ h(x^*x) \leq 1\} < \infty.$$

We denote by $\hat{\eta}(\phi)$ the unique vector in $\mathfrak{H}$ such that $\phi(x^*) = (\hat{\eta}(\phi) \mid \Lambda_h(x))$ for $x \in \mathfrak{n}_h$. For ϕ, $\psi \in \mathcal{M}_*$, define an element $\phi * \psi$ in $\mathcal{M}_*$ by

$$(\phi * \psi)(x) := (\phi \otimes \psi)(\delta(x)) \qquad (x \in \mathcal{M}).$$

This operation $*$ turns $\mathcal{M}_*$ into a Banach algebra. Let $(\mathcal{M}_*)^\infty_\tau$ be the set of analytic elements in $\mathcal{M}_*$ with respect to the action $\phi \mapsto \phi \circ \tau_t$ of the deformation automorphism on $\mathcal{M}_*$. For $\phi \in (\mathcal{M}_*)^\infty_\tau$, put $\phi^\sharp := \phi^* \circ \tau_{-i/2} \circ R$. This defines an involution on the subalgebra $(\mathcal{M}_*)^\infty_\tau$. Thanks to quasi left invariance, the equation

$$W\Lambda_{h\otimes h}(x \otimes y) = \Lambda_{h\otimes h}(\delta(y)(x \otimes 1)) \qquad (x, y \in \mathfrak{n}_h)$$

defines an isometry (in fact, a unitary) on $\mathfrak{H} \otimes \mathfrak{H}$. This unitary W is called the *Kac-Takesaki operator* of $\mathbb{W}$ and satisfies

$$W_{12}W_{23} = W_{23}W_{13}W_{12}, \quad \delta(x) = W(1 \otimes x)W^* \quad (x \in \mathcal{M}).$$

With W, the equation

$$\hat{\pi}(\phi) := (\phi \otimes id)(W^*) \qquad (\phi \in \mathcal{M}_*)$$

defines a homomorphism (resp. $*$-homomorphism) of $\mathcal{M}_*$ (resp. $(\mathcal{M}_*)^\infty_\tau$) into the set $\mathcal{B}(\mathfrak{H})$ of all bounded operators on $\mathfrak{H}$. The mapping $\hat{\pi}$ is called the *Fourier representation* of $\mathbb{W}$. Let $\widehat{\mathcal{M}}$ stand for the von Neumann algebra generated by $\hat{\pi}(\phi)$ $(\phi \in \mathcal{M}_*)$. By [BS, Proposition 3.5], $\widehat{\mathcal{M}}$ is the σ-strong* closure of the subalgebra $\hat{\pi}(\mathcal{M}_*)$ (or the $*$-subalgebra $\hat{\pi}((\mathcal{M}_*)_\tau^\infty)$). It is possible to equip $\widehat{\mathcal{M}}$ with a quasi Woronowicz algebra structure as follows:

$$\begin{aligned} &\text{coproduct}: && \hat{\delta}(y) := \widehat{W}(1 \otimes y)\widehat{W}^* \qquad (y \in \widehat{\mathcal{M}}) \\ &\text{unitary antipode}: && \hat{R}(y) := Jy^*J \\ &\text{deformation automorphism}: && \hat{\tau}_t := \operatorname{Ad} H^{it} \\ &\text{Haar measure}: && \hat{h}(x) := \begin{cases} \|\xi\|^2, & \text{if } x^{1/2} = \hat{\pi}_\ell(\xi) \text{ for } \xi \in \hat{\mathfrak{A}}'', \\ \infty, & \text{otherwise,} \end{cases} \end{aligned}$$

where $\widehat{W} = \Sigma W^* \Sigma$ and Σ is the flip on $\mathfrak{H} \otimes \mathfrak{H}$. $\hat{\mathfrak{A}}$ is a left Hilbert algebra obtained as the image of some suitable $*$-subalgebra in $(\mathcal{M}_*)_\tau^\infty$ under the map $\hat{\eta}$. In particular, we have

$$\hat{h}(\hat{\pi}(\omega)^*\hat{\pi}(\phi)) = (\hat{\eta}(\phi) \mid \hat{\eta}(\omega))$$

for $L^2(h)$-bounded functionals ϕ, ω. We denote this quasi Woronowicz algebra by $\widehat{\mathbb{W}}$ and call it the quasi Woronowicz algebra dual to $\mathbb{W}$. The Kac-Takesaki operator of $\widehat{\mathbb{W}}$ is $\widehat{W}$. The linear mapping $\mathcal{F}$ defined by

$$\mathcal{F}\Lambda_{\hat{h}}(\hat{\pi}(\phi)) := \hat{\eta}(\phi) \qquad (\phi : L^2(h)\text{-bounded})$$

extends to a unitary, still denoted by $\mathcal{F}$, from $\mathfrak{H}_{\hat{h}}$ onto $\mathfrak{H}$. We call this unitary the *Fourier transform.* Note that $\{\widehat{\mathcal{M}}, \mathfrak{H}\}$ is a standard representation. Thus we regard $\hat{\Delta} := \Delta_{\hat{h}}$ and $\hat{J} := J_{\hat{h}}$ as acting on the Hilbert space $\mathfrak{H}$. We have $R(x) = \hat{J}x^*\hat{J}$ $(x \in \mathcal{M})$.

In general, if ϕ and ψ are faithful normal semifinite weights on a von Neumann algebra $\mathcal{P}$ satisfying $\psi \circ \sigma_t^\phi = \psi$ (or, equivalently, $\phi \circ \sigma_t^\psi = \phi$), then, by [PT], there exists a unique non-singular positive self-adjoint operator K affiliated with the centralizer $\mathcal{P}_\phi = \{x \in \mathcal{P} : \sigma_t^\phi(x) = x \ (t \in \mathbf{R})\}$ of ϕ such that the Connes' Radon Nikodym derivative $(D\psi : D\phi)_t$ satisfies $(D\psi : D\phi)_t = K^{it}$ for $t \in \mathbf{R}$. For any $\varepsilon > 0$, set $K_\varepsilon := K(1 + \varepsilon K)^{-1}$. With this notation, it follows from [PT, Theorem 5.12] that we have

$$\psi(x) = \lim_{\varepsilon \downarrow 0} \phi(K_\varepsilon^{1/2} x K_\varepsilon^{1/2}). \qquad (x \in \mathcal{P}_+)$$

In this case, following the notation in [PT], we write $\psi = \phi(K\,\cdot)$. Finally, for a linear operator T on a Hilbert space, let $\mathfrak{D}(T)$ designate the domain of T.

Let G_1, G_2 be locally compact groups with left Haar measures μ_1, μ_2. The identities of G_1 and G_2 are both denoted by the letter e. By G_1 acting on the topological space G_2, we mean a continuous map $\alpha : G_1 \times G_2 \longrightarrow G_2$ such that

$$\alpha_g(\alpha_h(s)) = \alpha_{gh}(s), \qquad \alpha_e(s) = s \qquad (g, h \in G_1,\ s \in G_2)$$

together with a regularity condition: we assume that, for each $g \in G_1$, the measure $\mu_2 \circ \alpha_g$ is equivalent to μ_2 in the sense of absolute continuity, and that, with the Radon Nikodym derivative $d\mu_2 \circ \alpha_g / d\mu_2$, the map

$$(g, s) \in G_1 \times G_2 \mapsto \frac{d\mu_2 \circ \alpha_g}{d\mu_2}(s)$$

is jointly continuous.

Definition A system $(G_1, G_2, \alpha, \beta)$ is called a *matched pair* if:
(1) G_1, G_2 are locally compact groups;
(2) α is an action of G_1 on the topological space G_2, and β is an action of G_2 on the topological space G_1;
(3) the following identities (*the matched pair condition*) holds:

$$\text{(MP)} \quad \begin{cases} \alpha_g(e) = e, \qquad \beta_s(e) = e \\ \alpha_g(st) = \alpha_{\beta_t(g)}(s)\alpha_g(t), \quad \beta_s(gh) = \beta_{\alpha_h(s)}(g)\beta_s(h) \\ \qquad\qquad\qquad\qquad\qquad\qquad (g, h \in G_1,\ s, t \in G_2) \end{cases}$$

Throughout this note, we fix a matched pair $(G_1, G_2, \alpha, \beta)$ with left Haar measures μ_i for G_i $(i = 1, 2)$. We denote by $\chi(g, s)$ (resp. $\Psi(s, g)$) the Radon Nikodym derivative $d\mu_2 \circ \alpha_g / d\mu_2(s)$ (resp. $d\mu_1 \circ \beta_s / d\mu_1(g)$). Besides these cocycles, we introduce the following continuous "bicocycle" $\zeta : G_1 \times G_2 \longrightarrow (0, \infty)$:

$$\zeta(g, s) := \frac{\chi(g, s)}{\chi(g, e)} = \frac{\Psi(s, g)}{\Psi(s, e)} \qquad (g \in G_1,\ s \in G_2).$$

The important properties which the functions χ, Ψ, ζ enjoy are listed in [M, Lemma 2.2]. Examples of matched pairs are discussed in [M], [M1], [LW], [BS]. The action α induces an action, denoted by α again, of G_1 on the abelian von Neumann algebra $L^\infty(G_2)$ by $\alpha_g(k) := k \circ \alpha_g^{-1}$ $(k \in L^\infty(G_2))$. Similarly, β induces an action β of G_2 on $L^\infty(G_1)$. The crossed products $L^\infty(G_2) \rtimes_\alpha G_1$ and $L^\infty(G_1) \rtimes_\beta G_2$ are called the *bicrossproduct Hopf-von*

Neumann algebras associated with the matched pair $(G_1, G_2, \alpha, \beta)$. The coinvolutive Hopf-von Neumann algebra structure of $\mathcal{M} = L^\infty(G_2) \rtimes_\alpha G_1$ is described as follows (see [M]). Let $\mathfrak{H} = L^2(G_1) \otimes L^2(G_2) = L^2(G_1 \times G_2)$. Then define an operator W on $\mathfrak{H} \otimes \mathfrak{H}$ by

$$\{W\xi\}(g, s; h, t) := \xi(\beta_t(h)^{-1}g, s; h, \alpha_{\beta_t(h)^{-1}g}(s)t) \qquad (\xi \in \mathfrak{H} \otimes \mathfrak{H},\ g, h \in G_1,\ s, t \in G_2).$$

It turns out that W is a unitary, and that the map δ given by

$$\delta(x) := W(1 \otimes x)W^* \qquad (x \in \mathcal{M})$$

defines a coproduct on $\mathcal{M}$. Moreover, if we define an operator $\hat{J}$ on $\mathfrak{H}$ by

$$\{\hat{J}\xi\}(g, s) := \Delta(s)^{-1/2}\Psi(s, g)^{1/2}\overline{\xi(\beta_s(g), s^{-1})} \qquad (\xi \in \mathfrak{H},\ g \in G_1,\ s \in G_2),$$

where Δ is the modular function on G_2, then $\hat{J}$ is a unitary involution, and the map R defined by

$$R(x) := \hat{J}x^*\hat{J} \qquad (x \in \mathcal{M})$$

is shown to be a coinvolution (i.e., a unitary antipode) on $\mathcal{M}$.

