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Existence of δ_m-periodic points for smooth maps of compact manifold*

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Abstract. For a smooth self-map f of a compact manifold M we examine the connection between topological conditions put on M and differentials of a map f at periodic points.

Key words: periodic points, Lefschetz number, cohomological algebra.

1. Introduction

A classical example of the connection between global and local properties of a compact manifold M is Poincaré theorem: $\sum_{x\in C}\operatorname{ind}(T,x)=\chi(M)$, where $\chi(M)$ denotes the Euler characteristic of M, C is the set of critical points of the vector field T, and $\operatorname{ind}(T,x)$ the local index of T.

In 1983 Chow, Mallet-Paret and Yorke ([CMY]) proved that the sequence $\operatorname{ind}(f^n,x_0)$ of isolated fixed point indices of iterated C^1-map f is an integral linear combination of elementary periodic sequences with the periods determined by the spectrum of the derivative $Df(x_0)$ of f at x_0.

Basing on this fact Matsuoka and Shiraki ([MS]) formulated for self-maps of a compact manifold M with finitely many periodic points a global homological condition on M that forces an existence of a periodic point (so called a δ_m-periodic point) which satisfies a certain degeneracy condition.

On the other hand Marzantowicz and Przygodzki ([MP]) expressed a formula for $i_m(f)=\sum_{k|m}\mu(k)I(f^{m/k})$, where $I(f)$ is the fixed point index of f, in terms of periodic points of a compact manifold. If $i_m(f)\neq 0$ then we say that m is an algebraic period of f.

The aim of this paper is to prove the theorem analogous to given in [MS] but formulated in the language of algebraic periods. This approach is more general: we show that both theorems are equivalent for the class of maps with finitely many periodic points, but by a use of algebraic periods it

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is possible to find a δ_m-periodic point for maps with infinitely many periodic points as well.

We give an application of that observation to rational exterior spaces. For self-maps of such spaces the formula for Lefschetz number is known (cf. [H]), which allows to draw additional information about algebraic periods (cf. [G]).

2. Algebraic periods and periodic points

Let f be a self-map of a topological space X. For $n \geq 1$ we define $P^n(f) = Fix(f^n)$ and $P_n(f) = P^n(f) \setminus \bigcup_{k<n} P^k(f)$ called the set of n-periodic points. If $P_n(f) \neq 0$ then n is called a minimal period of f. The set of all minimal periods of f is denoted by $\mathrm{Per}(f)$.

Throughout the paper we assume that if $X = M$ is a compact manifold, then for every natural n, $P^n(f) \subset \mathrm{Int}\, X$ and $P^n(f)$ consists of isolated points only.

We begin with formulation of the results from [MS].

Definition 2.1 ([MS]) A periodic point x of f with minimal period n is said to be a *δ_m-periodic point* if $Df^n(x)$, the differential of f^n at x, has an eigenvalue which is an m'-th primitive root of unity for some multiple m' of m. $\diamond$

For integers $i \geq 0$, $n > 0$, let $e_i(n)$ be the number of eigenvalues of $f_{*i} : H_i(M;Q) \to H_i(M;Q)$, which are n-th primitive roots of unity (counting multiplicity). Define

$$e(n) = \sum_{i=0}^{\infty} (-1)^i e_i(n).$$

Theorem 2.2 ([MS]) *Let $f : M \to M$ be a C^1-map on a compact manifold M with finitely many periodic points. Let m be an odd prime number such that:*

(i) *$e(n) \neq 0$ for some multiple n of m*

(ii) *the period of any periodic point is not a multiple of m.*

Then f has a δ_m-periodic point. $\diamond$

Let us introduce the basic fact and results connected with algebraic periods. Let f be a self map of a compact manifold M and $I(f) = I(f, M)$

denotes the fixed point index of f, which is equal to $L(f)$ - the Lefschetz number of f. For every $n \in N$ let us define:

$$i_n(f) = \sum_{k|n} \mu(k) I(f^{n/k})$$

where $\mu(k)$ denotes the classical Möbius function, (cf. [Ch]).

