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On the modified Newton's approximation method for the solution of non-linear singular integral equations

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Abstract. This paper produces sufficient conditions for the convergence of the modified Newton-Kantorovich method applied to a class of nonlinear singular integral equations with Cauchy kernel in generalized Holder space.

Key words: Cauchy singular integral equations, modified Newton-Kantorovich method, index of integral equations.

1. Introduction

There is a large literature on nonlinear singular integral equations with Hilbert and Cauchy kernel and on related Riemann-Hilbert boundary value problems for analytic functions, cf. the monograph by Pogorzelski [11], the other by Guseinov A.I. and Mukhtarov Kh. Sh. [4]. The approximate solution of singular integral equations on closed curves has been intensively investigated by many approximation methods, specially the method of modified Newton-Kantorovich, of reduction, of collocation and of mechanical quadratures, (see, [2], [3], [5], [7], [9], [12], [13] and others). For the singular integral equations on an interval mention, Musaev, [10]; Junghanns, et al. [5], [6] and Wolfersdorf [15]. Consider the following nonlinear singular integral equation (NSIE):

$$(P(u))(s) = F(s, u(s)) - B[G(\cdot, u(\cdot))](s) = 0, \tag{1.1}$$

where

$$B[G(\sigma, u(\sigma))](s) = \frac{1}{\pi}\int_a^b \frac{G(\sigma, u(\sigma))}{\sigma - s} d\sigma$$

is a Cauchy principle value and u(s) is unknown function and the functions $F_{u^i}[s, u(s)]$, $G_{u^i}[s, u(s)]$ are defined and continuous in the region

$$D = \{a \leq s \leq b;\ u \in (-\infty, \infty)\}, \quad i = 0, 1, \ldots, m-1.$$

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The integral equation (1.1) is equivalent to the following Riemann-Hilbert problem: Find holomorphic function $w(z) = u(z) + iv(z)$, $z = x + iy$ in the upper half-plane $y > 0$ of the complex z-plane which is continuous in $y \geq 0$ and satisfies:

$$F(s, u(s)) + v(s) = 0, \qquad \text{for} \quad s \in [a, b]$$
$$u(s) = 0 \qquad \text{for} \quad s \notin [a, b].$$

where

$$w(z) = \frac{1}{\pi i} \int_a^b \frac{G(\xi, u(\xi))}{\xi - z} d\xi$$

(cf. Gakhov, [1], Pogorzelski [11]. Wegert, [14] and wolfersdrof, [15]).

Definition 1.1

(i) We denote by $\Phi(a, \frac{b-a}{2}]$ to be the class of all continuous monotonic increasing functions ϕ defined on the interval $(a, \frac{b-a}{2}]$ such that $\lim_{\delta \to 0^+} \phi(\delta) = 0$, and $\phi(\delta)\delta^{-1}$ is a nondecreasing function.

(ii) the class Φ^m is the class of all functions $\phi \in \Phi$ such that $a < t_1 < t_2 < \frac{b-a}{2}$ implies $t_1^m \phi(t_2) \leq c(m) t_2^m \phi(t_1)$, where m is a natural number.

(iii) we denote by $C = C[a, b]$ be the Banach space of all (real or complex-valued) continuous functions on $[a, b]$ with $\|u\|_C = \max_{s \in [a,b]} |u(s)|$.

(iv) For natural number m we define the generalized Holder space $H_{\phi,m}$ to be the set of all functions $u \in C$ such that $\omega_u^m(\delta) = O(\phi(\delta))$; $\omega_u^m(\delta)$ is the modulus of continuity of order m of u, and $\phi \in \Phi^m$. (cf. [4], [8], [12]).

