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Higher derivatives of holomorphic function with positive real part

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Abstract. Upper estimates of $|f^{(n)}(z)|/\operatorname{Re} f(z)$, $n \geq 2$, of f holomorphic and $\operatorname{Re} f > 0$ in a plane domain are proposed; the equality conditions are considered in detail.

Key words: function with positive real part, Schwarz-Pick's lemma, hyperbolic domain, Poincaré density, radius of univalency.

1. Introduction

Let $\mathcal{P}(\Omega)$ be the family of functions f holomorphic with positive real part $\operatorname{Re} f > 0$ in a domain Ω in the complex plane $\mathbf{C} = \{z; |z| < +\infty\}$. We shall prove some sharp upper estimates of the quotient $|f^{(n)}(z)|/\operatorname{Re} f(z)$ for $f \in \mathcal{P}(\Omega)$ at $z \in \Omega$ for $n \geq 2$, together with the detailed equality conditions.

The specified case $n = 1$ and $\Omega = D \equiv \{z;\ |z| < 1\}$ is well known. For $f \in \mathcal{P}(D)$,

$$\frac{|f'(z)|}{\operatorname{Re} f(z)} \leq \frac{2}{1 - |z|^2} \tag{1.1}$$

at each $z \in D$. The extremal functions are essentially $\ell_\alpha(z) = (1+\alpha z)/(1-\alpha z)$, where $\alpha \in \partial D \equiv \{z;\ |z| = 1\}$. More precisely, if the equality holds in (1.1) at a point $z \in D$, then

$$f(w) \equiv \frac{1 - \bar{a}w + \beta(w-a)}{1 - \bar{a}w - \beta(w-a)},$$

where $\beta \in \partial D$ and $a \in D$, so that the equality holds in (1.1) everywhere in D; one can prove that $f = A\ell_\gamma + iB$, where

$$A = \frac{1 - |a|^2}{|1 + a\beta|^2} > 0, \quad B = \frac{-2\ \operatorname{Im}(a\beta)}{|1 + a\beta|^2}, \quad \text{and} \quad \gamma = \frac{\beta + \bar{a}}{1 + a\beta} \in \partial D.$$

See Section 4 for the details on (1.1).

We begin with the case $\Omega = D$ and $n \geq 2$.

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Theorem 1 *For $f \in \mathcal{P}(D)$ the estimate*

$$\frac{|f^{(n)}(z)|}{\operatorname{Re} f(z)} \leq \frac{\ell_1^{(n)}(|z|)}{\operatorname{Re} \ell_1(|z|)} = \frac{n!2}{(1-|z|^2)(1-|z|)^{n-1}} \tag{1.2}$$

holds at each point $z \in D$ and for all $n \geq 2$. If the equality holds in (1.2) at a point $z \in D$ and for an $n \geq 2$, then $f = A\ell_\alpha + iB$, for an $\alpha \in \partial D$, and for $A > 0$ and B both real constants. Conversely, if $f = A\ell_\alpha + iB$, $\alpha \in \partial D$; $A > 0$ and B both real constants, then the equality holds in (1.2) at each point of the radius

$$\mathcal{R}(\alpha) = \{\overline{\alpha}t; 0 \leq t < 1\}$$

and for each $n \geq 2$, whereas the inequality (1.2) is strict at each point of $D \setminus \mathcal{R}(\alpha)$ and for each $n \geq 2$.

In Section 5 we shall prove Theorem 2 which is proposed in Section 4 and is a version of Theorem 1 for a hyperbolic domain Ω again with the detailed equality conditions.

2. Proof of Theorem 1

We begin with a lemma.