2. Technical results

This section is devoted to establishing preliminariy results that will be applied in Section 3, where we shall show that the dual weight on a bicrossproduct Hopf-von Neumann algebra always satisfies quasi left invariance, strong left invariance and commutativity. It is, however, usually difficult to show that a weight satisfies the equality in the definition of strong left invariance, as it stands. Thus our goal of this section is to establish a condition, equivalent to strong left invariance, which fits our purpose.

Let us assume for the time being that

(1) $(\mathcal{M}, \delta, R)$ is a coinvolutive Hopf-von Neumann algebra;
(2) τ is a one-parameter automorphism group on $\mathcal{M}$ satisfying condition (2) in the definition of a quasi Woronowicz algebra;
(3) h is a faithful normal semifinite weight on $\mathcal{M}$ satisfying $h \circ \tau_t = h$ for any $t \in \mathbf{R}$.

For a one-papameter automorphism group $\{\alpha_t\}_{t\in\mathbf{R}}$ on $\mathcal{M}$, we denote by $\mathcal{M}_\alpha^\infty$ the set of analytic elements in $\mathcal{M}$ with respect to $\{\alpha_t\}_{t\in\mathbf{R}}$, i.e., the set of all elements $x \in \mathcal{M}$ for which the function $t \in \mathbf{R} \mapsto \alpha_t(x)$ can be extended to an entire function on $\mathbf{C}$. With this notation, we define $\mathcal{S}_\tau$ to be the set of elements $x \in \mathfrak{a}_h \cap \mathcal{M}_\tau^\infty$ with $\tau_z(x) \in \mathfrak{a}_h$ for any $z \in \mathbf{C}$.

As in Section 1, we introduce a nonsingular positive self-adjoint operator H on $\mathfrak{H}$ which is the analytic generator of the (strongly continuous) one-parameter unitary group $\{U(t)\}$ given by

$$U(t)\Lambda_h(x) = \Lambda_h(\tau_t(x)) \qquad (t \in \mathbf{R},\ x \in \mathfrak{n}_h).$$

Thus $U(t) = H^{it}$ for all $t \in \mathbf{R}$.

The next lemma can be easily shown, so we leave its proof to readers as an exercise.

Lemma 2.1 *Let x be in $\mathcal{S}_\tau$. Then the vector $\Lambda_h(x)$ belongs to $\mathfrak{D}(H^{iz})$ for any $z \in \mathbf{C}$, and we have*

$$H^{iz}\Lambda_h(x) = \Lambda_h(\tau_z(x)).$$

Lemma 2.2 *The subspace $\Lambda_h(\mathcal{S}_\tau)$ is dense in $\mathfrak{H}$. Moreover, $\mathcal{S}_\tau$ is σ-weakly dense in $\mathcal{M}$.*

Proof. Let x be in $\mathfrak{a}_h$. We set

$$x_n = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \tau_t(x)dt, \qquad (n = 1, 2, \ldots).$$

Since the $\{\tau_t\}$-invariance of the weight h, $\tau_t(x)$ still lies in $\mathfrak{a}_h$. Thus, if $\xi' \in \mathfrak{A}'_h$ and $\eta \in \mathfrak{H}$, then we have

$$\begin{aligned}(x_n\xi' \mid \eta) &= \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} (\tau_t(x)\xi' \mid \eta)dt \\ &= (\pi_r(\xi')\Big(\sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} H^{it}\Lambda_h(x)dt\Big) \mid \eta),\end{aligned}$$

where π_r indicates the right multiplication of the right Hilbert algebra $\mathfrak{A}'_h$. Hence we obtain

$$\pi_r(\xi')\Big(\sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} H^{it}\Lambda_h(x)dt\Big) = x_n\xi'.$$

This shows that the vector $\sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} H^{it}\Lambda_h(x)dt$ is left bounded, and that

$$\pi_\ell\Big(\sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} e^{-nt^2} H^{it}\Lambda_h(x)dt\Big) = x_n.$$

It follows that $x_n \in \mathfrak{n}_h$. Similarly, one can show that $x_n^* \in \mathfrak{n}_h$. Consequently, x_n belongs to $\mathfrak{a}_h$.

In the meantime, it is known in general that x_n is in $\mathcal{M}_\tau^\infty$, and that

$$\tau_z(x_n) = \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} e^{-n(t-z)^2}\tau_t(x)dt, \qquad (z \in \mathbf{C}).$$

So, we may argue as in the previous paragraph in order to conclude that $\tau_z(x_n)$ belongs to $\mathfrak{a}_h$. Therefore, x_n lies in $\mathcal{S}_\tau$. Since

$$\lim_{n\to\infty} \|\Lambda_h(x_n) - \Lambda_h(x)\| = 0, \qquad \sigma\text{-weak-}\lim_{n\to\infty} x_n = x$$

by the Lebesgue dominated convergence theorem, it follows that $\mathfrak{A}_h \subseteq \overline{\Lambda_h(\mathcal{S}_\tau)}$ and $\mathfrak{a}_h \subseteq \overline{\mathcal{S}_\tau}^{\sigma\text{-weak}}$. This completes the proof. □

Next we set

$$\mathcal{S}_{\sigma,\tau} = \{x \in \mathfrak{a}_h \cap \mathcal{M}_\sigma^\infty \cap \mathcal{M}_\tau^\infty : \ \sigma_z^h(x),\ \tau_z(x) \in \mathfrak{a}_h\ (z \in \mathbf{C})\}.$$

With this notation, we have

Lemma 2.3 *The subspace $\Lambda_h(\mathcal{S}_{\sigma,\tau})$ is dense in $\mathfrak{H}$. In addition, $\mathcal{S}_{\sigma,\tau}$ is σ-weakly dense in $\mathcal{M}$.*

Proof. Let $x \in \mathcal{S}_\tau$, and put

$$x_n = \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} e^{-nt^2}\sigma_t^h(x)dt, \qquad (n \geq 1).$$

Then we have $x_n \in \mathfrak{a}_h \cap \mathcal{M}_\sigma^\infty$. Fix an $n \in \mathbf{N}$ for the moment. Since h is $\{\tau_t\}$-invariant, we find that

$$\tau_s(x_n) = \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} e^{-nt^2}\sigma_t^h(\tau_s(x))dt, \qquad (s \in \mathbf{R}).$$

Hence, if we define a function f on $\mathbf{C}$ by

$$f(z) = \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} e^{-nt^2}\sigma_t^h(\tau_z(x))dt, \qquad (z \in \mathbf{C}),$$

then $f(s) = \tau_s(x_n)$ for any $s \in \mathbf{R}$. Moeover, since $x \in \mathcal{S}_\tau$, i.e., $\tau_z(x) \in \mathfrak{a}_h$, $f(z)$ belongs to $\mathfrak{a}_h \cap \mathcal{M}_\sigma^\infty$. Since $f(z)$ is clearly an entire function, x_n is in $\mathcal{M}_\tau^\infty$, and we have $\tau_z(x_n) = f(z)$ for $z \in \mathbf{C}$. This proves that x_n lies in $\mathfrak{a}_h \cap \mathcal{M}_\sigma^\infty \cap \mathcal{M}_\tau^\infty$. Furthermore, since both of the following elements

$$\sigma_z^h(x_n) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-n(t-z)^2} \sigma_t^h(x) dt$$

and

$$\tau_z(x_n) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \sigma_t^h(\tau_z(x)) dt \tag{2.3.1}$$

belong to $\mathfrak{a}_h$ for any $z \in \mathbf{C}$, it follows that $x_n \in \mathcal{S}_{\sigma,\tau}$. As in the preceding lemma, since

$$\lim_{n\to\infty} \|\Lambda_h(x_n) - \Lambda_h(x)\| = 0, \qquad \sigma\text{-weak-}\lim_{n\to\infty} x_n = x,$$

the assertion now follows from Lemma 2.2. □

The lemma that follows can be found in [VD, Lemma 4.2]. Van Daele proved this lemma in the case where the operator K is the modular operator associated with a left Hilbert algebra. But, as he remarked at the beginning of Section 4 of [VD], the lemma still holds true for any nonsingular positive self-adjoint operator.

Lemma 2.4 *Let K be a nonsingular positive self-adjoint operator on $\mathfrak{H}$. If $r > 0$, then one has*

$$K^{-1/2}(K^{-1} + r)^{-1} = \int_{-\infty}^{\infty} \frac{r^{it-1/2}}{e^{\pi t} + e^{-\pi t}} K^{it} dt$$

in the strong-operator topology.

Lemma 2.5 *Let $s \in \mathbf{R}$. The subspace $\Lambda_h(\mathcal{S}_{\sigma,\tau})$ is a core for the operator H^s.*

Proof. Let $s \in \mathbf{R}$. Take any $x \in \mathcal{S}_\tau$. As we have shown in the proof of Lemma 2.3, the element

$$x_n = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \sigma_t^h(x) dt, \qquad (n \geq 1)$$

belongs to $\mathcal{S}_{\sigma,\tau}$.