(v) For $u \in H_{\phi,m}$, we define

$$\|u\|_{\phi,m} = \|u\|_C + \sup_{a < \delta \leq \frac{b-a}{2}} \frac{\omega_u^m(\delta)}{\phi(\delta)}.$$

In [4], [10] and others, the modified Newton-Kantorovich method is used to find the approximate solution for some classes of NSIE in Holder space H_α, $(0 < \alpha < 1)$. In the present paper we shall study the application of modified Newton-Kantorovich method to the solution of NSIE (1.1) with different cases of the index χ ($\chi = 0$, $\chi > 0$ and $\chi < 0$) in the space $H_{\phi,m}$. For this aim, we introduce the following:

Lemma 1.1 [4], [12] *Let the functions $F(s, u(s))$ and $G(s, u(s))$ are defined and continuous in the region D, have all partial derivatives up to order*

$(m-1)$ *and satisfy the following conditions respectively*:

$$\left|\frac{\partial^k F(s_2,u_2)}{\partial s^i \partial u^j} - \frac{\partial^k F(s_1,u_1)}{\partial s^i \partial u^j}\right| \leq c_0(k)\phi(|s_2-s_1|) + |u_2-u_1|, \tag{1.2}$$

$$\left|\frac{\partial^k G(s_2,u_2)}{\partial s^i \partial u^j} - \frac{\partial^k G(s_1,u_1)}{\partial s^i \partial u^j}\right| \leq \eta_0(k)\phi(|s_2-s_1|) + |u_2-u_1|, \tag{1.3}$$

for arbitrary $(s_l,u_l) \in D$ $(l=1,2)$, $i+j=k$, $k=0,1,\ldots,m-1$, *where* $\phi \in \Phi^m$, $c_0(k)$ *and* $\eta_0(k)$ *are constants. If* $u(s) \in H_{\phi,m}$ *then* $F(s,u)$ *and* $G(s,u)$ *belong to* $H_{\phi,m}$.

2. First Case: $(\chi = 0)$

Lemma 2.1 *If the functions* $F(s,u(s))$ *and* $G(s,u(s))$ *satisfy the conditions of Lemma* 1.1 *and*

$$G_{u^i}(a,u(a)) = G_{u^i}(b,u(b)) = 0, \quad i=0,1,2.$$

Then the operator $P(u)$ *is Frechet differentiable in the space* $H_{\phi,m}$ *and its derivative is given by*:

$$P'(u)h(s) = F'_u(s,u(s))h(s) - \frac{1}{\pi}\int_a^b \frac{G'_u(\sigma,u(\sigma))}{\sigma - s}h(\sigma)d\sigma \tag{2.1}$$

and satisfies Lipschitz condition:

$$\|P'(u_2) - P'(u_1)\|_{\phi,m} \leq \xi_0 \|u_2 - u_1\|_{\phi,m}$$

in the sphere

$$N_{\phi,m}(u_0,\rho) = (u \in H_{\phi,m}, \|u-u_0\|_{\phi,m} < \rho)$$

where ξ_0 *is a constant.*

Proof. Let $u(s)$ be any a fixed point in the space $H_{\phi,m}[a,b]$ and $h(s)$ be an arbitrary element in $H_{\phi,m}[a,b]$, then we obtain

$$P(u+h) - P(u) = P'(u)h(s) + \Omega_1(s) + \Omega_2(s),$$

where

$$\Omega_1(s) = \int_0^1 (1-t)F_u''(s, u(s) + th(s))h^2(s)dt$$

and

$$\Omega_2(s) = -\frac{1}{\pi}\int_a^b \int_0^1 (1-t)G_u''(\sigma, u(\sigma) + th(\sigma))h^2(\sigma)dt\frac{d\sigma}{\sigma - s}.$$

If $\psi(\sigma) \in H_{\phi,m}[a,b]$ and $\psi(a) = \psi(b) = 0$, then

$$\left\| \frac{1}{\pi}\int_a^b \frac{\psi(\sigma)}{\sigma - s}d\sigma \right\| \leq R\|\psi\|_{\phi,m}, [4]. \tag{2.2}$$