Lemma 1 *Let f be holomorphic in D, let $0 < \rho \leq 1$, and let $A \neq 0$ and B both be complex constants. Set for a fixed $z \in D$,*

$$\frac{f\left(\dfrac{\rho w + z}{1 + \overline{z}\rho w}\right) - B}{A} = \sum_{k=0}^{\infty} b_k w^k, \quad w \in D. \tag{2.1}$$

Then for each $n \geq 1$,

$$\frac{f^{(n)}(z)}{n!} = \frac{A}{\rho^n(1-|z|^2)^n} \sum_{k=0}^{n-1} \binom{n-1}{k} (\overline{z}\rho)^{n-1-k} b_{k+1}. \tag{2.2}$$

Proof. Let $g(w)$ be the left-hand side function in (2.1) of the variable $w \in D$. Set

$$\zeta = \frac{\rho w + z}{1 + \overline{z}\rho w}, \quad w \in D, \quad \text{so that} \quad d\zeta = \frac{\rho(1-|z|^2)}{(1+\overline{z}\rho w)^2} dw.$$

Observe that

$$\frac{(1+\bar{z}\rho w)^{n-1}}{w^{n+1}} = \sum_{k=0}^{n-1} \binom{n-1}{k} (\bar{z}\rho)^{n-1-k} w^{-k-2} \quad \text{for} \quad w \neq 0.$$

Then

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i} \int\limits_{\left|\frac{\zeta - z}{1-\bar{z}\zeta}\right| = \frac{\rho}{2}} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

$$= \frac{A}{\rho^n (1-|z|^2)^n} \sum_{k=0}^{n-1} \binom{n-1}{k} (\bar{z}\rho)^{n-1-k} \frac{1}{2\pi i} \int\limits_{|w|=\frac{1}{2}} \frac{g(w)}{w^{k+2}} dw.$$

This is (2.2).

Let $\mathcal{P}_o$ be the family of functions $f \in \mathcal{P}(D)$ with $f(0) = 1$. A typical member of $\mathcal{P}_o$ is ℓ_α, $\alpha \in \partial D$. For

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$$

of $\mathcal{P}_o$ we have the estimate $|a_k| \leq 2$ for all $k \geq 1$ and furthermore, $|a_1| = 2$ if and only if $f = \ell_\alpha$ for an $\alpha \in \partial D$. See [G, p. 80]; the estimate $|a_1| \leq 2$ follows from the Schwarz inequality: $|g'(0)| \leq 1$ for $g = (f-1)/(f+1)$; the equality holds if and only if $g(z) \equiv \alpha z$ for an $\alpha \in \partial D$ or $f = \ell_\alpha$.

Simple computation shows that

$$\ell_\alpha\left(\frac{w-b}{1-\bar{b}w}\right) \equiv \frac{1-|b|}{1+|b|} \ell_\alpha(w) \tag{2.3}$$

in D for all $\alpha \in \partial D$ and all $b \in \mathcal{R}(\alpha)$. □

Proof of Theorem 1. Fix $z \in D$ and let

$$g(w) = \frac{f\left(\dfrac{w+z}{1+\bar{z}w}\right) - i \operatorname{Im} f(z)}{\operatorname{Re} f(z)} = 1 + \sum_{k=1}^{\infty} b_k w^k. \tag{2.4}$$

We apply Lemma 1 to g with $\rho = 1$, $A = \operatorname{Re} f(z)$, and $B = i \operatorname{Im} f(z)$. Then

$$\frac{f^{(n)}(z)}{n!} = \frac{\operatorname{Re} f(z)}{(1-|z|^2)^n} \sum_{k=0}^{n-1} \binom{n-1}{k} \bar{z}^{n-1-k} b_{k+1}, \tag{2.5}$$

which, together with $g \in \mathcal{P}_o$, yields that

$$\frac{|f^{(n)}(z)|}{n!} \leq \frac{2(1+|z|)^{n-1}\operatorname{Re} f(z)}{(1-|z|^2)^n}; \tag{2.6}$$

this is equivalent to (1.2). □

Suppose that the equality holds in (1.2) or in (2.6) at z and for an $n \geq 2$. Then $|b_{k+1}| = 2$ for $0 \leq k \leq n-1$, so that $|b_1| = 2$. Hence $g = \ell_\alpha$ for an $\alpha \in \partial D$. Since $b_{k+1} = 2\alpha^{k+1}$ for all $k \geq 0$, it follows that

