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Global attractivity of a nonautonomous discrete logistic model

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Abstract. In this paper we consider the nonautonomous discrete logistic model

$$x_{n+1}=x_{n}\exp[r_{n}(1-x_{n})], \quad n\in N, \tag{1.1}$$

where $\{r_{n}\}$ is a sequence of nonnegative numbers. We obtain some sufficient conditions for an arbitrary solution $\{x_{n}\}$ satisfying the initial condition

$$x_{0}=a>0, \tag{1.2}$$

to converge to 1 as $n\rightarrow\infty$. Under appropriate hypotheses, the necessary and sufficient conditions for any solution of (1.1) with (1.2) tending to 1 as $n\rightarrow\infty$ have also been obtained.

Key words: discrete nonautonomous logistic model, global attractivity.

1. Introduction

Consider the discrete nonautonomous logistic model

$$x_{n+1}=x_{n}\exp[r_{n}(1-x_{n})], \quad n\in N, \tag{1.1}$$

where $\{r_{n}\}$ is a sequence of nonnegative numbers. It is easy to see that, for any given initial condition

$$x_{0}=a>0, \tag{1.2}$$

Eq. (1.1) has an unique solution $\{x_{n}\}$ which is positive for all $n\in N$ and satisfies (1.2). In [1], it was proved that every solution of (1.1) with (1.2) tends to 1 if $r_{n}\leq 3/2$ and $\sum_{n=0}^{\infty}r_{n}=\infty$.

When $r_{n}\equiv r>0$, Eq. (1.1) reduces to

$$x_{n+1}=x_{n}\exp[r(1-x_{n})], \quad n\in N, \tag{1.3}$$

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which has been studied in the literature in its own right as a discrete population model of a single species with non-overlapping generations. It was shown in [2, 3] that for some values of the parameter r, solutions of Eq. (1.3) are "chaotic". It was also proved in [4] that any solution of Eq. (1.3) with (1.2) converges to 1 as $n \to \infty$ if and only if $r \leq 2$.

In this paper, we discuss Eq. (1.1) and obtain the following results.

Theorem 1.1 *If*

$$\sum_{n=0}^{\infty} r_n = \infty, \tag{1.4}$$

and

$$\limsup_{n\to\infty} r_n \leq 2. \tag{1.5}$$

Then any solution $\{x_n\}$ *of Eq.* (1.1) *with* (1.2) *converges to* 1 *as* $n \to \infty$.

Theorem 1.2 *Assume that* $\{r_n\}$ *is bounded. If*

$$\liminf_{n\to\infty} r_n > 2, \tag{1.6}$$

then every nontrivial solution $\{x_n\}$ *of Eq.* (1.1) *with* (1.2) *cannot converge to* 1 *as* $n \to \infty$.

Combining Theorem 1.1 and 1.2, we obtain the following necessary and sufficient conditions, that is

Corollary 1.1 *Assume that* (1.4) *holds and the limit* $\lim_{n\to\infty} r_n$ *exists. Then any solution* $\{x_n\}$ *of Eq.* (1.1) *with* (1.2) *converges to* 1 *as* $n \to \infty$ *if and only if*

$$\lim_{n\to\infty} r_n \leq 2. \tag{1.7}$$

2. Proofs of Theorem 1.1 and 1.2

First, we establish the following lemma.

Lemma 2.1 *Assume that* r *is a nonnegative constant, let*

$$f(x) = x(\exp[r(1-x)] + 1) - 2.$$

If there exists a constant x^* *such that*

$$f(x^*)(x^* - 1) < 0, \tag{2.1}$$

then

$$(x^* - 1)^2 < \frac{3}{2}(r - 2). \tag{2.2}$$

Proof. Clearly, $f(1) = 0$, $f(x) \le -2$ for $x \le 0$, and $f(x) \ge 0$ for $x \ge 2$. Since (2.1) holds, we see that either $x^* \in (0,1)$ or $x^* \in (1,2)$.

If $x^* \in (0,1)$, by (2.1), we know that $f(x^*) > 0$, this implies

$$\begin{aligned} r &> \frac{1}{1-x^*} \ln \frac{2-x^*}{x^*} \\ &= \frac{1}{1-x^*} \ln \frac{1+(1-x^*)}{1-(1-x^*)} \\ &= \frac{1}{1-x^*} \left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(1-x^*)^k}{k} + \sum_{k=1}^{\infty} \frac{(1-x^*)^k}{k} \right) \\ &\ge \frac{2}{1-x^*} \left((1-x^*) + \frac{(1-x^*)^3}{3} \right) \\ &= 2 + \frac{2}{3}(1-x^*)^2, \end{aligned}$$

which leads to (2.2).