If ψ, $\tilde{\psi} \in H_{\phi,m}[a,b]$, then

$$\|\psi\tilde{\psi}\|_{\phi,m} \leq R\|\psi\|_{\phi,m}\|\tilde{\psi}\|_{\phi,m}, [4], \tag{2.3}$$

where R is a constant. Hence from (2.2) and (2.3) we obtain

$$\lim_{\|h\|\to 0}\frac{\|\Omega_1(s)\|}{\|h\|} = 0 \quad \text{and} \quad \lim_{\|h\|\to 0}\frac{\|\Omega_2(s)\|}{\|h\|} = 0,$$

which proves the differentiability of $P(u)$ in the sense of Frechet and its derivative is given by (2.1). Moreover, the Frechet derivative $P'(u)$ satisfies Lipschitz condition:

$$\begin{aligned} P'(u_2) - P'(u_1) &= (F_u'(s, u_2(s)) - F_u'(s, u_1(s)))h(s) \\ &\quad - \frac{1}{\pi}\int_a^b (G_u'(\sigma, u_2) - G_u'(\sigma, u_1))h(\sigma)\frac{d\sigma}{\sigma - s} \\ &= E(s)(u_2 - u_1)h(s) \\ &\quad - \frac{1}{\pi}\int_a^b Y(\sigma)(u_2(\sigma) - u_1(\sigma))h(\sigma)\frac{d\sigma}{\sigma - s}. \end{aligned}$$

where

$$E(s) = \int_0^1 F_u''(s, u_1 + t(u_2 - u_1))dt \quad \text{and}$$

$$Y(\sigma) = \int_0^1 G_u''(\sigma, u_1 + t(u_2 - u_1))dt.$$

Obviously $E(s)$ and $Y(s)$ belong to $H_{\phi,m}$ hence, using inequalities (2.2) and

(2.3) we have

$$\|P'(u_2) - P'(u_1)\|_{\phi,m}$$
$$= \sup_{\|h\|_{\phi,m}=1} \|E(s)(u_2 - u_1)$$
$$- \frac{1}{\pi}\int_a^b Y(\sigma)(u_2 - u_1)h(\sigma)\frac{d\sigma}{\sigma - s}\|_{\phi,m}$$
$$\leq R(\|E(s)\|_{\phi,m} + R\|Y(s)\|_{\phi,m})\|u_2 - u_1\|_{\phi,m},$$

where

$$\|E(s)\|_{\phi,m} \leq c_0(2)(\|u_1\|_c + \|u_2 - u_1\|_c) + \|F_u''(s,0)\|_c + c(u_1, m)$$

and

$$\|Y(s)\|_{\phi,m} \leq \eta_0(2)(\|u_1\|_c + \|u_2 - u_1\|_c) + \|G_u''(s,0)\|_c + \eta(u_1, m)$$

where $c(u_1, m)$ and $h(u_1, m)$ are constants. Hence;

$$\|P'(u_2) - P'(u_1)\|_{\phi,m} \leq \xi_0\|u_2 - u_1\|_{\phi,m},$$

where

$$\xi_0 = R(R\eta_0(2) + c_0(2))(\|u_1\|_c + \|u_2 - u_1\|_c) + R\|F_u''(s,0)\|_c$$
$$+ R^2\|G_u''(s,0)\|_c + Rc(u_1, m) + R^2\eta(u_1, m)$$

then the lemma be valid. □

Theorem 2.1 *If the functions $F(s, u(s))$ and $G(s, u(s))$ satisfy the conditions of Lemma 2.1, ${F_u'}^2(s, u(s)) \neq 0$ everywhere on $[a, b]$ and ${F_u'}^2(s, u(s)) + {G_u'}^2(s, u(s)) \neq 0$. Then the linear operator*

$$L_0h = F_u'(s, u_0(s))h(s) - \frac{1}{\pi}\int_a^b \frac{G_u'(\sigma, u_0(\sigma))}{\sigma - s}h(\sigma)d\sigma \tag{2.4}$$

has a bounded inverse L_0^{-1}, for any fixed point $u_0 \in H_{\phi,m}$.