$$\frac{f^{(m)}(z)}{m!} = \frac{2\alpha(\overline{z}+\alpha)^{m-1}\operatorname{Re} f(z)}{(1-|z|^2)^m} \tag{2.7}$$

for all $m \geq 1$. Since the equality holds in (2.6) we then have

$$|\overline{z}+\alpha|^{n-1} = (1+|z|)^{n-1},$$

so that $z \in \mathcal{R}(\alpha)$. It then follows from (2.3) for $b = z$ that

$$f(w) = \left(\operatorname{Re} f(z)\right)\ell_\alpha\left(\frac{w-z}{1-\overline{z}w}\right) + i\operatorname{Im} f(z) = A\ell_\alpha(w) + iB,$$

where

$$A = \frac{1-|z|}{1+|z|}\operatorname{Re} f(z) > 0 \quad \text{and} \quad B = \operatorname{Im} f(z).$$

Conversely given $f = A\ell_\alpha + iB$, $\alpha \in \partial D$, $A > 0$, B both real constants, and given $n \geq 2$, we have the chain of identities

$$\frac{n!2}{(1-|z|^2)|1-\alpha z|^{n-1}} = \frac{|f^{(n)}(z)|}{\operatorname{Re} f(z)} = \frac{n!2}{(1-|z|^2)(1-|z|)^{n-1}}$$

if and only if $z \in \mathcal{R}(\alpha)$.

3. Application of Theorem 1

Suppose that $h > 0$ is harmonic in D. Then we have a holomorphic function f with $\operatorname{Re} f = h$ in D. Since

$$f^{(n)}(z) = 2\frac{\partial^n h(z)}{\partial z^n},$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

for $z = x + iy$, $n \geq 1$, we have

$$\frac{|f^{(n)}|}{\operatorname{Re} f} = \frac{2}{h}\left|\frac{\partial^n h}{\partial z^n}\right|. \tag{3.1}$$

We then have the estimate of the right-hand side of (3.1) with the aid of (1.1) and (1.2), together with the detailed equality conditions. Notice that, in the case where $n = 1$,

$$\frac{|f'|}{\operatorname{Re} f} = \Big|\operatorname{grad}(\log h)\Big|,$$

where for $g = \log h$,

$$|\operatorname{grad} g| = \sqrt{g_x^2 + g_y^2}.$$

Let Γ be the family of f holomorphic in D such that $f(z) + f(w) \neq 0$ for all $z, w \in D$. In particular, $f \in \Gamma$ never vanishes in D and $\mathcal{P}(D) \subset \Gamma$. We call $f \in \Gamma$ a Gel'fer function. In [Y1, Theorem 5, p. 254] we proved that if $f(0) = 1$ for $f \in \Gamma$, then $\operatorname{Re} f(z) > 0$ in the disk $\{|z| < 1/\sqrt{2}\}$. The constant $1/\sqrt{2}$ is sharp. For $p = (1+i)\,/\sqrt{2}$, the function $f(z) = (1 - \overline{p}z)/(1 + pz)$ is in Γ, $f(0) = 1$, and further $\operatorname{Re} f(i/\sqrt{2}) = 0$.

If $f \in \Gamma$ and $z \in D$, then for a constant ρ, $0 < \rho \leq 1$, the function

$$g(w) = \frac{f\left(\dfrac{\rho w + z}{1 + \overline{z}\rho w}\right)}{f(z)}$$

of $w \in D$ is in Γ with $g(0) = 1$. Since $|g'(0)| \leq 2$ (see, for example, [Y1, (G8)]), it follows that

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{2}{1 - |z|^2};$$

the equality holds for $f = \ell_\alpha$ at each point $z = \overline{\alpha}t$, $-1 < t < 1$.

We now obtain

Corollary to Theorem 1 *For $f \in \Gamma$ the strict inequality*

$$\left|\frac{f^{(n)}(z)}{f(z)}\right| < \frac{n!2\sqrt{2}(|z|+\sqrt{2})^{n-1}}{(1-|z|^2)^n} \tag{3.2}$$

holds for all $n \geq 2$ and at each $z \in D$.