Meanwhile, if we apply Lemma 2.4 to the operator $K = H^{-s}$ and $r = 1$, then we obtain

$$H^{s/2}(H^s + 1)^{-1} = \int_{-\infty}^{\infty} \frac{1}{e^{\pi t} + e^{-\pi t}} H^{-its} dt.$$

By Lemma 2.1, the vector $\Lambda_h(x_n)$ is in the domain of H^z for any $z \in \mathbf{C}$, in particular, of $H^{-s/2}(H^s + 1)$, so that we have

$$\begin{aligned} \Lambda_h(x_n) &= H^{s/2}(H^s+1)^{-1} \cdot H^{-s/2}(H^s+1)\Lambda_h(x_n) \\ &= \int_{-\infty}^{\infty} \frac{1}{e^{\pi t} + e^{-\pi t}} H^{-its}(H^{-s/2}(H^s+1)\Lambda_h(x_n))dt \\ &= \int_{-\infty}^{\infty} \frac{1}{e^{\pi t} + e^{-\pi t}} (1+H^s)H^{-its-s/2}\Lambda_h(x_n)dt \qquad (2.5.1) \end{aligned}$$

By Lemma 2.1 again, we have $H^{-its-s/2}\Lambda_h(x_n) = \Lambda_h(\tau_{-ts+(s/2)i}(x_n))$. From the definition of $\mathcal{S}_{\sigma,\tau}$ and Equation (2.3.1), it follows that $\tau_{-ts+(s/2)i}(x_n)$ still belongs to $\mathcal{S}_{\sigma,\tau}$. Hence, from the identity above, we have $H^{-its-s/2}\Lambda_h(x_n) \in \Lambda_h(\mathcal{S}_{\sigma,\tau})$. Consequently, the vector $(1 + H^s)H^{-its-s/2}\Lambda_h(x_n)$ is in the subspace $(1 + H^s)\Lambda_h(\mathcal{S}_{\sigma,\tau})$ for any $t \in \mathbf{R}$. From this, together with (2.5.1), it follows that $\Lambda_h(x_n)$ lies in the closure of $(1 + H^s)\Lambda_h(\mathcal{S}_{\sigma,\tau})$. Since $\Lambda_h(x_n)$ converges to $\Lambda_h(x)$, $\Lambda_h(x)$ still belongs to this closure. It results from Lemma 2.2 that $(1 + H^s)\Lambda_h(\mathcal{S}_{\sigma,\tau})$ is dense in $\mathfrak{H}$. Therefore, by [T, Lemma 1.1], the subspace $\Lambda_h(\mathcal{S}_{\sigma,\tau})$ is a core for the operator H^s. □

From this point on, we assume further that the weight h is quasi left invariant. With this assumption, one may prove the following lemma exactly the same way as in [S, Lemme II.12].

Lemma 2.6 *Let $(\mathcal{M}, \delta, R, \tau, h)$ be as above. Suppose that $x, y \in \mathfrak{n}_h$. Then, for any $\omega \in \mathcal{M}_*$, both $(1 \otimes y^*)\delta(x)$ and $\delta(y^*)(1 \otimes x)$ belong to $\mathfrak{m}_{\omega \otimes h}$, and one has*

(i) $\langle W, \omega \otimes \omega_{\Lambda_h(x),\Lambda_h(y)} \rangle = (\omega \otimes h)((1 \otimes y^*)\delta(x))$;

(ii) $\langle W^*, \omega \otimes \omega_{\Lambda_h(x),\Lambda_h(y)} \rangle = (\omega \otimes h)(\delta(y^*)(1 \otimes x))$,

where W of course stands for the Kac-Takesaki operator determined by the system $(\mathcal{M}, \delta, R, h)$.

As an easy consequence of the previous lemma, we obtain the next.

Lemma 2.7 *Let $(\mathcal{M}, \delta, R, \tau, h)$ be as above. Then the weight h is strong-*

ly left invariant if and only if it satisfies

$$(\phi\circ\tau_{-i/2}\circ R\otimes id)(W^*)=(\phi\otimes id)(W) \tag{2.7.1}$$

for any $\phi\in(\mathcal{M}_*)^\infty_\tau$.

Proof. Let x, $y\in\mathfrak{n}_h$ and $\phi\in(\mathcal{M}_*)^\infty_\tau$. Then, with the aid of Lemma 2.6, it is readily verified that

$$\begin{aligned}&\langle(\phi\circ\tau_{-i/2}\circ R\otimes id)(W^*),\,\omega_{\Lambda_h(x),\Lambda_h(y)}\rangle\\&\qquad=(\phi\circ\tau_{-i/2}\circ R\otimes h)(\delta(y^*)(1\otimes x))\\&\langle(\phi\otimes id)(W),\,\omega_{\Lambda_h(x),\Lambda_h(y)}\rangle\\&\qquad=(\phi\otimes h)((1\otimes y^*)\delta(x)).\end{aligned}$$

From these identities, it easily follows that the strong left invariance of h is equivalent to the equality:

$$\begin{aligned}&\langle(\phi\circ\tau_{-i/2}\circ R\otimes id)(W^*),\,\omega_{\Lambda_h(x),\Lambda_h(y)}\rangle\\&\qquad=\langle(\phi\otimes id)(W),\,\omega_{\Lambda_h(x),\Lambda_h(y)}\rangle,\quad(x,y\in\mathfrak{n}_h,\ \phi\in(\mathcal{M}_*)^\infty_\tau).\end{aligned}$$

The assertion now follows, since $\Lambda_h(\mathfrak{n}_h)$ is dense in $\mathfrak{H}$. □

Lemma 2.8 *Suppose that h is strongly left invariant. If V is a conjugate-linear isometric involution on $\mathfrak{H}$ implementing the unitary antipode R, then we have*

$$(VH^{1/2}\zeta_1\otimes\xi\mid W(VH^{-1/2}\zeta_2\otimes\eta))=(W(\zeta_2\otimes\xi)\mid\zeta_1\otimes\eta)$$

for any $\zeta_1\in\mathfrak{D}(H^{1/2})$, $\zeta_2\in\mathfrak{D}(H^{-1/2})$ and $\xi,\eta\in\mathfrak{H}$.

Proof. With the previous notation, let x, $y\in\mathcal{S}_{\sigma,\tau}$. Then put $\phi=\omega_{\Lambda_h(x),\Lambda_h(y)}$. Note that, with the notation of [MN, Definition 2.1], we have $\hat\eta(\phi)=\Lambda_h(x\sigma^h_{-i}(y^*))$ (see also the paragraph preceding Lemma 2.2 of [MN]).

We first claim that ϕ is in $(\mathcal{M}_*)^\infty_\tau$, and that $\phi\circ\tau_{-i/2}=\omega_{\Lambda_h(\tau_{i/2}(x)),\Lambda_h(\tau_{-i/2}(y))}$. In fact, suppose that $a\in\mathcal{S}_{\sigma,\tau}$. Then the function $t\in\mathbf{R}\mapsto\phi(\tau_t(a))$ certainly has an extension to an entire function $\phi(\tau_z(a))$ on $\mathbf{C}$. Meanwhile, by the definition of ϕ and Lemma 2.1, we have

$$\begin{aligned}\phi(\tau_z(a))&=(\hat\eta(\phi)\mid\Lambda_h(\tau_{\bar z}(a^*)))=(\Lambda_h(x\sigma^h_{-i}(y^*))\mid\Lambda_h(\tau_{\bar z}(a^*)))\\&=(\Lambda_h(\sigma^h_{-i}(y^*))\mid x^*H^{i\bar z}\Lambda_h(a^*))\\&=(\Lambda_h(\sigma^h_{-i}(y^*))\mid H^{i\bar z}\tau_{-\bar z}(x^*)\Lambda_h(a^*))\end{aligned}$$

$$
\begin{aligned}
&= (\Lambda_h(\sigma^h_{-i}(y^*)) \mid H^{i\bar{z}}\tau_{-\bar{z}}(x^*)\Delta^{-1/2}J\Lambda_h(a)) \\
&= (\Lambda_h(\sigma^h_{-i}(y^*)) \mid H^{i\bar{z}}\Delta^{-1/2}J \cdot J\sigma^h_{-i/2}(\tau_{-\bar{z}}(x^*))J\Lambda_h(a)) \\
&= (H^{-iz}\Lambda_h(\sigma^h_{-i}(y^*)) \mid \Delta^{-1/2}J\Lambda_h(a\tau_{-z}(x))) \\
&= (\Lambda_h(\tau_{-z} \circ \sigma^h_{-i}(y^*)) \mid \Delta^{-1/2}Ja\Lambda_h(\tau_{-z}(x))) \\
&= (\Delta^{-1/2}\Lambda_h(\sigma^h_{-i} \circ \tau_{-z}(y^*)) \mid Ja\Lambda_h(\tau_{-z}(x))) \\
&= (\Delta^{1/2}\Lambda_h(\tau_{-z}(y^*)) \mid Ja\Lambda_h(\tau_{-z}(x))) \\
&= (a\Lambda_h(\tau_{-z}(x)) \mid J\Delta^{1/2}\Lambda_h(\tau_{-z}(y^*))) \\
&= (a\Lambda_h(\tau_{-z}(x)) \mid \Lambda_h(\tau_{-\bar{z}}(y))).
\end{aligned}
$$

Since this computation is valid for any $a \in \mathcal{S}_{\sigma,\tau}$, it follows from Lemma 2.3 that ϕ belongs to $(\mathcal{M}_*)^\infty_\tau$, and that $\phi \circ \tau_z = \omega_{\Lambda_h(\tau_{-z}(x)),\Lambda_h(\tau_{-\bar{z}}(y))}$. This proves the claim.