If $x^* \in (1,2)$, by (2.1), we know that $f(x^*) < 0$, this implies that

$$\begin{aligned} r &> \frac{1}{x^*-1} \ln \frac{x^*}{2-x^*} \\ &= \frac{1}{x^*-1} \ln \frac{1+(x^*-1)}{1-(x^*-1)} \\ &\ge 2 + \frac{2}{3}(x^*-1)^2. \end{aligned}$$

So, (2.2) holds.

The proof of Lemma 1.1 is now complete. □

Proof of Theorem 1.1. Assume that $\{x_n\}$ is a solution of Eq. (1.1) with (1.2). Let

$$V(n) = (x_n - 1)^2, \quad n \in N. \tag{2.3}$$

Then

$$\begin{aligned}\triangle V(n) &= (x_{n+1}-1)^2-(x_n-1)^2\\ &= (x_{n+1}-x_n)(x_{n+1}+x_n-2) \qquad (2.4)\\ &= x_n(\exp[r_n(1-x_n)]-1)(x_n(\exp[r_n(1-x_n)]+1)-2),\end{aligned}$$

here $\triangle$ denotes the forward difference operator defined by $\triangle V(n) = V(n+1) - V(n)$.

Since

$$(\exp[r_n(1-x_n)]-1)(x_n-1) \leq 0, \quad n \in N. \qquad (2.5)$$

We claim that, for any $m \in N$,

$$\triangle V(m) > 0 \quad \text{implies} \quad V(m) < \frac{3}{2}(r_m-2). \qquad (2.6)$$

In fact, if $\triangle V(m) > 0$, by (2.4) and (2.5), we have

$$(x_m(\exp[r_m(1-x_m)]+1)-2)(x_m-1) < 0,$$

this leads to, by Lemma 1.1, that

$$(x_m-1)^2 < \frac{3}{2}(r_m-2).$$

So (2.6) holds.

We consider three possible cases.

Case 1: There is a $n^* \in N$, such that $\triangle V(n) > 0$ for $n \geq n^*$.

In this case, by (2.6), we have

$$V(n) < \frac{3}{2}(r_n-2) \quad \text{for} \quad n \geq n^*. \qquad (2.7)$$

By this and (1.5), we know that $\limsup_{n\to\infty} V(n) \leq 0$. Since $V(n) \geq 0$ for $n \in N$, we see that $\lim_{n\to\infty} V(n) = 0$, which is equivalent to $\lim_{n\to\infty} x_n = 1$.

Case 2: There is a $n^* \in N$, such that $\triangle V(n) \leq 0$ for $n \geq n^*$.

In this case, $\{V(n)\}$ is nonincreasing for $n \geq n^*$. Since $V(n) \geq 0$, we see that $\lim_{n\to\infty} V(n)$ exists. Let

$$\alpha = \lim_{n\to\infty} V(n),$$

then

$$\lim_{n\to\infty} |x_n - 1| = \sqrt{\alpha}. \tag{2.8}$$

Denote $\beta = \sqrt{\alpha}$, we shall prove that $\beta = 0$.

In fact, if $\beta > 0$, we consider three subcases.

Subcase a: $x_n - 1 > 0$ for large n.

In this subcase, by (2.8), we have

$$\lim_{n\to\infty} x_n = 1 + \beta. \tag{2.9}$$

There is a sufficient large integer m_1, such that

$$x_n - 1 \geq \frac{\beta}{2} \quad \text{for} \quad n \geq m_1. \tag{2.10}$$

By (1.1), we get

$$x_{n+1} = x_n \exp[r_n(1 - x_n)] \leq x_n \exp\left[-\frac{\beta}{2} r_n\right] \quad \text{for } n \geq m_1,$$

this leads to, for $p \in N$, that

$$x_{m_1+p+1} \leq x_{m_1} \exp\left[-\frac{\beta}{2} \sum_{i=m_1}^{m_1+p} r_i\right]. \tag{2.11}$$

Let $p \to \infty$ in (2.11), and noting (1.4), we get

$$\lim_{p\to\infty} x_{m_1+p+1} = 0.$$

This contradicts (2.9), so subcase a is impossible.

Subcase b: $x_n - 1 < 0$ for large n.