Proof. To find the operator L_0^{-1}, we investigate the solvability of the equation,

$$F_u'(s, u_0(s))h(s) - \frac{1}{\pi}\int_a^b \frac{G_u'(\sigma, u_0(\sigma))}{\sigma - s}h(\sigma)d\sigma = g(s) \tag{2.5}$$

where $u_0 \in H_{\phi,m}[a,b]$ be a fixed point and $g(s) \in H_{\phi,m}[a,b]$ be an arbitrary element. We introduce the following piecewise holomorphic function

$$V(z) = \frac{1}{2\pi i}\int_a^b \frac{G'_u(\sigma, u_0(\sigma))}{\sigma - z} h(\sigma) d\sigma, \quad V^{\pm}(\infty) = 0.$$

Then according to Sokhotski-plemelj Formula [1] equation (2.5) leads to the following Riemann boundary value problem

$$V^+(s) = B(s)V^-(s) + A(s) \tag{2.6}$$

where

$$B(s) = \frac{F'_u(s, u_0(s)) + iG'_u(s, u_0(s))}{F'_u(s, u_0(s)) - iG'_u(s, u_0(s))},$$

$B(s) \neq 0$ everywhere on $[a,b]$ and belongs to $H_{\phi,m}$,
and

$$A(s) = \frac{ig(s)G'_u(s, u_0(s))}{F'_u(s, u_0(s)) - iG'_u(s, u_0(s))}.$$

The index $\chi = -(\lambda_1 + \lambda_2)$, where λ_1 and λ_2 are integers which defined from the following relations:

$$-1 < \lambda_1 + \theta(a) < 1, \quad -1 < \lambda_2 + \theta(b) < 1,$$

where $\theta(s) = \mp \frac{1}{2\pi i} \ln B(s)$. Putting

$$B(s) = \frac{X_0^+(s)}{X_0^-(s)} \tag{2.7}$$

where

$$X_0(z) = \exp(\Gamma(z)) = \exp\left(\frac{1}{2\pi i}\int_a^b \frac{\ln B(\sigma)}{\sigma - z} d\sigma\right).$$

From the equation (2.7) the boundary condition (2.6) has the form

$$\frac{V^+(s)}{X_0^+(s)} - \frac{V^-(s)}{X_0^-(s)} = \frac{A(s)}{X_0^+(s)}$$

here, we obtain

$$V(z) = \frac{X_0(z)}{2\pi i}\int_a^b \frac{A(\sigma)}{X_0^+(\sigma)(\sigma - z)} d\sigma.$$

Hence the solution of the equation (2.5) has the following form

$$\begin{aligned} h(s) &= \frac{1}{K_0(s)}\left(F_u'(s,u_0)g(s) + W_0(s)\frac{1}{\pi}\int_a^b \frac{G_u'(\sigma,u_0(\sigma))}{W_0(\sigma)(\sigma-s)}\,g(\sigma)d\sigma\right) \\ &= L_0^{-1}\left(g(s)\right), \end{aligned}$$

where

$$K_0(s) = F_u'^{\,2}(s,u_0(s)) + G_u'^{\,2}(s,u_0(s))$$

and

$$W_0(s) = X_0^+(s)\left(F_u'(s,u_0(s)) - G_u'(s,u_0(s))\right)$$

From inequalities (2.2) and (2.3), we have

$$\|L_0^{-1}\|_{\phi,m} \le D_0 \quad \text{and} \quad \|P_0(u)\| \le N_0,$$

where D_0 and N_0 are constants. Hence all the conditions of applicability and convergence of modified Newton's method are satisfied, thus the following theorem is valid. □