As we saw in the proof of Lemma 2.7, the strong left invariance of h ensures that we have

$$\langle W^*, \phi \circ \tau_{-i/2} \circ R \otimes \omega_{\xi,\eta}\rangle = \langle W, \phi \otimes \omega_{\xi,\eta}\rangle, \qquad (\xi, \eta \in \mathfrak{H}).$$

From this, together with the preceding paragraph, we obtain

$$\langle W^*, \omega_{\Lambda_h(\tau_{i/2}(x)),\Lambda_h(\tau_{-i/2}(y))} \circ R \otimes \omega_{\xi,\eta}\rangle = \langle W, \omega_{\Lambda_h(x),\Lambda_h(y)} \otimes \omega_{\xi,\eta}\rangle.$$

Since $\omega_{\Lambda_h(\tau_{i/2}(x)),\Lambda_h(\tau_{-i/2}(y))} \circ R = \omega_{V\Lambda_h(\tau_{-i/2}(y)),V\Lambda_h(\tau_{i/2}(x))}$ with V the implementation of R on $\mathfrak{H}$, it results that

$$
\begin{aligned}
&(W^*(V\Lambda_h(\tau_{-i/2}(y)) \otimes \xi) \mid V\Lambda_h(\tau_{i/2}(x)) \otimes \eta) \\
&\qquad = (W(\Lambda_h(x) \otimes \xi) \mid \Lambda_h(y) \otimes \eta).
\end{aligned}
$$

Namely, we have

$$
\begin{aligned}
&(VH^{1/2}\Lambda_h(y) \otimes \xi \mid W(VH^{-1/2}\Lambda_h(x) \otimes \eta)) \\
&\qquad = (W(\Lambda_h(x) \otimes \xi) \mid \Lambda_h(y) \otimes \eta), \quad (x, y \in \mathcal{S}_{\sigma,\tau},\ \xi, \eta \in \mathfrak{H}).
\end{aligned}
$$

Since the subspace $\Lambda_h(\mathcal{S}_{\sigma,\tau})$ is a core for both $H^{1/2}$ and $H^{-1/2}$ by Lemma 2.5, we can easily deduce the asserted identity. □

Remark 2.9 (1) From the proof of Lemma 2.8, we find that, conversely, if the assertion of Lemma 2.8 is the case, then equation (2.7.1) holds true for any functional ϕ in the linear span of elements in $\mathcal{M}_*$ of the form $\omega_{\Lambda_h(x),\Lambda_h(y)}$, where x, $y \in \mathcal{S}_{\sigma,\tau}$. We shall soon show that equation (2.7.1) is actually valid for <u>any</u> ϕ in $(\mathcal{M}_*)_\tau^\infty$.

(2) Note that the linear span which appeared in (1) is a dense subspace of $(\mathcal{M}_*)_\tau^\infty$. In fact, if $\phi \in (\mathcal{M}_*)_\tau^\infty$, then, since $\{\mathcal{M}, \mathfrak{H}\}$ is a standard representation, there are vectors ζ_1, $\zeta_2 \in \mathfrak{H}$ such that $\phi = \omega_{\zeta_1,\zeta_2}$. By Lemma 2.3, we can choose sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{S}_{\sigma,\tau}$ so that $\lim_{n\to\infty} \|\Lambda_h(x_n) - \zeta_1\| = \lim_{n\to\infty} \|\Lambda_h(y_n) - \zeta_2\| = 0$. It is now easy to see that $\lim_{n\to\infty} \|\phi - \omega_{\Lambda_h(x_n),\Lambda_h(y_n)}\| = 0$.

In the next lemma, $\hat{\pi}$ stands for the Fourier representation associated with the system $(\mathcal{M}, \delta, R, \tau, h)$ as defined in the preceding section.

Lemma 2.10 *Suppose that we have*

$$(VH^{1/2}\zeta_1 \otimes \xi \mid W(VH^{-1/2}\zeta_2 \otimes \eta)) = (W(\zeta_2 \otimes \xi) \mid \zeta_1 \otimes \eta)$$

for any $\zeta_1 \in \mathfrak{D}(H^{1/2})$, $\zeta_2 \in \mathfrak{D}(H^{-1/2})$ *and* $\xi, \eta \in \mathfrak{H}$, *where* V *is any conjugate-linear isometric involution implementing the antipode* R *on* $\mathfrak{H}$. *Let* $x, y \in \mathcal{S}_{\sigma,\tau}$, *and put* $\phi = \omega_{\Lambda_h(x),\Lambda_h(y)}$. *Then* $\hat{\pi}(\phi)^* = \hat{\pi}(\phi^\sharp)$.

Proof. A proof has been already obtained in the calculation of the proof of Lemma 2.8. In fact, by that computation, we see that, if $\xi, \eta \in \mathfrak{H}$, then, since $\phi^* = \omega_{\Lambda_h(y),\Lambda_h(x)}$, we get

$$(\hat{\pi}(\phi^\sharp)\xi \mid \eta) = (VH^{1/2}\Lambda_h(x) \otimes \xi \mid W(VH^{-1/2}\Lambda_h(y) \otimes \eta)).$$

In the meantime, we have

$$(\hat{\pi}(\phi)^*\xi \mid \eta) = \overline{(\phi \otimes \omega_{\eta,\xi})(W^*)} = (W(\Lambda_h(y) \otimes \xi) \mid \Lambda_h(x) \otimes \eta).$$

Hence, by assumption, $(\hat{\pi}(\phi^\sharp)\xi \mid \eta) = (\hat{\pi}(\phi)^*\xi \mid \eta)$. This proves the lemma. □

Lemma 2.11 *Let* $\phi \in (\mathcal{M}_*)_\tau^\infty$, $\xi \in \mathfrak{D}(H^{1/2})$ *and* $\eta \in \mathfrak{D}(H^{-1/2})$. *Then we have*

$$(\hat{\pi}(\phi \circ R)H^{1/2}\xi \mid H^{-1/2}\eta) = (\hat{\pi}(\phi \circ \tau_{-i/2} \circ R)\xi \mid \eta).$$

Proof. Let us consider the two functions F, G below defined on the strip $\mathbf{D} = \{z \in \mathbf{C} : -1/2 \leq \operatorname{Im} z \leq 0\}$, which are continuous on $\mathbf{D}$ and analytic on the interior of $\mathbf{D}$:

$$\begin{aligned} F(z) &= (\hat{\pi}(\phi \circ R)H^{iz}\xi \mid H^{i\bar{z}}\eta) \qquad (-1/2 \leq \operatorname{Im} z \leq 0), \\ G(z) &= (\hat{\pi}(\phi \circ \tau_z \circ R)\xi \mid \eta). \end{aligned}$$

Since $\{\mathcal{M}, \mathfrak{H}\}$ is standard, there are vectors $\zeta_1, \zeta_2 \in \mathfrak{H}$ such that $\phi = \omega_{\zeta_1,\zeta_2}$. Let V be any conjugate-linear isometric involution implementing R on $\mathfrak{H}$ (there is at least one such involution, i.e., the canonical implementation of R, since $\{\mathcal{M}, \mathfrak{H}\}$ is standard). Then, for any $t \in \mathbf{R}$, we have

$$\begin{aligned} F(t) &= (\hat{\pi}(\phi \circ R)H^{it}\xi \mid H^{it}\eta) = (\hat{\pi}(\omega_{V\zeta_2,V\zeta_1})H^{it}\xi \mid H^{it}\eta) \\ &= (W^*(V\zeta_2 \otimes H^{it}\xi) \mid V\zeta_1 \otimes H^{it}\eta). \end{aligned}$$

Since $(\tau_t \otimes \tau_t) \circ \delta = \delta \circ \tau_t$, it follows from the definition of the Kac-Takesaki operator that we have $(H^{it} \otimes H^{it})W = W(H^{it} \otimes H^{it})$ for any $t \in \mathbf{R}$. Hence

$$\begin{aligned} F(t) &= (W^*(H^{-it}V\zeta_2 \otimes \xi) \mid H^{-it}V\zeta_1 \otimes \eta) \\ &= (\hat{\pi}(\omega_{H^{-it}V\zeta_2,H^{-it}V\zeta_1})\xi \mid \eta). \end{aligned}$$

But an simple computation shows that $\omega_{H^{-it}V\zeta_2,H^{-it}V\zeta_1} = \phi \circ \tau_t \circ R$. Thus we find that $F(t) = G(t)$. By the unicity theorem, it follows that $F(-i/2) = G(-i/2)$, which proves the assertion. □

For the next proposition, which is our main result of this section, we refer readers to [DC] and [W].

Proposition 2.12 *Suppose that V is a conjugate-linear isometric involution on $\mathfrak{H}$ implementing the unitary antipode R. The weight h is strongly left invariant if and only if we have*

$$(VH^{1/2}\zeta_1 \otimes \xi \mid W(VH^{-1/2}\zeta_2 \otimes \eta)) = (W(\zeta_2 \otimes \xi) \mid \zeta_1 \otimes \eta) \tag{2.12.1}$$

for any $\zeta_1 \in \mathfrak{D}(H^{1/2})$, $\zeta_2 \in \mathfrak{D}(H^{-1/2})$ and $\xi, \eta \in \mathfrak{H}$.

Proof. We have already shown the "only if" part in Lemma 2.8. Thus it remains to prove, by virtue of Lemma 2.7, that the equality (2.12.1) ensures that the identity (2.7.1) holds for any $\phi \in (\mathcal{M}_*)_\tau^\infty$.