In this case, by (2.8), we have

$$\lim_{n\to\infty} x_n = 1 - \beta. \tag{2.12}$$

There is a sufficient large integer m_2 such that

$$1 - x_n \geq \frac{\beta}{2} \quad \text{for} \quad n \geq m_2. \tag{2.13}$$

By (1.1), we get

$$x_{n+1} = x_n \exp[r_n(1 - x_n)] \geq x_n \exp\left[\frac{\beta}{2} r_n\right] \quad \text{for } n \geq m_2.$$

So, for $p \in N$, that

$$x_{m_2+p+1} \geq x_{m_2} \exp\left[\frac{\beta}{2}\sum_{i=m_2}^{m_2+p} r_i\right]. \tag{2.14}$$

Let $p \to \infty$ in (2.14), and noting (1.4), we are led to

$$\lim_{p\to\infty} x_{m_2+p+1} = \infty,$$

which contradicts (2.12). So subcase b is impossible.

Subcase c: There is a sequence $\{n_i\}$ of positive integers, such that

$$x_{n_i} - 1 < 0, \quad x_{n_i+1} - 1 > 0.$$

Thus

$$\lim_{i\to\infty} x_{n_i} = 1 - \beta, \quad \lim_{i\to\infty} x_{n_i+1} = 1 + \beta. \tag{2.15}$$

Since

$$x_{n_i+1} = x_{n_i} \exp[r_{n_i}(1 - x_{n_i})],$$

by (2.15), we have

$$\begin{aligned}
\lim_{i\to\infty} r_{n_i} &= \lim_{i\to\infty} \frac{1}{1 - x_{n_i}} \ln \frac{x_{n_i+1}}{x_{n_i}} \\
&= \frac{1}{\beta} \ln \frac{1+\beta}{1-\beta} \\
&= \frac{1}{\beta}\left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\beta^k}{k} + \sum_{k=1}^{\infty}\frac{\beta^k}{k}\right) \\
&= 2\sum_{k=0}^{\infty}\frac{\beta^{2k}}{2k+1} \\
&> 2.
\end{aligned}$$

This contradicts (1.5), and subcase c is impossible.

According to the above discussions, we know that $\beta = 0$ and so $\lim_{n\to\infty} x_n = 1$ for case 2.

Case 3: There exists a sequence $\{n_j\}$ of integers, such that

$$\triangle V(n_1) \leq 0, \quad \triangle V(n) > 0 \text{ for } n_{2k-1} + 1 \leq n \leq n_{2k},$$

$$\Delta V(n) \le 0 \quad \text{for} \quad n_{2k}+1 \le n \le n_{2k+1}, \quad k = 1, 2, \dots .$$

In this case, we see that

$$V(n) \le V(n_{2k}+1) \quad \text{for} \quad n_{2k-1}+1 \le n \le n_{2k+1}, \quad k = 1, 2, \dots . \tag{2.16}$$

Since $\Delta V(n_{2k}) > 0$ for $k = 1, 2, \dots$, by (2.6), we get

$$V(n_{2k}) < \frac{3}{2}(r_{n_{2k}} - 2), \quad k = 1, 2, \dots . \tag{2.17}$$

By (1.5), this implies that $\lim_{k\to\infty} V(n_{2k}) = 0$, that is,

$$\lim_{k\to\infty} x_{n_{2k}} = 1. \tag{2.18}$$

By (1.1),

$$x_{n_{2k}+1} = x_{n_{2k}} \exp[r_{n_{2k}}(1 - x_{n_{2k}})],$$

so

$$\lim_{k\to\infty} x_{n_{2k}+1} = 1 \quad \text{and} \quad \lim_{k\to\infty} V(n_{2k}+1) = 0.$$

Noting (2.16), it is obvious that

$$\lim_{n\to\infty} V(n) = 0, \quad \text{i.e.} \quad \lim_{n\to\infty} x_n = 1.$$

In view of the above three cases, we know that Theorem 1.1 holds. This completes the proof. □

Proof of Theorem 1.2. Assume, for the sake of contradiction, that (1.1) has a nontrivial solution $\{x_n\}$ such that $\lim_{n\to\infty} x_n = 1$.

By (1.1), we have

$$x_{n+1} - 1 = (x_n - 1)\exp[r_n(1 - x_n)] + \exp[r_n(1 - x_n)] - 1,$$

so,

$$\frac{|x_{n+1} - 1|}{|x_n - 1|} \ge \frac{|\exp[r_n(1 - x_n)] - 1|}{|r_n(1 - x_n)|} r_n - \exp[r_n(1 - x_n)]. \tag{2.19}$$

Since $\{r_n\}$ is bounded, we know $\lim_{n\to\infty} r_n(1 - x_n) = 0$. By (2.19) and (1.6), we get

$$\liminf_{n\to\infty} \frac{|x_{n+1} - 1|}{|x_n - 1|} \ge \liminf_{n\to\infty} r_n - 1 > 1,$$

which leads to

$$\lim_{n\to\infty} |x_n - 1| = \infty.$$

This is a contradiction. Therefore, Theorem 1.2 holds, the proof is complete. □

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