Definition 2.3 A natural number n is called an algebraic period if $i_n(f) \neq 0$. $\diamond$

The following congruence (called Dold's relations) holds (cf. [D]):

Proposition 2.4 *For every $n \in N$ we have $i_n(f) \equiv 0 \pmod{n}$.*

This formula has a clear interpretation for a self-map f of a discrete countable set X. We have in that case: $|P_n(f)| = i_n(f)$ and the congruence (2.4) result from the fact that $P_n(f)$ consists of n-orbits (cf. [D]).

The numbers $i_n(f)$ for C^1 self-maps of a compact manifold M may be expressed by differentials at periodic points.

Define the subset of natural numbers $O(x)$ for $x \in P_d(f)$ as $O(x) = \mathrm{Per}(Df^d(x))$. Let σ_- denote the number of eigenvalues of $Df^d(x)$ (counted with multiplicity) smaller than -1.

Theorem 2.5 (cf. [MP]) *Let $f : M \to M$ be a C^1 map of a compact manifold M. Then there exist integers $c_k(x)$ such that*

$$i_n(f) = \sum_{dk=n} \sum_{x \in P_d(f)} c_k(x) + \sum_{2dk=n} \sum_{x \in P_d(f)} [(-1)^{\sigma_-(x)k} - 1] c_k(x)$$

with the convention that $c_k(x) = 0$ if $k \notin O(x)$. $\diamond$

Lemma 2.6 *The structure of the set $O(x)$ is as follows* (cf. [CMY]), [MP]):

$$O(x) = \{\mathrm{lcm}(K) : K \subset \sigma_{(1)}(Df^d(x))\} \cup \{1\}$$

where $\sigma_{(1)}(Df^d(x))$ is the set of degrees of primitive roots of unity contained in $\sigma(Df^d(x))$-the spectrum of derivative at x.

Now we are in a position to use algebraic periods for finding δ_m-periodic points.

Theorem 2.7 *Let $f : M \to M$ be a C^1-map of a compact manifold M.*

Let m be an odd prime number such that:
(i) *n is an algebraic period for some multiple n of m*
(ii) *the period of any periodic point is not a multiple of m.*
Then f has a δ_m-periodic point.

Proof. By Theorem 2.5 we have:

$$i_n(f) = \sum_{dk=n} \sum_{x \in P_d(f)} c_k(x) + \sum_{2dk=n} \sum_{x \in P_d(f)} \alpha_k(x) c_k(x),$$

where $\alpha_k(x) = (-1)^{\sigma_-(x)k} - 1$ ($k \in O(x)$) is an integer.

Let n be a multiple of m: $n = ms$. The first sum above extends over all $dk = ms$, the second over all $2dk = ms$. It follows from (ii) that d is not a multiple of m thus $m|k$, because m is a prime number different from 2.

Clearly, $i_n(f) \neq 0$ implies that there exists such k that $c_k(x) \neq 0$. Since $m|k$ and $k \in O(x)$, among elements of $\sigma_1(Df^d(x))$ there is multiplicity of m: $m' = ml$. This is equivalent that x is a δ_m-periodic point. □

Roughtly speaking the formula of Theorem 2.5 says that the coefficient $i_n(f)$ is the sum of two kinds of components: one that comes from n-periodic points and one from δ_m-periodic points, where $m \,|\, n$ and m is a prime number.

In order to establish the relation between Theorems 2.2 and Theorem 2.7 we need some lemmas.

Let ϕ be the Euler function. If $\varepsilon_1, \ldots, \varepsilon_{\phi(d)}$ are all d-th primitive roots of unity then define

$$L^d = \varepsilon_1 + \cdots + \varepsilon_{\phi(d)}.$$

Lemma 2.8 $L^d = \mu(d)$.

Proof. Induction by the number of primes in decomposition of d. The statement is true for $d = q$, where q is prime. Inductively we assume that the proposition is true for $d = p_1 \cdots p_r$, where $p_1 \cdots p_r$ are prime numbers (not necessarily different). Consider now the number $w = dp$. We have:

$$L^{dp} = \varepsilon_1 + \cdots + \varepsilon_{\phi(dp)}.$$

On the other hand

$$\varepsilon_1 + \cdots + \varepsilon_{\phi(dp)} + \varepsilon_{\phi(dp)+1} \cdots + \varepsilon_{dp} = 0,$$

where the sum above extends over all roots of unity of degree dp.