Theorem 2.2 *Let the conditions of Theorem* 2.1 *are satisfied and* $u_0 \in H_{\phi,m}$ *be the initial approximation of equation* (1.1), *then if* $\|L_0^{-1}P(u_0)\| \le M_0$, $\epsilon_0 = M_0 D_0 \xi_0 < \frac{1}{2}$. *Then the equation* (1.1) *has a unique solution* u^* *in the sphere*

$$\|u-u_0\|_{\phi,m} \le \rho_0, \quad \rho > \rho_0 = M_0\left(1-\sqrt{1-2\epsilon_0}\right)/\epsilon_0$$

to which the successive approximations:

$$u_{n+1} = u_n - L_0^{-1}P(u_n)$$

of modified Newton's method converges and the rate of convergence is given by the inequality

$$\|u_n - u^*\|_{\phi,m} \le \frac{M_0\left(1-\sqrt{1-2\epsilon_0}\right)^n}{\sqrt{1-2\epsilon_0}}.$$

3. Second Case: $(\chi > 0)$

Definition 3.1 We denote by $H^*_{\phi,m}$ to the class of all functions $u(s)$, represented in the form $u(s) = |s-c|^{-\alpha_0}u_*(s)$ in the neighborhood of the end points a and b, where $-1 \leq \alpha_0 < 1$, $c = a$ or b and $u_*(s) \in H_{\phi,m}[a,b]$. The set of all possible solutions of equation (1.1) can be divided into the following subclasses;

- $H^*_{\phi,m}(0)$ is the subclass of the functions from $H^*_{\phi,m}[a,b]$ not limiting near the end points a and b.
- $H^*_{\phi,m}(a)(H^*_{\phi,m}(b))$ is the subclass of the functions from $H^*_{\phi,m}[a,b]$ bounded near the end point $a(b)$.
- $H^*_{\phi,m}(a,b)$ is the subclass of the functions from $H^*_{\phi,m}[a,b]$ bounded near the end points a and b, vanishing at these points.

Now we are looking for the solution of equation (1.1) in the class $H^*_{\phi,m}(a,b)$.

Lemma 3.1 *If the functions $F(s,u)$ and $G(s,u)$ satisfy the conditions of Lemma 1.1 and $G(a,u(a))$=$G(b,u(b)) = 0$, Then the operator $P(u)$ is Frechet differentiable in the space $H^*_{\phi,m}(a,b)$ and its derivative is given by:*

$$P'(u)h(s) = F'_u(s,u(s))h(s) - \frac{1}{\pi}\int_a^b \frac{G'_u(\sigma,u(\sigma))}{\sigma - s}h(\sigma)d\sigma, \qquad (3.1)$$

where $h(s)$ be an arbitrary element in $H^*_{\phi,m}(a,b)$, and satisfies Lipschitz condition

$$\|P'(u_2) - P'(u_1)\|_{H^*_{\phi,m}} \leq \xi_0 \|u_2 - u_1\|_{H^*_{\phi,m}}$$

in the sphere

$$N_{H^*_{\phi,m}}(u_0,\rho) = (u \in H^*_{\phi,m}, \|u - u_0\|_{H^*_{\phi,m}} < \rho).$$

Proof. Similarly as Lemma 2.1. □

Theorem 3.1 *If the functions $F(s,u(s))$ and $G(s,u(s))$ satisfy the conditions of Lemma 3.1, $\frac{G'_u(c,u(c))}{F'_u(c,u(c))} > 0$ and $F'^2_u(s,u(s)) + G'^2_u(s,u(s)) \neq 0$. Then the linear operator*

$$L_0 h = F'_u(s,u_0(s))h(s) - \frac{1}{\pi}\int_a^b \frac{G'_u(\sigma,u_0(\sigma))}{\sigma - s}h(\sigma)d\sigma \qquad (3.2)$$

has a bounded inverse L_0^{-1} for any fixed point $u_0 \in H^*_{\phi,m}(a,b)$.