First we note that, as we discussed in Remark 11,(1), Equation (2.7.1) is true for any functional in the linear span of elements of the form $\omega_{\Lambda_h(x),\Lambda_h(y)}$, where $x, y \in \mathcal{S}_{\sigma,\tau}$. Let ϕ be in $(\mathcal{M}_*)_\tau^\infty$. Take any $\xi \in \mathfrak{D}(H^{1/2})$ and $\eta \in \mathfrak{D}(H^{-1/2})$. By Remark 2.9,(2), we may choose a sequence $\{\phi_n\}$ inside the linear span appearing in Remark 2.9 so that $\lim_{n\to\infty} \|\phi_n - \phi\| = 0$. Then, by Lemma 2.10 and Lemma 2.11,

$$\begin{aligned}
&((\phi\circ\tau_{-i/2}\circ R\otimes id)(W^*)\xi \mid \eta)\\
&\quad = (\hat{\pi}(\phi\circ\tau_{-i/2}\circ R)\xi \mid \eta) = (\hat{\pi}(\phi\circ R)H^{1/2}\xi \mid H^{-1/2}\eta)\\
&\quad = \lim_{n\to\infty}(\hat{\pi}(\phi_n\circ R)H^{1/2}\xi \mid H^{-1/2}\eta)\\
&\quad = \lim_{n\to\infty}(\hat{\pi}(\phi_n\circ\tau_{-i/2}\circ R)\xi \mid \eta) = \lim_{n\to\infty}(\hat{\pi}(\phi_n^*)^*\xi \mid \eta)\\
&\quad = (\hat{\pi}(\phi^*)^*\xi \mid \eta) = (\phi\otimes\omega_{\xi,\eta})(W) = ((\phi\otimes id)(W)\xi \mid \eta).
\end{aligned}$$

The density argument now proves that $(\phi\circ\tau_{-i/2}\circ R\otimes id)(W^*) = (\phi\otimes id)(W)$. □

3. Bicrossproduct Hopf-von Neumann algebras

Throughout this section, we fix a matched pair $(G_1,\ G_2,\ \alpha,\ \beta)$ with μ_i a left Haar measure of G_i $(i=1,2)$. Let us denote by $\mathcal{M}$ the bicrossproduct Hopf-von Neumann algebra $L^\infty(G_2)\rtimes_\alpha G_1$ associated with this matched pair. The coproduct and the unitary antipode (coinvolution) of $\mathcal{M}$ will be denoted respectively by δ and R. By [M, Proposition 2.7], the bicocycle ζ defined in [M, Lemma 2.2] induces a one-parameter automorphism group $\{\tau_t\}$ on $\mathcal{M}$ which commutes with δ and R: $(\tau_t\otimes\tau_t)\circ\delta = \delta\circ\tau_t$, $\tau_t\circ R = R\circ\tau_t$ $(t\in\mathbf{R})$. Finally, we denote by h the dual weight (on $\mathcal{M}$) of the canonical Haar measure, denoted by μ_2 again, on $L^\infty(G_2)$. Readers should note that our notation for these objects is different from Majid's in [M]; he uses Γ for the coproduct, κ for the antipode, ζ_t for the one-parameter automorphims group and ϕ for the dual weight. Our goal of this section is to prove that the system $(\mathcal{M},\ \delta,\ R,\ \tau,\ h)$ is a quasi Woronowicz algebra. For this purpose, since we already know that this system satisfies conditions (1) and (2) in the definition of a quasi Woronowicz algebra, we need to show that the weight h enjoys the properties listed in (3) there.

Let $\mathfrak{H} = L^2(G_1)\otimes L^2(G_2) = L^2(G_1\times G_2)$. Following the convention, we denote by π_α the embedding of $L^\infty(G_2)$ into the crossed product $\mathcal{M} = L^\infty(G_2)\rtimes_\alpha G_1$. So, for any $k\in L^\infty(G_2)$, we have

$$\{\pi_\alpha(k)\xi\}(g,s) = k(\alpha_g(s))\xi(g,s), \qquad (\xi\in\mathfrak{H},\ g\in G_1,\ s\in G_2).$$

We use the symbol $\lambda^{(i)}$ for the left regular representation of G_i $(i=1,2)$, but, if there is no danger of confusion, we will suppress the superscript "(i)" in $\lambda^{(i)}$, and simply write λ. With the notation just introduced, the von Neumann algebra $\mathcal{M}$ is generated by $\pi_\alpha(L^\infty(G_2))$ and $\{\lambda_g\otimes 1 : g\in G_1\}$.

Let $K(G_1, L^\infty(G_2))$ be the space of σ-strong* continuous functions from G_1 to $L^\infty(G_2)$ with compact support. As in [H], this space can be turned into an involutive algebra (see [H, Lemma 2.3 (a)]) with product

$$(X * Y)(g) = \int_{G_1} \alpha_h(X(gh))Y(h^{-1})dh, \quad (X, Y \in K(G_1, L^\infty(G_2))),$$

and involution

$$X^\sharp(g) = \Delta(g)^{-1}\alpha_g^{-1}(X(g^{-1})^*), \qquad (X \in K(G_1, L^\infty(G_2))).$$

The algebra $K(G_1, L^\infty(G_2))$ has a natural $*$-representation μ on $\mathfrak{H}$ defined by

$$\mu(X) = \int_{G_1} (\lambda_g \otimes 1)\pi_\alpha(X(g))dg \qquad (X \in K(G_1, L^\infty(G_2))),$$

whose image generates the crossed product $\mathcal{M} = L^\infty(G_2) \rtimes_\alpha G_1$ (see [H, Lemma 2.3 (e)]). Inside $K(G_1, L^\infty(G_2))$, there is a $\sharp$-subalgebra, which is, under the notation of [H], $B_{\mu_2} \cap B^\sharp_{\mu_2}$, that induces a left Hilbert algebra in $\mathfrak{H}$ whose left von Neumann algebra is $\mathcal{M}$ ([H, Lemma 2.12]). In our special situation, we are interestred more in $K(G_1 \times G_2)$, the space of compactly supported continuous functions on $G_1 \times G_2$, rather than this subalgebra. Observe that $K(G_1 \times G_2)$ can be naturally considered as a subspace of $K(G_1, L^\infty(G_2))$. In fact, it is easy to see that $K(G_1 \times G_2)$ is a $\sharp$-subalgebra of the $\sharp$-subalgebra mentioned above. Moreover, it can be shown without difficulty that $K(G_1 \times G_2)$ is dense in $B_{\mu_2} \cap B^\sharp_{\mu_2}$, as left Hilbert algebras, with respect to the $\sharp$-graph norm. Hence $K(G_1 \times G_2)$ ($\subseteq \mathfrak{H} = L^2(G_1 \times G_2)$) induces a left Hilbert algebra $\mathfrak{T}$ equivalent to the one induced by $B_{\mu_2} \cap B^\sharp_{\mu_2}$, where the multiplication $*$ and the involution $\sharp$ are given by

$$\begin{aligned}(X * Y)(g, s) &= \int_{G_1} X(gh, \alpha_h^{-1}(s))Y(h^{-1}, s)dh, \\ &\qquad (X, Y \in \mathfrak{T} = K(G_1 \times G_2),\ g \in G_1,\ s \in G_2); \\ X^\sharp(g, s) &= \Delta(g)^{-1}\overline{X(g^{-1}, \alpha_g(s))}, \\ &\qquad (X \in K(G_1 \times G_2),\ g \in G_1,\ s \in G_2).\end{aligned}$$

Then $\mu(X)$ is just the left multiplication by X. In particular, the corresponding left von Neumann algebra is again $\mathcal{M}$, and the weight associated with it is the dual weight h. It then follows from Theorem 3.2 of [H] that the weight h satisfies

$$\begin{aligned} h(\mu(X)^*\mu(X)) &= \int_{G_1} \mu_2(X(g,\,\cdot)^*X(g,\,\cdot))dg \\ &= \int_{G_1}\left(\int_{G_2}|X(g,s)|^2ds\right)dg, \\ & \qquad\qquad (X\in\mathfrak{T}=K(G_1\times G_2)). \qquad (3.1)\end{aligned}$$

Next we quickly review the construction of the one-parameter automorphism group $\{\tau_t\}$. For this, we first introduce a (possibly unbounded) linear operator H on $\mathfrak{H}$ by using the bicocycle ζ as follows:

$$\{H\xi\}(g,s)=\zeta(g,s)\xi(g,s), \qquad (\xi\in\mathfrak{D}(H)),$$

where $\mathfrak{D}(H)=\{\xi\in\mathfrak{H}:\zeta\xi\in\mathfrak{H}\}$. It is easily verified that H is a densely defined, nonsingular, positive, self-adjoint linear operator. By [M, Proposition 2.7], the restriction τ_t of the automorphism $\operatorname{Ad}H^{it}$ to $\mathcal{M}$ is a coinvolutive Hopf-von Neumann algebra automorphism.

Lemma 3.2 *The one-parameter automorphism group $\{\tau_t\}$ satisfies*

$$\begin{aligned}\tau_t(\pi_\alpha(k)) &= \pi_\alpha(k), \qquad (k\in L^\infty(G_2),\,t\in\mathbf{R}); \\ \tau_t(\lambda_g\otimes 1) &= \pi_\alpha(\zeta(g^{-1},\,\cdot)^{-it})(\lambda_g\otimes 1), \qquad (g\in G_1,\,t\in\mathbf{R}).\end{aligned}$$

Moreover, $\{\tau_t\}$ preserves the dual weight h: $h\circ\tau_t=h$ $(t\in\mathbf{R})$.

Proof. It is only a matter of computation to verfiy the first two identities, so we leave it to readers.