Thus by our inductive hypothesis:

$$\varepsilon_1 + \cdots + \varepsilon_{\phi(dp)} + \sum_{l|dp, l \neq dp} \mu(l) = 0.$$

As $\sum_{l|dp} \mu(l) = 0$ we obtain finally:

$$\varepsilon_1 + \cdots + \varepsilon_{\phi(dp)} - \mu(dp) = 0,$$

which ends the proof. □

Let $\varepsilon_1, \ldots, \varepsilon_{\phi(d)}$ be all d-th primitive roots of unity. Define

$$i_n^d = \sum_{l|n} \mu(n/l)(\varepsilon_1^l + \cdots + \varepsilon_{\phi(d)}^l).$$

Lemma 2.9 $L^d = \mu(d)$. *The following equality holds:*

$$i_n^d = \begin{cases} 0 & \text{if } n \not| d \\ \sum_{k|n} \mu(d/k)\mu(n/k)\dfrac{\phi(d)}{\phi(d/k)} & \text{if } n \mid d. \end{cases}$$

Proof.

$$i_n^d = \sum_{l|n} \mu(n/l)(\varepsilon_1^l + \cdots + \varepsilon_{\phi(d)}^l) = \sum_{l|n} \mu(n/l)\mu(d/(l,d))\frac{\phi(d)}{\phi(d/(l,d))}.$$

The last equality results from Lemma 2.8 and the fact that for $l|n$ the sum $\varepsilon_1^l + \cdots + \varepsilon_{\phi(d)}^l$ consists of $d/(l,d)$-primitive roots of unity, each taken $\frac{\phi(d)}{\phi(d/(l,d))}$ times. Observe that if $(n,d) = 1$, $n > 1$ then $i_n^d = \mu(d)\sum_{l|n} \mu(n/l) = 0$ (cf. [Ch]), otherwise

$$\begin{aligned} i_n^d &= \sum_{k|(n,d)} \sum_{\{l|n:(l,d)=k\}} \mu(n/l)\mu(d/(l,d))\frac{\phi(d)}{\phi(d/(l,d))} \\ &= \sum_{k|(n,d)} \mu(d/k)\frac{\phi(d)}{\phi(d/k))} \sum_{\{l|n:(l,d)=k\}} \mu(n/l). \end{aligned}$$

Let us calculate the sum: $\sum_{\{l|n:(l,d)=k\}} \mu(n/l)$. Notice that:

$$\sum_{\{l|n:(l,d)=k\}} \mu(n/l) = \sum_{\{l|n:(n/l,d)=k\}} \mu(l).$$

Let us now consider two cases (a) $n \not| d$ and (b) $n \,|\, d$.

(a) If $n \not| d$ then there exist: a prime number q and a natural number α such that $q^\alpha | n$ and $q^\alpha \not| d$. We have in this case:

$$\sum_{\{l|n:(n/l,d)=k\}} \mu(l) = \sum_{\{\tilde{l}:q\not|\tilde{l}|n,(n/\tilde{l},d)=k\}} \mu(\tilde{l}) + \sum_{\{l':q|l'|n,(n/l',d)=k\}} \mu(l').$$

Define the following function:

$b: \{\tilde{l} : q \not| \tilde{l}|n, (n/\tilde{l}, d) = k, \mu(\tilde{l}) \neq 0\} \to \{l' : q|l'|n, (n/l', d) = k, \mu(l') \neq 0\}$, $b(\tilde{l}) = q\tilde{l}$. Then b is bijection and $\mu(\tilde{l}) = -\mu(b(\tilde{l}))$. As a consequence we obtain $\sum_{\{l|n:(n/l,d)=k\}} \mu(l) = 0$.