Proof. To find the operator L_0^{-1}, we investigate the solvability of the equation,

$$F'_u(s,u_0(s))h(s) - \frac{1}{\pi}\int_a^b \frac{G'_u(\sigma,u_0(\sigma))}{\sigma - s}h(\sigma)d\sigma = g(s) \tag{3.3}$$

where $u_0 \in H^*_{\phi,m}(a,b)$ be a fixed point and $g(s) \in H^*_{\phi,m}(a,b)$ be an arbitrary element. As the above case, we obtain the boundary condition (2.6). The canonical function $X_1(z)$ which is the solution of the homogeneous Riemann problem of equation (2.6), near and at c is bounded and having finite degree at infinity has the form:

$$X_1(z) = (a-z)^{\lambda_1}(b-z)^{\lambda_2}\exp\left(\int_a^b \frac{\theta(\sigma)}{\sigma - z}d\sigma\right),$$

where

$$\theta(s) = \frac{1}{\pi}\arctan\frac{G'_u(s,u_0(s))}{F'_u(s,u_0(s))}$$

and λ_1, λ_2 are selecting integers satisfying the conditions

$$0 < \lambda_1 - \theta(a) < 1, \quad 0 < \lambda_2 + \theta(b) < 1$$

The number $\chi = -(\lambda_1 + \lambda_2)$ is called the index of the equation (3.3). Hence

$$\begin{aligned} X_1(s) &= (a-s)^{\alpha}(b-s)^{\beta}\exp\Big\{\Big(\theta(a) - \theta(s)\ln(a-s) \\ &\quad + (\theta(s) - \theta(b))\ln(b-s) + \int_a^b \frac{\theta(\sigma)-\theta(s)}{\sigma - s}d\sigma\Big)\Big\} \end{aligned}$$

where $\alpha = \lambda_1 - \theta(a)$ and $\beta = \lambda_2 + \theta(b)$.

The unique solution of equation (3.3) is obtained in the subclass $\tilde{H}{}^*_{\phi,m}(a,b)$ of $H^*_{\phi,m}(a,b)$ where

$$\begin{aligned} \tilde{H}{}^*_{\phi,m}(a,b) &= \Big\{h \in H^*_{\phi,m}(a,b) : \int_a^b \sigma^{k-1}G'_u(\sigma,u_0(\sigma))h(\sigma)d\sigma = 0, \\ &\qquad k = 1,\dots,\chi - 1\Big\}, \end{aligned}$$

and this solution has the form

$$\begin{aligned} h(s) &= \frac{1}{K_1(s)}\Big(F'_u(s,u_0)g(s) + W_1(s)\frac{1}{\pi}\int_a^b \frac{G'_u(\sigma,u_0(\sigma))}{W_1(\sigma)(\sigma - s)}g(\sigma)d\sigma\Big) \\ &= [L_0^{-1}(u_0)]g(s), \end{aligned}$$

where

$$K_1(s) = F_u^{'2}(s, u_0(s)) + G_u^{'2}(s, u_0(s))$$

and

$$W_1(s) = X_1^+(s)\left(F_u^{'2}(s, u_0(s)) - G_u^{'2}(s, u_0(s))\right).$$

As the preceding case we have $\|L_0^{-1}\|_{\phi,m} \le D_1$ and $\|P_0(u)\| \le N_1$. Thus the following theorem is valid. □

Theorem 3.2 *Let the conditions of Theorem* 3.1 *are satisfied and* $u_0 \in \overset{\sim *}{H}_{\phi,m}(a,b)$ *be the initial approximation of equation* (1.1)*, then, if*

$$\|L_0^{-1}P(u_0)\| \le M_1, \quad \epsilon_1 = M_1 D_1 \xi_0 < \frac{1}{2}.$$

Then the equation (1.1) *has a unique solution* u^{**} *in the sphere*

$$\|u - u_0\|_{\phi,m} \le \rho_1, \quad \rho > \rho_1 = M_1\left(1 - \sqrt{1 - 2\epsilon_1}\right)/\epsilon_1$$

to which the successive approximations: $u_{n+1} = u_n - L_0^{-1}P(u_n)$ *of modified Newton's method converges and the rate of convergence is given by the inequality*

$$\|u_n - u^{**}\|_{\phi,m} \le \frac{M_1\left(1 - \sqrt{1 - 2\epsilon_1}\right)^n}{\sqrt{1 - 2\epsilon_1}}.$$