Proof. Set $\rho = 1/\sqrt{2}$ and fix $z \in D$. Then the function

$$g(w) = \frac{f\left(\dfrac{\rho w + z}{1+\bar{z}\rho w}\right)}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k w^k$$

is in $\mathcal{P}_o$. It then follows from (2.2) with $A = f(z)$ that

$$\left|\frac{f^{(n)}(z)}{f(z)}\right| \leq \frac{n!2(1+\rho|z|)^{n-1}}{\rho^n(1-|z|^2)^n}. \tag{3.3}$$

If the equality would hold in (3.3) for an $n \geq 2$, then $|b_{k+1}| = 2$ for $0 \leq k \leq n-1$. Hence $|b_1| = 2$, so that there exists $\alpha \in \partial D$ with $g = \ell_\alpha$. Then f has $(\rho\bar{\alpha} + z)/\ (1 + \bar{z}\rho\bar{\alpha}) \in D$ as a pole. This is a contradiction. □

4. Hyperbolic domain

A domain Ω in the plane $\mathbf{C}$ is called hyperbolic if $\mathbf{C} \setminus \Omega$ contains at least two points. Let ϕ be a universal covering projection from D onto a hyperbolic domain Ω (a projection $\phi : D \to \Omega$, for short); ϕ is holomorphic and ϕ' is zero-free in D. The Poincaré density Π_Ω is then the function in Ω defined by

$$\Pi_\Omega(z) = \frac{1}{(1-|w|^2)|\phi'(w)|}, \quad z \in \Omega,$$

where $z = \phi(w)$; the choice of ϕ and w is immaterial as far as $z = \phi(w)$ is satisfied.

Let Ω and Σ both be hyperbolic domains and let $f : \Omega \to \Sigma$ be holomorphic. The Schwarz-Pick lemma is the estimate of $|f'|$ in terms of Π_Ω and Π_Σ, namely,

$$\Pi_\Sigma(f(z))|f'(z)| \leq \Pi_\Omega(z) \tag{4.1}$$

at each point $z \in \Omega$; see, for example, [Y2, p. 304]. If the equality holds at a point $z \in \Omega$, then $f \circ \phi : D \to \Sigma$ is a projection for each projection ϕ :

$D \to \Omega$, and, moreover, the equality holds in (4.1) everywhere in Ω. See, for example, [Y2] and [Y3], for recent researches on the Schwarz-Pick lemma.

The case where $\Omega = D$ and Σ is the half-plane $H = \{w; \text{Re } w > 0\}$ is of our main interest. Since

$$\Pi_D(z) = \frac{1}{1-|z|^2} \quad \text{and} \quad \Pi_H(z) = \frac{1}{2 \text{ Re } z},$$

(4.1) for $f \in \mathcal{P}(D)$ is reduced to (1.1).

To consider the higher derivatives we need a device. For a projection $\phi : D \to \Omega$ we suppose that $z = \phi(w)$. Let $\rho_\Omega(z)$ be the greatest r such that $0 < r \leq 1$ and ϕ is univalent in

$$\left\{\zeta; \left|\frac{\zeta - w}{1 - \overline{w}\zeta}\right| < r\right\}$$

which is the non-Euclidean disk of center w and the non-Euclidean radius arctanh r, and also is the disk of

$$\text{center} \quad \frac{w(1-r^2)}{1-r^2|w|^2} \in D \quad \text{and radius} \quad \frac{r(1-|w|^2)}{1-r^2|w|^2} \leq 1.$$

Again $\rho_\Omega(z)$ is independent of the particular choice of ϕ and w as far as $z = \phi(w)$ is satisfied. We may therefore call $\rho_\Omega(z)$ the radius of univalency of Ω at $z \in \Omega$. In particular, the set

$$\Delta(z) = \left\{\phi(\zeta); \left|\frac{\zeta - w}{1 - \overline{w}\zeta}\right| < \rho_\Omega(z)\right\}, \quad z = \phi(w), \tag{4.2}$$

is a simply connected domain depending only on $z \in \Omega$; $\Delta(z)$ will be considered in Section 6.