(b) If $n \,|\, d$ then $(n/l, d) = n/l$, but the sum is taken over l such that $(n/l, d) = k$, thus $n/l = k$ and

$$\sum_{\{l|n:(n/l,d)=k\}} \mu(l) = \sum_{l=n/k} \mu(l) = \mu(n/k).$$

We now return to the calculation of i_n^d. We have: if $n \not| d$ then by (a) $i_n^d = 0$, if $n \,|\, d$ then $(n, d) = n$ so by (b) $i_n^d = \sum_{k|n} \mu(d/k)\mu(n/k)\frac{\phi(d)}{\phi(d/k)}$. This completes the proof. □

Lemma 2.10 $i_n^n = n$.

Proof. We have by Lemma 2.9:

$$i_n^n = \sum_{k|n} \mu(n/k)\mu(n/k)\frac{\phi(n)}{\phi(n/k)} = \phi(n)\sum_{k|n}\frac{\mu^2(k)}{\phi(k)}.$$

Let $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ then $\phi(n) = n(1 - \frac{1}{p_1}) \cdots (1 - \frac{1}{p_r}) = n\frac{(p_1-1)\cdots(p_r-1)}{p_1\cdots p_r}$

$$\begin{aligned} i_n^n &= n\frac{(p_1-1)\cdots(p_r-1)}{p_1\cdots p_r}\left(1 + \sum_{1\le l_1<\cdots<l_h\le r}\frac{1}{(p_{l_1}-1)\cdots(p_{l_h}-1)}\right) \\ &= \frac{n}{p_1\cdots p_r}\left(1 + \sum_{1\le l_1<\cdots<l_h\le r}(p_{l_1}-1)\cdots(p_{l_h}-1)\right) \end{aligned}$$

$$= \frac{n}{p_1 \cdots p_r} \sum_{k|p_1 \cdots p_r} \phi(k) = n.$$

The last equality is the consequence of well known fact: $\sum_{k|s} \phi(k) = s$ (cf. [Ch]). □

Proposition 2.11 *Theorem 2.2 and Theorem 2.7 are equivalent for smooth maps with finitely many periodic points.*

Proof. Define

$$L_C(f) = \sum_{\lambda \in C \cap \sigma(f)} (-1)^{\dim \lambda} \lambda,$$

where C is the set of all roots of unity, $\sigma(f)$ is the spectrum of the map induced by f on homology, $\dim \lambda = i$ if λ is an eigenvalue for $H_i(M; Q)$.

Let us notice now that if $\{L(f^n)\}_{n=1}^{\infty}$ is bounded then $L(f^n) = L_C(f^n)$, (cf. [BB], [Ma]). On the other hand for smooth maps with finitely many periodic points, $\{L(f^n)\}_{n=1}^{\infty}$ is bounded (cf. [SS], [CMY]).

Thus, using our terminology we obtain for maps with finitely many periodic points: $L(f) = \sum_d \frac{e(d)}{\phi(d)} L^d$, where the sum extends over the degrees of all primitive roots of unity in $C \cap \sigma(f)$.

As a consequence we have:

$$i_n(f) = \sum_d \frac{e(d)}{\phi(d)} \sum_{l|n} \mu(n/l)(\varepsilon_1^l + \cdots + \varepsilon_{\phi(d)}^l) = \sum_d \frac{e(d)}{\phi(d)} i_n^d.$$

Let us assume now that $e(km) \neq 0$ and m is an odd prime number. Define $n_0 = \max\{nm : e(nm) \neq 0\}$. Consider $i_{n_0}(f) = \sum_d \frac{e(d)}{\phi(d)} i_{n_0}^d$. By Lemma 2.9 $i_{n_0}^d = 0$ if $d < n_0$. This implies that $i_{n_0}(f) = \frac{e(n_0)}{\phi(n_0)} i_{n_0}^{n_0}$. Now Lemma 2.10 gives: $i_{n_0}(f) = \frac{e(n_0)}{\phi(n_0)} n_0 \neq 0$. This ends the proof of the first part of the equivalence. To prove the adverse implication let us assume that for some n, multiplicity of prime odd m we have: $i_n(f) = \sum_d \frac{e(d)}{\phi(d)} i_n^d \neq 0$. Then there exists d_0 such that $\frac{e(d_0)}{\phi(d_0)} i_n^{d_0} \neq 0$. From Lemma 2.9 we deduce that $n \mid d_0$, on the other hand $m \mid n$ finally $m|d_0$ and $e(d_0) \neq 0$ which ends the proof. □

3. δ_m-Periodic points on rational exterior spaces

For a given space X and an integer $r \geq 0$ let $H^r(X;Q)$ be the r-th singular cohomology space with rational coefficients. Let next $H^*(X;Q) = \bigoplus_0^s H^r(X;Q)$ be the algebra of cohomology with the multiplication given by the cup product.