4. Third Case: ($\chi < 0$)

Theorem 4.1 *If the functions* $F(s, u(s))$ *and* $G(s, u(s))$ *satisfy the conditions of Lemma* 3.1, $\frac{G_u'(c,u(c))}{F_u'(c,u(c))} < 0$ *and* $F_u^{'2}(s, u(s)) + G_u^{'2}(s, u(s)) \neq 0$. *Then the linear operator* (3.2) *has abounded inverse from the space* Q *into* $H_{\phi,m}^*(a,b)$*, where* $Q = \{q : q = (u, c_0, c_1, \ldots, c_{-\chi-1});\ u \in H_{\phi,m}^*(a,b)$ *and* $c_0, c_1, \ldots, c_{-\chi-1}$ *are complex numbers*$\}$.

Proof. In this case the function $X_1(z)$ has at infinity a pole of order $(-\chi)$, to obtain a solution of equation (3.3) we must using the following conditions:

$$\int_a^b \frac{\sigma^{m-1}A(\sigma)}{X_1^{\pm}(\sigma)}\, d\sigma = 0, \quad m = 1, \ldots, -\chi.$$

Then $L_0(u_0) : H^*_{\phi,m}(a,b) \to H^*_{\phi,m}(a,b)$ in general has no inverse for arbitrary element $g(s) \in H^*_{\phi,m}(a,b)$. From [10] consider the equation

$$T(q) = P(u) - \sum_{k=0}^{-\chi-1} c_k\, s^k, \tag{4.1}$$

where s^k, $k = 0, 1, \ldots, -\chi - 1$, are linear independence solutions of the equation $L_0^{-1} g = 0$. We define

$$\|q\|_Q = \|u\|_{H^*_{\phi,m}} + \sum_{k=0}^{-\chi-1} |c_k|$$

The Frechet derivative of the operator $T(q)$ at arbitrary point q is given by

$$T'(q)f = P'(u)h + \sum_{k=0}^{-\chi-1} d_k\, s^k, \quad f = (h,\, d_0, d_1, \ldots, d_{-\chi-1}).$$

Moreover, $T'(q)$ satisfies Lipschitz condition in the sphere $N(q_0, \delta^*)$ of the form

$$\|T'(q_1) - T'(q_2)\|_{Q \to H^*_{\phi,m}} \le \xi_0 \|q_1 - q_2\|; \quad q_1, q_2 \in N(q_0, \delta^*).$$

The linear singular integral equation

$$T'(q_0)f = L_0(h) + \sum_{k=0}^{-\chi-1} d_k\, s^k = g(s)$$

has a unique solution $f = (h,\, d_0, d_1, \ldots, d_{-\chi-1}) \in Q$ for arbitrary right part $g(s) \in H^*_{\phi,m}(a,b)$. Hence there exists inverse operator.

$$[T'(q_0)]^{-1} : H^*_{\phi,m}(a,b) \to Q.$$

The unknowns $d_0, d_1, \ldots, d_{-\chi-1}$ are defined from the following relation

$$d_K = \sum_{m=0}^{-\chi-1} \frac{\Delta_{k,m}(-\chi)}{\Delta(-\chi)} D_m, \quad k = 0, 1, \ldots, -\chi - 1$$

where

$$\begin{aligned} D_m &= \int_a^b \frac{G'_u(\sigma, u_0(\sigma))g(\sigma)}{W_1(\sigma)}\, \sigma^m d\sigma,\ \Delta(-\chi) \\ &= \det[\tau_{m,0}, \tau_{m,1}, \ldots, \tau_{m,-\chi-1}]_{m=0}^{-\chi-1} \ne 0 \end{aligned}$$

where

$$\tau_{m,k} = \int_a^b \frac{G_u'(\sigma, u_0(\sigma))\sigma^k}{W_1(\sigma)} \sigma^m d\sigma$$

and $\triangle_{k,m}(-\chi)$ be the cofactor of the element in the kth row and mth column in the determinant $\triangle(-\chi)$. Therefore the following theorem is valid. □