From the asserted two equalities, it follows that, for any $X\in K(G_1, L^\infty(G_2))$,

$$\tau_t(\mu(X))=\int_{G_1}(\lambda_g\otimes 1)\pi_\alpha(\bar{\tau}_t(X)(g))dg, \qquad (3.2.1)$$

where $\bar{\tau}_t(X)\in K(G_1,L^\infty(G_2))$ is defined by $\bar{\tau}_t(X)(g)=\alpha_g^{-1}(\zeta(g^{-1},\,\cdot)^{-it})$ $X(g)$ $(g\in G_1)$. If X, $Y\in K(G_1\times G_2)$, then we have $\bar{\tau}_t(X)(g,s)\overline{\bar{\tau}_t(Y)(g,s)}=X(g,s)\overline{Y(g,s)}$. From this and (3.1), we find that $h\circ\tau_t$ equals h on the σ-weakly dense $*$-subalgebra $\mu(K(G_1\times G_2))^*\mu(K(G_1\times G_2))$ contained in $\mathfrak{m}_h$. In the meantime, by [H, Theorem 3.2] (or by [M, Lemma 2.8]), one has $\sigma_t^h(\mu(X))=\mu(\bar{\sigma}_t^h(X))$ for any $X\in K(G_1,L^\infty(G_2))$, where

$$\bar{\sigma}_t^h(X)(g)=\Delta(g)^{it}\chi(g,\,\cdot)^{it}X(g), \qquad (g\in G_1). \qquad (3.2.2)$$

Hence it results that (i) the $*$-subalgebra $\mu(K(G_1 \times G_2))^*\mu(K(G_1 \times G_2))$ is left stable under the modular automorphism group σ^h; (ii) $\bar{\tau}_t$ and $\bar{\sigma}_s^h$ commute, which in turn implies that τ_t commutes with σ_s^h. Consequently, $h \circ \tau_t$ is σ^h-invariant. Therefore, by [PT, Proposition 5.9], it follows that $h \circ \tau_t = h$. □

Lemma 3.3 *The dual weight h satisfies*

$$(\tau_t \otimes \sigma_t^h) \circ \delta = \delta \circ \sigma_t^h, \qquad (t \in \mathbf{R}).$$

Proof. By [M, Lemma 2.9], the one-parameter automorphism group $\{\tau_t\}$ enjoys the following property:

$$(id \otimes \tau_{-t} \circ \sigma_t^h) \circ \delta = \delta \circ \tau_{-t} \circ \sigma_t^h, \qquad (t \in \mathbf{R}).$$

Since τ_t "commutes" with the coproduct δ, the equality above is equivalent to the asserted identity. □

Lemma 3.4 *The dual weight h is quasi left invariant.*

Proof. By the proof of [M, Lemma 2.10], we see that, for any $x, y \in \mu(K(G_1 \times G_2))$,

$$(h \otimes h)((x^* \otimes 1)\delta(y^*y)(x \otimes 1)) = h(x^*x)h(y^*y).$$

It follows from this that $\delta(y)(x \otimes 1) \in \mathfrak{n}_{h\otimes h}$, and that the Kac-Takesaki operator W_h in the sense of [MN] given by

$$W_h(\Lambda_h(x) \otimes \Lambda_h(y)) = \Lambda_{h\otimes h}(\delta(y)(x \otimes 1)) \qquad (x, y \in \mu(K(G_1 \times G_2)))$$

is precisely the unitary W introduced in [M, Theorem 2.6] to define the coproduct δ. Hence, if $\zeta_1, \zeta_2 \in \mathfrak{T}'$ and $x, y \in \mu(K(G_1 \times G_2))$, then

$$(\pi_r(\zeta_1) \otimes \pi_r(\zeta_2))W_h(\Lambda_h(x) \otimes \Lambda_h(y)) = \delta(y)(x \otimes 1)(\zeta_1 \otimes \zeta_2), \tag{3.4.1}$$

where π_r in general denotes the right multiplication associated with a left Hilbert algebra. Let $b \in \mathfrak{n}_h$ and set $\xi = \Lambda_h(b)$. The vector ξ is then left bounded with respect to $\mathfrak{T}$. So, by [H1, Theorem 5], there exists a sequence $\{y_n\}$ in $\mu(K(G_1 \otimes G_2))$ such that $\|y_n\| \leq \|b\|$, $\lim_{n\to\infty} \|\Lambda_h(y_n) - \xi\| = 0$. In particular, we have s-$\lim_{n\to\infty} y_n = b$. Thus, if $\zeta_1, \zeta_2 \in \mathfrak{T}'$ and $x \in$

$\mu(K(G_1 \times G_2))$, then, by (3.4.1),

$$\begin{aligned}(\pi_r(\zeta_1) \otimes \pi_r(\zeta_2))W_h(\Lambda_h(x) \otimes \Lambda_h(b)) &= \lim_{n\to\infty} (\pi_r(\zeta_1) \otimes \pi_r(\zeta_2))W_h(\Lambda_h(x) \otimes \Lambda_h(y_n)) \\ &= \lim_{n\to\infty} \delta(y_n)(x \otimes 1)(\zeta_1 \otimes \zeta_2) = \delta(b)(x \otimes 1)(\zeta_1 \otimes \zeta_2).\end{aligned}$$

By a similar argument as above, we may show that

$$\begin{aligned}(\pi_r(\zeta_1) \otimes \pi_r(\zeta_2))W_h(\Lambda_h(a) \otimes \Lambda_h(b)) &= \delta(b)(a \otimes 1)(\zeta_1 \otimes \zeta_2), \qquad (a, b \in \mathfrak{n}_h,\ \zeta_1,\ \zeta_2 \in \mathfrak{T}').\end{aligned}$$

This implies that $W_h(\Lambda_h(a) \otimes \Lambda_h(b))$ is a left bounded vector associated with the left Hilbert algebra $\mathfrak{T} \otimes \mathfrak{T}$, and that $(\mu \otimes \mu)(W_h(\Lambda_h(a) \otimes \Lambda_h(b))) = \delta(b)(a \otimes 1)$. In particular, $\delta(b)(a \otimes 1)$ belongs to $\mathfrak{n}_{h\otimes h}$ and

$$\begin{aligned}(\pi_r(\zeta_1) \otimes \pi_r(\zeta_2))W_h(\Lambda_h(a) \otimes \Lambda_h(b)) &= (\pi_r(\zeta_1) \otimes \pi_r(\zeta_2))\Lambda_{h\otimes h}(\delta(b)(a \otimes 1)).\end{aligned}$$

It is now easy to see that

$$W_h(\Lambda_h(a) \otimes \Lambda_h(b)) = \Lambda_{h\otimes h}(\delta(b)(a \otimes 1)).$$

Since W_h is an isometry, it results that

$$(h \otimes h)((a^* \otimes 1)\delta(b^*b)(a \otimes 1)) = h(a^*a)h(b^*b), \qquad (a, b \in \mathfrak{n}_h).$$

By the same argument as in the proof of [S, Lemme II.8], we have

$$(id \otimes h) \circ \delta(a) = h(a) \cdot 1$$

for any $a \in \mathfrak{m}_h^+$. This proves the lemma. □

Lemma 3.5 *The dual weight h is strongly left invariant.*

Proof. By the second claimed equality of [M, Lemma 2.9], we have

$$(W_h(\hat{J}\delta \otimes H^{-1/2}\eta) \mid \hat{J}\gamma \otimes H^{1/2}\xi) = (\gamma \otimes \eta \mid W_h(\delta \otimes \xi))$$

for any $\delta, \gamma, \xi, \eta \in \mathfrak{T} = K(G_1 \times G_2) \subseteq \mathfrak{H}$, where $\hat{J}$ is the unitary involution on $\mathfrak{H}$ introduced in [M, Theorem 2.6]. Since $\mathfrak{T}$ is dense in $\mathfrak{H}$, this equality is still valid even if both δ and γ are arbitrary vectors in $\mathfrak{H}$. So let us replace δ and γ in the equation by $H^{-1/2}\delta$ and $H^{1/2}\gamma$ $(\delta, \gamma \in \mathfrak{T})$, respectively. Moreove, since the bicocyle $\zeta > 0$ is continuous, the vectors ξ, η can be respectively replaced by $\zeta^{-1/2}\xi$ and $\zeta^{1/2}\eta$ $(\xi, \eta \in \mathfrak{T})$. Consequently, we get

$$\begin{aligned}(W_h(\hat{J}H^{-1/2}\delta \otimes \eta) \mid \hat{J}H^{1/2}\gamma \otimes \xi) \\ = (\gamma \otimes \eta \mid W_h(\delta \otimes \xi)). \qquad (\delta, \gamma, \xi, \eta \in \mathfrak{T})\end{aligned}$$

Meanwhile, it is plain to see that the subspace $\mathfrak{T}$ is a core for both $H^{1/2}$ and $H^{-1/2}$. From this, together with density of $\mathfrak{T}$ in $\mathfrak{H}$, we finally obtain

$$(W_h(\hat{J}H^{-1/2}\zeta_1 \otimes \xi) \mid \hat{J}H^{1/2}\zeta_2 \otimes \eta) = (\zeta_2 \otimes \xi \mid W_h(\zeta_1 \otimes \eta))$$

for any $\zeta_1 \in \mathfrak{D}(H^{-1/2})$, $\zeta_2 \in \mathfrak{D}(H^{1/2})$ and $\xi, \eta \in \mathfrak{H}$. From Proposition 2.12 and the fact that $\hat{J}$ implements the coinvolution R, it follows that the weight h is strongly left invariant. □

Lemma 3.6 *The weight h is $\sigma^{h\circ R}$-invariant, i.e., we have*

$$h \circ \sigma_t^{h\circ R} = h \qquad (t \in \mathbf{R}).$$

Proof. First, as shown in the proof of [M, Theorem 2.6], we have

$$R(\mu(X)) = \mu(\bar{R}(X)) \qquad (X \in K(G_1 \times G_2)),$$

where $\bar{R}(X) \in K(G_1 \times G_2)$ is defined as follows:

$$\bar{R}(X)(g, s) := \frac{\Psi(s, g)}{\Delta(\beta_s(g))}\zeta(g, s)^{-1/2}X(\beta_s(g)^{-1}, \alpha_g(s)^{-1}).$$

Let X be in $K(G_1 \times G_2)$. Using the identity above and (3.2.2), we get

$$\begin{aligned}&\bar{R}(\bar{\sigma}^h_{-t}(\bar{R}(X)))(g, s) \\ &\quad = \chi(\beta_s(g)^{-1}, \alpha_g(s)^{-1})^{-it}\Delta(\beta_s(g))\zeta(g, s)^{-1/2} \\ &\qquad\qquad \times \frac{\Psi(s, g)}{\Delta(\beta_s(g))}\bar{R}(X)(\beta_s(g)^{-1}, \alpha_g(s)^{-1}).\end{aligned}$$