Theorem 2 *For $f \in \mathcal{P}(\Omega)$ of a hyperbolic domain $\Omega \subset \mathbf{C}$ the inequality*

$$\frac{|f^{(n)}(z)|}{\text{Re } f(z)} \leq 2 \cdot \frac{(2n-1)!}{(n-1)!} \left(\frac{\Pi_\Omega(z)}{\rho_\Omega(z)}\right)^n \tag{4.3}$$

holds for each $n \geq 2$ and at each $z \in \Omega$. If the equality holds in (4.3) at a point $z \in \Omega$ and for an $n \geq 2$, then the following two items hold.

(I) *There exist complex constants $Q \neq 0$ and R such that Ω is the slit domain*

$$\Omega = \mathbf{C} \setminus \left\{Qt + R; \, t \leq -\frac{1}{4}\right\}; \tag{4.4}$$

in particular, $\rho_\Omega(z) \equiv 1$.

(II) *The function* f *is of the form*

$$f(w) = A\sqrt{\frac{Q}{4w+Q-4R}} + iB, \tag{4.5}$$

where $A > 0$ *and* B *are real constants and the branch of* $\sqrt{\ }$ *is chosen so that* $f(R) = A + iB$.

Conversely suppose that f *of* (4.5) *is given in* Ω *of* (4.4). *Then the equality holds in* (4.3) *at each point of the half-line*

$$\mathcal{L} = \left\{Qt + R;\, t > -\frac{1}{4}\right\} \tag{4.6}$$

and for each $n \geq 2$, *whereas the inequality* (4.3) *is strict at each point of* $\Omega \setminus \mathcal{L}$ *and for each* $n \geq 2$.

The function of (4.5) maps Ω of (4.4) univalently onto H.

The inequality (4.3) in the specified case $\Omega = D$ reads that

$$\frac{|f^{(n)}(z)|}{\operatorname{Re} f(z)} \leq 2 \cdot \frac{(2n-1)!}{(n-1)!} \cdot \frac{1}{(1-|z|^2)^n} \tag{4.7}$$

at each $z \in D$ and for each $n \geq 2$. Since

$$2^{n-1} < \frac{(2n-1)!}{n!(n-1)!} \quad \text{for} \quad n \geq 2,$$

(4.7) is worse than (1.2). Hence Theorem 2 is not an extension of Theorem 1.

As preparation for the proof of Theorem 2 we begin with the class $\mathcal{S}$ of functions f holomorphic and univalent in D with $f(0) = f'(0) - 1 = 0$. Typical members of $\mathcal{S}$ are the rotations of the Koebe function $K = K_1$, namely, $K_\alpha(z) = z/(1-\alpha z)^2$, $\alpha \in \partial D$. K.S. Chua's coefficient theorem [**C**, Theorem 2] for the inverse function f^* of $f \in \mathcal{S}$ in $f(D)$ is the following. Let f^{*k} be the k-th power of f^* ($k = 1, 2, \cdots$) having the expansion

$$f^{*k}(w) = \sum_{n=k}^{\infty} B_{nk}(f) w^n$$

in a neighborhood of 0. Note that $B_{kk}(f) = 1$. Then

$$|B_{nk}(f)| \leq |B_{nk}(K)| \tag{4.8}$$

for $n \geq k \geq 1$. If $n \geq 2$ and if the equality holds in (4.8) for a pair n, k with $n > k$, then $f = K_\alpha$ for an $\alpha \in \partial D$, so that the equality holds in (4.8) for all $n \geq k \geq 1$. Chua observed that [**C**, (8) and (16)]