Theorem 4.2 *Let the conditions of Theorem* 4.1 *are satisfied and* $u_0 \in H^*_{\phi,m}(a,b)$, $q_0 = (u_0, c_{0,0}, \ldots, c_{-\chi-1,0})$,

$$\|T'(q_0)^{-1}\|_{H^*_{\phi,m}(a,b)\to Q} < D_2 \quad \textit{and}$$
$$\|[T'(q_0)]^{-1}T(q_0)\|_{Q\to Q} \leq M_2.$$

If $\epsilon_2 = M_2 D_2 \xi_0 < \frac{1}{2}$. *Then the equation* (4.1) *has a unique solution* $q^* = (u^{**}, c_0^*, \ldots, c_{-\chi-1}^*) \in N(q_0, \delta_0)$ *to which the iteration process converges*

$$q_{n+1} = q_n - [T'(q_0)]^{-1}T(q_n), \quad n = 0, 1, \ldots,$$

where

$$\delta^* \geq \delta_0 = M_2(1 - \sqrt{1 - 2\epsilon_2})/\epsilon_2$$

Moreover, the rate of convergence of $q_n \in Q$ *to* $q^* \in Q$ *given by the inequality*

$$\|q_n - q^*\|_Q \leq \frac{M_2\,(1 - \sqrt{1 - 2\epsilon_2})^n}{\sqrt{1 - 2\epsilon_2}}. \tag{4.2}$$

Lemma 4.1 *Let the conditions of Theorem* 4.2 *are satisfied then the values* $c_{0,n}, \ldots, c_{-\chi-1,n}$ *tends to zero as* n *tends to infinity iff when* $u^{**} \in H^*_{\phi,m}(a,b)$ *is the solution of equation* (1.1).

Proof. Let $q^* = (u^{**}, c_0^*, \ldots, c_{-\chi-1}^*) \in Q$ is the solution of equation (4.1), if u^{**} is the solution of equation (1.1), then $\sum\limits_{k=0}^{-\chi-1} c_k^* s^k = 0$ from here $c_0^* = c_1^* = \cdots = c_{-\chi-1}^* = 0$ by the linearly independence of the functions s^k, $k = 0, 1, \ldots, -\chi - 1$. Then it follows from the inequality (4.2) that the values $c_{0,n}, \ldots, c_{-\chi-1,n}$ tend to zero as n tends to infinity. If the values

$c_{0,n}, \ldots, c_{-\chi-1,n}$ tend to zero as n tends to infinity, then by the inequality

$$\sum_{k=0}^{-\chi-1} |c_{k,n} - c_k^*| \le \frac{M_2\,(1-\sqrt{1-2\epsilon_2})^n}{\sqrt{1-2\epsilon_2}}.$$

it follows that $c_0^* = c_1^* = \cdots = c_{-\chi-1}^* = 0$. Then u^{**} is the solution of equation (1.1). From Theorem 4.2 and Lemma 4.1 we have q^* is the solution of equation (4.1), $q_n = (u_n, c_{0,n}, \ldots, c_{-\chi-1,n})$ is its approximate solution and u^{**} is the solution of equation (1.1), then the sequence $\{u^n\}$ is naturally taken as the approximate solution of equation (1.1). Moreover, the following inequality is valid

$$\|u_n - u^{**}\| \le \frac{M_2\,(1-\sqrt{1-2\epsilon_2})^n}{\sqrt{1-2\epsilon_2}}.$$

□

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