Meanwhile, we have

$$\begin{aligned}&\bar{R}(X)(\beta_s(g)^{-1}, \alpha_g(s)^{-1}) \\ &\quad = \frac{\Psi(\alpha_g(s)^{-1}, \beta_s(g)^{-1})}{\Delta(\beta_{\alpha_g(s)^{-1}}(\beta_s(g)^{-1}))}\zeta(\beta_s(g)^{-1}, \alpha_s(g)^{-1})^{-1/2} \\ &\qquad\qquad \times X(\beta_{\alpha_g(s)^{-1}}(\beta_s(g)^{-1})^{-1}, \alpha_{\beta_s(g)^{-1}}(\alpha_g(s)^{-1})^{-1}).\end{aligned}$$

Thanks to the matched pair condition (MP) and [M, Lemma 2.2], one finds

$$\beta_{\alpha_g(s)^{-1}}(\beta_s(g)^{-1})^{-1} = g, \qquad \alpha_{\beta_s(g)^{-1}}(\alpha_g(s)^{-1})^{-1} = s$$

$$\Psi(\alpha_g(s)^{-1}, \beta_s(g)^{-1}) = \frac{\Delta(\beta_s(g))}{\Delta(g)}\Psi(s^{-1}, e),$$
$$\chi(\beta_s(g)^{-1}, \alpha_g(s)^{-1}) = \frac{\Delta(\alpha_g(s))}{\Delta(s)}\chi(g^{-1}, e),$$
$$\zeta(\beta_s(g)^{-1}, \alpha_g(s)^{-1}) = \zeta(g, s).$$

These identities yield

$$\bar{R}(\bar{\sigma}^h_{-t}(\bar{R}(X)))(g,s) = \left[\frac{\Delta(\alpha_g(s))}{\Delta(s)}\right]^{-it} \chi(g,e)^{it}\Delta(\beta_s(g))^{it}X(g,s).$$

So, if, for each $t \in \mathbf{R}$, we define an element $\bar{\sigma}^{h\circ R}_t(X) \in K(G_1 \times G_2)$ by

$$\bar{\sigma}^{h\circ R}_t(X)(g,s) := \left[\frac{\Delta(\alpha_g(s))}{\Delta(s)}\right]^{-it} \chi(g,e)^{it}\Delta(\beta_s(g))^{it}X(g,s),$$

then, since $\sigma^{h\circ R}_t = R \circ \sigma^h_{-t} \circ R$, we have

$$\sigma^{h\circ R}_t(\mu(X)) = \mu(\bar{\sigma}^{h\circ R}_t(X)). \tag{3.6.1}$$

Let $r, t \in \mathbf{R}$. From (3.2.2) and (3.6.1), it can be easily verified that

$$\bar{\sigma}^{h\circ R}_{-t}(\bar{\sigma}^h_r(\bar{\sigma}^{h\circ R}_t(X))) = \bar{\sigma}^h_r(X).$$

From this, it follows that $\sigma^{h\circ R}_{-t} \circ \sigma^h_r \circ \sigma^{h\circ R}_t(\mu(X)) = \sigma^h_r(\mu(X))$, which implies

$$\sigma^{h\circ R}_t \circ \sigma^h_r = \sigma^h_r \circ \sigma^{h\circ R}_t \qquad (r, t \in \mathbf{R}). \tag{3.6.2}$$

Now let us take an arbitrary $r \in \mathbf{R}$ and fix it. Set $\psi = h \circ \sigma^{h\circ R}_r$. By (3.6.2), it is obvious that $\sigma^\psi = \sigma^h$. Moreover, for any $X, Y \in K(G_1 \times G_2)$, one finds

$$\begin{aligned}
&\psi(\mu(Y)^*\mu(X)) \\
&\quad = h(\mu(\bar{\sigma}^{h\circ R}_r(Y))^*\mu(\bar{\sigma}^{h\circ R}_r(X))) \qquad \text{by (10)} \\
&\quad = \int_{G_1}\left(\int_{G_2} \bar{\sigma}^{h\circ R}_r(X)(g,s)\overline{\bar{\sigma}^{h\circ R}_r(Y)(g,s)}ds\right)dg \qquad \text{by (3.1)} \\
&\quad = \int_{G_1}\left(\int_{G_2} X(g,s)\overline{Y(g,s)}ds\right)dg = h(\mu(Y)^*\mu(X)).
\end{aligned}$$

This shows that ψ equals h on the σ-weakly dense $*$-subalgebra $\mu(K(G_1 \times G_2))^*\mu(K(G_1 \times G_2))$ of $\mathfrak{m}_h$, stable under the modular automorphism group σ^h. Therefore, by [PT, Proposition 5.9], we conclude that $\psi = h$. □

We summarize what we have established so far in the theorem that follows.

Theorem 3.7 *Suppose that $(G_1, G_2, \alpha, \beta)$ is a matched pair. Then the bicrossproduct Hopf-von Neumann algebras $L^\infty(G_2) \rtimes_\alpha G_1$ and $L^\infty(G_1) \rtimes_\beta G_2$ associated with it are quasi Woronowicz algebras. Moreover, these quasi Woronowicz algebras are dual to each other.*

Proof. From Lemmas 3.2, 3.4, 3.5, 3.6, it follows that $(L^\infty(G_2) \rtimes_\alpha G_1, \delta, R, \tau, h)$ is a quasi Woronowicz algebra. The discussion in the case of $L^\infty(G_1) \rtimes_\beta G_2$ goes parallel to the one made so far in this section. The assertion that these Woronowicz algebras are dual to each other can be verified from the argument given in Section 3 of [M]. The details are left to readers. □

Remark. If the quasi Woronowicz algebra $\mathcal{M} = L^\infty(G_2) \rtimes_\alpha G_1$ is compact, i.e., the Haar measure h is bounded, then, since h is the dual weight of the Haar measure of G_2, it entails that G_1 is discrete, and that G_2 is compact. From this, it follows that the matched pair is modular in the sense of [M]. Thus the quasi Woronowicz algebra $\mathcal{M}$ becomes a compact Kac algebra.

Appendix – The Radon Nikodym derivative $(D(h \circ R) : Dh)_t$

The purpose of this appendix is to find the formula for the Radon Nikodym derivative $(D(h \circ R) : Dh)_t$ for the quasi Woronowicz algebra $(L^\infty(G_2) \rtimes_\alpha G_1, \delta, R, \tau, h)$ associated with a matched pair $(G_1, G_2, \alpha, \beta)$. This, together with the results in the preceding section, more or less completes listing all the relevant information on the bicrossproduct quasi Woronowicz algebra arising from a matched pair of groups. To give an explicit description of this Radon Nikodym derivative, we need some preparatory results. In the following discussion, we retain the notation established so far.

Let us introduce a one-parameter unitary group $V(t)$ on $\mathfrak{H}$ by

$$\{V(t)\xi\}(g, s) := \left[\frac{\Delta(s)}{\Delta(\alpha_g(s))} \cdot \frac{\Delta(\beta_s(g))}{\Delta(g)}\right]^{it} \Delta(s)^{-it} \Psi(s, g)^{-it} \xi(g, s)$$

$$(\xi \in \mathfrak{H}, \ g \in G_1, \ s \in G_2).$$

With Q a nonsingular positive self-adjoint operator on $\mathfrak{H}$ given by

$$\{Q\xi\}(g,s) := \left[\frac{\Delta(s)}{\Delta(\alpha_g(s))}\cdot\frac{\Delta(\beta_s(g))}{\Delta(g)}\right]\Delta(s)^{-1}\Psi(s,g)^{-1}\xi(g,s) \qquad (\xi\in\mathfrak{D}(Q)),$$

where $\mathfrak{D}(Q)=\{\xi\in\mathfrak{H}:(g,s)\mapsto\left[\frac{\Delta(s)}{\Delta(\alpha_g(s))}\cdot\frac{\Delta(\beta_s(g))}{\Delta(g)}\right]\Delta(s)^{-1}\Psi(s,g)^{-1}\xi(g,s)$ is in $\mathfrak{H}\}$, it is easy to see that $V(t)=Q^{it}$ for any $t\in\mathbf{R}$.

We start with the next lemma.

Lemma A.1 *For each $t\in\mathbf{R}$, $V(t)$ belongs to $\mathcal{M}$.*

Proof. Let $u(\cdot)$ be the unitary representation of G_1 on $L^2(G_2)$ defined in [M, Proposition 2.4]. This is a representation that implements the action α of G_1 on $L^\infty(G_2)$. Next let $\rho(\cdot)$ stand for the right regular representation of G_1. Then, by the commutation theorem for crossed products (see [H, Theorem 2.1] for example), we know that $\mathcal{M}'$ is generated by $\mathbf{C}\otimes L^\infty(G_2)$ and $\{\rho(g)\otimes u(g): g\in G_1\}''$. It is easily checked that $[V(t),\, 1\otimes k]=0$ for any $k\in L^\infty(G_2)$. Thus it remains to show that $[V(t),\,\rho(g)\otimes u(g)]=0$ for all $g\in G$. Let $\xi\in\mathfrak{H}$. Then

$$\begin{aligned}&\{V(t)(\rho(g)\otimes u(g))\xi\}(h,s)\\&\quad=\left[\frac{\Delta(s)}{\Delta(\alpha_h(s))}\cdot\frac{\Delta(\beta_s(h))}{\Delta(h)}\right]^{it}\\&\qquad\times\Delta(s)^{-it}\Psi(s,h)^{-it}\Delta(g)^{1/2}\chi(g^{-1},s)^{1/2}\xi(hg,\alpha_{g^{-1}}(s)).\end{aligned}$$

In the meantime, we have

$$\begin{aligned}&\{(\rho(g)\otimes u(g))V(t)\xi\}(h,s)\\&\quad=\Delta(g)^{1/2}\chi(g^{-1},s)^{1/2}\left[\frac{\Delta(\alpha_{g^{-1}}(s))}{\Delta(\alpha_{hg}(\alpha_{g^{-1}}(s)))}\cdot\frac{\Delta(\beta_{\alpha_{g^{-1}}(s)}(hg))}{\Delta(hg)}\right]^{it}\\&\qquad\times\Delta(\alpha_{g^{-1}}(s))^{-it}\Psi(\alpha_{g^{-1}}(s),hg)^{-it}\xi(hg,\alpha_{g^{-1}}(s)).\end{aligned}$$

From condition (MP) and [M, Lemma 2.2],

$$\begin{aligned}&\beta_{\alpha_{g^{-1}}(s)}(hg)=\beta_s(h)\beta_s(g^{-1})^{-1},\\&\Psi(\alpha_{g^{-1}}(s),hg)=\frac{\Delta(g^{-1})}{\Delta(\beta_s(g^{-1}))}\Psi(s,h).\end{aligned}$$

Consequently, one obtains

$$\begin{aligned}
&\{(\rho(g) \otimes u(g))V(t)\xi\}(h, s) \\
&\quad = \left[\frac{\Delta(s)}{\Delta(\alpha_h(s))} \cdot \frac{\Delta(\beta_s(h))}{\Delta(h)}\right]^{it} \Delta(s)^{-it}\Psi(s, h)^{-it} \\
&\qquad\qquad \times \Delta(g)^{1/2}\chi(g^{-1}, s)^{1/2}\xi(hg, \alpha_{g^{-1}}(s)) \\
&\quad = \{V(t)(\rho(g) \otimes u(g))\xi\}(h, s).
\end{aligned}$$

This completes the proof. □

The next lemma follows from a straightforward calculation, using the identity on the modular operator Δ_h of the weight h deduced in [M, Lemma 2.8]. Hence we leave the verification to readers.