$$B_{nk}(K) = (-1)^{n-k}\frac{k}{n}\binom{2n}{n-k}, \quad 1 \leq k \leq n,$$

and further that

$$\sum_{k=1}^{n} |B_{nk}(K)| = \binom{2n-1}{n}. \tag{4.9}$$

For later use we remark that

$$\frac{1-K^*(w)}{1+K^*(w)} = \frac{1}{\sqrt{4w+1}} \tag{4.10}$$

because

$$K^*(w) = \frac{2w+1-\sqrt{4w+1}}{2w}$$

for $w \in K(D)$. It follows from

$$(K_\alpha)^*(w) = \overline{\alpha}K^*(\alpha w), \quad w \in K_\alpha(D),$$

that

$$B_{nk}(K_\alpha) = B_{nk}(K)\alpha^{n-k}, \quad \text{for} \quad 1 \leq k \leq n \quad \text{and} \quad \alpha \in \partial D.$$

5. Proof of Theorem 2

Supposing first that $0 \in \Omega$ and $\phi(0) = \phi'(0) - 1 = 0$ for a projection $\phi : D \to \Omega$, and further that $f(0) = 1$, we shall prove that

$$\frac{\rho_\Omega(0)^n |f^{(n)}(0)|}{n!} \leq 2\binom{2n-1}{n} \tag{5.1}$$

for all $n \geq 2$. Furthermore we shall observe that if the equality holds in (5.1) for an $n \geq 2$, then there exists $\beta \in \partial D$ such that $\Omega = K_\beta(D)$ and

$$f(w) = \frac{1}{\sqrt{4\beta w+1}} \tag{5.2}$$

for $w \in \Omega$, the branch satisfying $f(0) = 1$.

Set $\rho = \rho_\Omega(0)$. Then $\Phi(z) = \rho^{-1}\phi(\rho z)$ is a member of $\mathcal{S}$ and

$$F(z) \equiv f(\rho\Phi(z)) = f(\phi(\rho z))$$

is of $\mathcal{P}_o$. Applying the composite function theorem [T, Theorem 1] to

$$F \circ \Phi^*(\zeta) = f(\rho\zeta), \quad \zeta = \Phi(z) \in \Phi(D),$$

we have

$$\rho^n f^{(n)}(\rho\zeta) = (F \circ \Phi^*)^{(n)}(\zeta) = \sum_{k=1}^{n} A_{nk}(\zeta) F^{(k)}(\Phi^*(\zeta)), \tag{5.3}$$

where

$$A_{nk}(\zeta) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}(\Phi^*)^{k-j}(\zeta)(\Phi^{*j})^{(n)}(\zeta),$$

and further, $(\Phi^*)^{k-j}$ is the $(k-j)$-th power of Φ^* with $(\Phi^*)^0 = 1$ and $(\Phi^{*j})^{(n)}$ is the n-th derivative of the j-th power of Φ^*, $1 \le j \le k \le n$. Setting $\zeta = 0$ in (5.3) one now has

$$\frac{\rho^n f^{(n)}(0)}{n!} = \sum_{k=1}^{n} B_{nk}(\Phi)\frac{F^{(k)}(0)}{k!}. \tag{5.4}$$

Since $|F^{(k)}(0)| \le k!2$ for all $k \ge 1$, and since (4.8) holds for $\Phi \in \mathcal{S}$, one immediately has (5.1) with the aid of (4.9).

Suppose that the equality holds in (5.1) for an $n \ge 2$. Then $|B_{21}(\Phi)| = |B_{21}(K)|$ and $|F'(0)| = 2$. Hence we have α and β of ∂D such that $F = \ell_\alpha$ and $\Phi = K_\beta$. If $\rho < 1$, then $f(\phi(\rho\overline{\alpha})) = \ell_\alpha(\overline{\alpha}) = \infty$, so that f is not holomorphic in Ω. Hence $\rho = 1$, so that

$$f = F \circ \Phi^* = \ell_\alpha \circ (K_\beta)^*,$$

and $\Omega = \phi(D) = \Phi(D) = K_\beta(D)$. To have (5.2) we next show that $\alpha\overline{\beta} = -1$. Set $\gamma = -\alpha\overline{\beta}$. Then (5.4) reads that

$$\begin{aligned}\frac{f^{(n)}(0)}{n!} &= \sum_{k=1}^{n} B_{nk}(K_\beta)\frac{(\ell_\alpha)^{(k)}(0)}{k!} \\ &= \sum_{k=1}^{n}(-\beta)^{n-k}|B_{nk}(K)|2\alpha^k\end{aligned}$$

$$= 2(-\beta)^n \sum_{k=1}^{n} \gamma^k |B_{nk}(K)|.$$

It then follows from (4.9) that

$$\sum_{k=1}^{n} |B_{nk}(K)| = \binom{2n-1}{n} = \frac{|f^{(n)}(0)|}{n!2} = \left| \sum_{k=1}^{n} \gamma^k |B_{nk}(K)| \right|.$$

Hence, squaring the left- and the right-most terms one has

$$\sum |B_{nk}(K)||B_{nj}(K)|(1-\gamma^{k-j}) = 0,$$

where the summation is taken over all k, j with $1 \le k \le n$, $1 \le j \le n$; note that $n \ge 2$. Since $\operatorname{Re}(1-\gamma^{k-j}) \ge 0$, it follows that $\operatorname{Re}(1-\gamma^{k-j}) = 0$ so that $\gamma^{k-j} = 1$ for $k \ne j$, $1 \le k \le n$, $1 \le j \le n$. Hence $\gamma = 1$. We thus have, with the aid of (4.10), that

$$f(w) = \ell_\alpha \circ (K_\beta)^*(w) = \frac{1 + \alpha\overline{\beta}K^*(\beta w)}{1 - \alpha\overline{\beta}K^*(\beta w)} = \frac{1 - K^*(\beta w)}{1 + K^*(\beta w)} = \frac{1}{\sqrt{4\beta w + 1}}.$$

Given f of (5.2) in $\Omega = K_\beta(D)$, $\beta \in \partial D$, we consider the set E of points $z \in \Omega$ where the equality holds in (4.3) for all $n \ge 2$. Since $w = K_\beta(\zeta) \in \Omega$, $\zeta \in D$, simple calculation yields that

$$\operatorname{Re} f(w) = \frac{1-|\zeta|^2}{|1+\beta\zeta|^2}, \quad |f^{(n)}(w)| = 2 \cdot \frac{(2n-1)!}{(n-1)!} \left| \frac{1-\beta\zeta}{1+\beta\zeta} \right|^{2n+1},$$

because

$$\prod_{k=0}^{n-1} \left(\frac{1}{2} + k \right) = 2^{1-2n} \cdot \frac{(2n-1)!}{(n-1)!},$$

and further, $\rho_\Omega(w) \equiv 1$, and

$$\frac{1}{\Pi_\Omega(w)} = \frac{(1-|\zeta|^2)|1+\beta\zeta|}{|1-\beta\zeta|^3}$$

for $\zeta \in D$. Hence

$$\frac{|f^{(n)}(w)|}{\Pi_\Omega(w)^n \operatorname{Re} f(w)} = 2 \cdot \frac{(2n-1)!}{(n-1)!} \left(\frac{1-|\zeta|^2}{|1-\beta^2\zeta^2|} \right)^{n-1} \tag{5.5}$$

for $\zeta \in D$. Consequently, $w = K_\beta(\zeta)$ is in E if and only if $\beta\zeta$ is on the real diameter $(-1,1)$ or equivalently, if and only if

$$w \in \Lambda \equiv \left\{ \overline{\beta} t;\ t > -\frac{1}{4} \right\}.$$

Hence $E = \Lambda$. It is easy to prove that the inequality (4.3) is strict at each point $z \in K_\beta(\Omega) \setminus \Lambda$ and for each $n \geq 2$.