Lemma A.2 *For each $t \in \mathbf{R}$, the unitary $V(t)$ belongs to the centralizer $\mathcal{M}_h$ of the weight h.*

Lemma A.3 *The one-parameter automorphism group $\operatorname{Ad} V(t)$ on $\mathcal{M}$ satisfies*

$$\begin{aligned}
\operatorname{Ad} V(t)(\pi_\alpha(k)) &= \pi_\alpha(k), \qquad (k \in L^\infty(G_2)); \\
\operatorname{Ad} V(t)(\lambda_g \otimes 1) &= \pi_\alpha(F_{t,g})(\lambda_g \otimes 1), \qquad (g \in G_1,\, t \in \mathbf{R}),
\end{aligned}$$

where, for each $t \in \mathbf{R}$ and $g \in G_1$, $F_{t,g}$ is a function in $L^\infty(G_2)$ given by

$$F_{t,g}(s) := \left[\frac{\Delta(s)}{\Delta(\alpha_{h^{-1}}(s))} \cdot \frac{\Delta(\beta_s(h^{-1}))}{\Delta(h^{-1})}\right]^{-it} \zeta(h^{-1}, s)^{it} \quad (s \in G_2).$$

Proof. The first identity is trivial. For the second identity, let $h \in G_1$. We first find from the matched pair condition (MP) that, for any $\xi \in \mathfrak{H}$,

$$\begin{aligned}
&\{V(t)(\lambda_h \otimes 1)V(-t)\xi\}(g, s) \\
&\quad = \left[\frac{\Delta(\alpha_g(s))}{\Delta(\alpha_{h^{-1}g}(s))} \cdot \frac{\Delta(\beta_{\alpha_g(s)}(h^{-1}))}{\Delta(h^{-1})}\right]^{-it} \\
&\qquad\qquad \times \Psi(s, g)^{-it}\Psi(s, h^{-1}g)^{it}\xi(h^{-1}g, s).
\end{aligned}$$

By [M, Lemma 2.2], we have

$$\Psi(s, g) = \frac{\Delta(\beta_s(g))}{\Delta(g)}\Psi(\alpha_g(s), e),$$

$$\Psi(s,h^{-1}g)=\frac{\Delta(\beta_s(g))}{\Delta(g)}\Psi(\alpha_g(s),h^{-1}).$$

From this, it follows that, with $F_{t,h}$ the function defined in the assertion of this lemma, we have

$$\begin{aligned}\{V(t)(\lambda_h\otimes 1)V(-t)\xi\}(g,s) &= F_{t,h}(\alpha_g(s))\xi(h^{-1}g,s)\\ &= \{\pi_\alpha(F_{t,h})(\lambda_h\otimes 1)\xi\}(g,s).\end{aligned}$$

This proves the lemma. □

Lemma A.4 *We have*

$$\operatorname{Ad}V(t)\circ\sigma_t^h=\sigma_t^{h\circ R}$$

for all $t\in\mathbf{R}$.

Proof. Let X be in $K(G_1\times G_2)$. We use the notation introduced in the preceding lemma. By (3.2.2) and Lemma A.3, we have

$$\begin{aligned}&\operatorname{Ad}V(t)\circ\sigma_t^h(\mu(X))\\ &=\operatorname{Ad}V(t)(\mu(\bar\sigma_t^h(X)))\\ &=\int_{G_1}\operatorname{Ad}V(t)(\lambda_g\otimes 1)\pi_\alpha(\bar\sigma_t^h(X)(g))dg\\ &=\int_{G_1}\Delta(g)^{it}\chi(g,\,\cdot)\pi_\alpha(F_{t,g})(\lambda_g\otimes 1)\pi_\alpha(X(g))dg\\ &=\int_{G_1}\Delta(g)^{it}\chi(g,\,\cdot)^{it}(\lambda_g\otimes 1)\pi_\alpha(\alpha_{g^{-1}}(F_{t,g})X(g))dg.\end{aligned}$$

As a (bounded) continuous function on G_2, one has, for any $s\in G_2$,

$$\begin{aligned}&\{\Delta(g)^{it}\chi(g,\,\cdot)^{it}\alpha_{g^{-1}}(F_{t,g})X(g)\}(s)\\ &=\Delta(g)^{it}\chi(g,s)^{it}F_{t,g}(\alpha_g(s))X(g,s)\\ &=\Delta(g)^{it}\chi(g,s)^{it}\left[\frac{\Delta(\alpha_g(s))}{\Delta(\alpha_{g^{-1}}(\alpha_g(s)))}\cdot\frac{\Delta(\beta_{\alpha_g(s)}(g^{-1}))}{\Delta(g^{-1})}\right]^{-it}\\ &\qquad\qquad\times\zeta(g^{-1},\alpha_g(s))^{it}X(g,s)\\ &=\left[\frac{\Delta(\alpha_g(s))}{\Delta(s)}\right]^{-it}\Delta(\beta_s(g))^{it}\chi(g,s)^{it}\zeta(g^{-1},\alpha_g(s))^{it}X(g,s)\\ &=\left[\frac{\Delta(\alpha_g(s))}{\Delta(s)}\right]^{-it}\Delta(\beta_s(g))^{it}\chi(g,e)^{it}X(g,s).\end{aligned}$$

This shows that

$$\{\Delta(g)^{it}\chi(g,\cdot)^{it}\alpha_{g^{-1}}(F_{t,g})X(g)\}(s) = \bar{\sigma}_t^{h\circ R}(X)(g,s).$$

From this, it follows that $\mathrm{Ad}\, V(t)\circ\sigma_t^h(\mu(X)) = \sigma_t^{h\circ R}(\mu(X))$. □

Proposition A.5 *The Radon Nikodym derivative $(D(h\circ R) : Dh)_t$ of $h\circ R$ with respect to h is the one-parameter unitary group $V(t) = Q^{it}$ defined above. In particular, we have $h\circ R = h(Q\,\cdot)$.*

Proof. It suffices by Lemma A.2 to prove the last assertion.

Note first that, by Lemma A.2, the operator Q is affiliated with the centralizer $\mathcal{M}_h$. Thus, with the notation in [PT], it makes sense to consider the weight $h(Q\,\cdot)$. Let $\psi = h(Q\,\cdot)$. From Lemma A.4, we have $\sigma^\psi = \sigma^{h\circ R}$. By using condition (MP) and the identity on $\bar{R}$ mentioned in the proof of Lemma 3.6, it can be verified that

$$h\circ R(\mu(Y)^*\mu(X)) = \int_{G_1}\int_{G_2}\Delta(s)^{-1}\Psi(s,e)^{-1}X(g,s)\overline{Y(g,s)}dsdg,$$
$$(X,Y\in K(G_1\times G_2)).$$

In the meantime, for any $\varepsilon > 0$, put $Q_\varepsilon := Q(1+\varepsilon Q)^{-1}$, which belongs to $\mathcal{M}_h$, due to Lemma A.2. Let $X\in K(G_1\times G_2)$. Note that, as an vector in $\mathfrak{H}$, X is in $\mathfrak{D}(Q)$. By definition, we have

$$\begin{aligned}\psi(\mu(X)^*\mu(X)) &= \lim_{\varepsilon\downarrow 0} h(\mu(X)^*\mu(X)Q_\varepsilon)\\ &= \lim_{\varepsilon\downarrow 0}(\Lambda_h(\mu(X)Q_\varepsilon)\mid \Lambda_h(\mu(X)))\\ &= \lim_{\varepsilon\downarrow 0}(J_hQ_\varepsilon J_h\Lambda_h(\mu(X))\mid \Lambda_h(\mu(X)))\\ &= \lim_{\varepsilon\downarrow 0}(J_hQ_\varepsilon J_hX\mid X)\\ &= (J_hQJ_hX\mid X),\end{aligned}$$

where J_h is the modular conjugation of the weight h. Since

$$\{J_h\xi\}(g,s) = \Delta(g)^{-1/2}\chi(g,s)\overline{\xi(g^{-1},\alpha_g(s))}, \qquad (\xi\in\mathfrak{H}),$$

we find from a direct computation that

$$(J_hQJ_hX\mid X) = \int_{G_1}\int_{G_2}\Delta(s)^{-1}\Psi(s,e)^{-1}|X(g,s)|^2dsdg.$$

Hence, by the polarization trick, we obtain

$$h \circ R(\mu(Y)^*\mu(X)) = \psi(\mu(Y)^*\mu(X)).$$

It follows that ψ equals $h \circ R$ on the σ-weakly dense $*$-subalgebra $\mu(K(G_1 \times G_2))^*\mu(K(G_1 \times G_2))$ of $\mathfrak{m}_{h \circ R}$, invariant under $\sigma^{h \circ R}$. Therefore, by [PT, Proposition 5.9], we conclude that ψ coincides with $h \circ R$. □

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