To prove (4.3) at $z = a \in \Omega$ we choose a projection $\phi : D \to \Omega$, with $\phi(0) = a$ and consider the domain

$$\Sigma = \left\{ \frac{\zeta - a}{\phi'(0)};\ \zeta \in \Omega \right\}$$

for which $0 \in \Sigma$ and $\psi = (\phi - a)/\phi'(0)$ is a projection $\psi : D \to \Sigma$ with $\psi(0) = \psi'(0) - 1 = 0$. Then the function

$$g(w) = \frac{f(a + \phi'(0)w) - i \operatorname{Im} f(a)}{\operatorname{Re} f(a)}$$

is in $\mathcal{P}(\Sigma)$ with $g(0) = 1$. Since $g^{(n)}(0) = \phi'(0)^n f^{(n)}(a)/\operatorname{Re} f(a)$, since $\rho_\Sigma(0) = \rho_\Omega(a)$, and since $|\phi'(0)| = 1/\Pi_\Omega(a)$, we may apply (5.1) to g in Σ to have

$$\left(\frac{\rho_\Omega(a)}{\Pi_\Omega(a)} \right)^n \frac{|f^{(n)}(a)|}{n! \operatorname{Re} f(a)} = \frac{\rho_\Sigma(0)^n |g^{(n)}(0)|}{n!} \leq 2 \binom{2n-1}{n}$$

for all $n \geq 2$. This is equivalent to (4.3) for $z = a$. If the equality holds in (4.3) at $z = a$, thenwe have (I) and (II) with

$$Q = \overline{\beta}\phi'(0), \quad R = a, \quad A = \operatorname{Re} f(a), \quad \text{and} \quad B = f(a).$$

The detailed proof is obvious.

Remark 1. How about the case $n = 1$ in Theorem 2? Since (4.1) for $\Sigma = H$ is valid, we have

$$\frac{|f'(z)|}{\operatorname{Re} f(z)} \leq 2\Pi_\Omega(z) \leq 2 \frac{\Pi_\Omega(z)}{\rho_\Omega(z)}. \tag{5.6}$$

Suppose that the left- and the right-most are the same in (5.6). Then $\rho_\Omega(z) = 1$, so that Ω must be simply connected, and furthermore, the equalities hold in (5.6) for every poit of Ω. The function F in the proof must be ℓ_α for some $\alpha \in \partial D$ because $|F'(0)| = 2$ (and we have no explicit form for $\Phi = \phi$.) We can also prove (the weaker result)

$$\frac{|f'(z)|}{\operatorname{Re} f(z)} \leq 2\,\frac{\Pi_\Omega(z)}{\rho_\Omega(z)}$$

by the same method as in the proof of Theorem 2.

Remark 2. It is known that $\mathcal{P}(\Omega) = \mathbf{C}$ if the closed set $\mathbf{C} \setminus \Omega$ is of logarithmic capacity zero. In other words, $\mathcal{P}(\Omega) = \mathbf{C}$ if $\Omega \in O_{HP} = O_G$; see [AS, p. 208]. More generally, g is holomorphic and bounded in modulus by $M > 0$ in Ω if and only if $f = (g+M)/(g-M) \in \mathcal{P}(\Omega)$. Hence one observes that $\mathcal{P}(\Omega) = \mathbf{C}$ if and only if $\Omega \in O_{AB}$.

6. Positive harmonic function in Ω

Let $h > 0$ be harmonic in a hyperbolic domain Ω. Then for each $\Delta(z)$ of (4.2) we have a holomorphic function f with $\operatorname{Re} f = h$ in $\Delta(z)$. Since the proof of Theorem 2 is "local" in its character, we have

$$\left|\frac{\partial^n h(z)}{\partial z^n}\right| \frac{1}{h(z)} \leq \frac{(2n-1)!}{(n-1)!} \left(\frac{\Pi_\Omega(z)}{\rho_\Omega(z)}\right)^n$$

at each $z \in \Omega$ and for each $n \geq 1$. Actually, (4.3) for the present f is valid, and the case $n = 1$ is obvious by $\rho_\Omega(z) \leq 1$. If the equality holds at a point $z \in D$, then $\rho_\Omega(z) = 1$, so that f can be defined in the whole Ω, the slit domain of (4.4). The equality conditions are different according as $n = 1$ or $n > 1$.

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