An element $x \in H^r(X;Q)$ is *decomposable* if there are some pairs of elements $(x_i, y_i) \in H^p(X;Q) \times H^q(X;Q)$ $p,q > 0$, $p+q = r > 0$ so that $x = \sum x_i \cup y_i$, where $\cup$ is the cup product in $H^*(X;Q)$. Let $A^r(X) = H^r(X)/D^r(X)$, where D^r is the subspace over Q consisting of all decomposable elements. Then $A^r(X)$ is a vector space over Q. For a continuous map $f : X \to X$ let f^* be the induced homomorphism on the cohomology spaces and $A(f)$ the induced homomorphism on $A(X)$.

Definition 3.1 A connected topological space X is called *rational exterior* if it is possible to find some homogeneous elements $x_i \in H^{odd}(X;Q)$, $i = 1, \ldots, k$ such that the inclusions $x_i \hookrightarrow H^*(X;Q)$ give rise to a ring isomorphism $\Lambda_Q(x_1, \ldots, x_k) = H^*(X;Q)$. ◇

One of the simplest example of a rational exterior space is T^2: if x_1, x_2 are generators of $H^1(T^2;Q)$ then $x_1 \cup x_2$ is a generator for $H^2(T^2;Q)$. Thus $H^*(T^2;Q) = \Lambda(x_1, x_2)$ - exterior algebra with two generators.

Among rational exterior spaces there are: finite H-spaces, including all finite dimensional compact Lie groups and some real Stiefel manifolds.

Definition 3.2 Let f be a self-map of a space X and let $I : A(X) \to A(X)$ be the identity morphism. The polynomial

$$A_f(t) = \det(tI - A(f)) = \prod_{r \geq 1} \det(tI - A^r(f))$$

will be called the *characteristic polynomial* of f. The zeros of this polynomial: $\lambda_1(f), \ldots, \lambda_k(f)$, $k = \operatorname{rank} X$, where $\operatorname{rank} X$ is the dimension of $A(X)$ over Q, will be called the *quotient eigenvalues* of f.

Theorem 3.3 ([H]) *Let f be a self-map of a rational exterior space, A denotes the matrix of $A(f)$, and let $\lambda_1, \ldots, \lambda_k$ be quotient eigenvalues of f. Then $L(f^n) = \det(I - A^n) = \prod_{i=1}^k (1 - \lambda_i^n)$.* ◇

Let us introduce the following definition:

Definition 3.4 A map f will be called *essential* providing it satisfies the conditions:

(a) 1 is not its quotient eigenvalue

(b) at least one quotient eigenvalue is neither zero nor a primitive root of unity.

We have the following characterization of essential maps:

Proposition 3.5 (cf. [G]) *A self-map f of a rational exterior space is essential iff $\{L(f^m)\}_{m=1}^{\infty}$ is unbounded.*

Basing on some nontrivial inequalities for algebraic numbers proved in [JL] it is possible to observe the presence of large algebraic periods for essential self-maps of rational exterior spaces.

Let $T_A = \{n \in N : \det(I - A^n) \neq 0\}$, A denotes the matrix of $A(f)$.

Theorem 3.6 ([G]) *Let X be a rational exterior space. Then there exists a number n_X which depends only on the space X, and is independent of the choice of f, such that for every essential self-map f of X and all $n > n_X$, $n \in T_A$, n is an algebraic period of f.*

Theorems 3.6 makes possible to find δ_m-periodic points of self-maps of rational exterior spaces.

Theorem 3.7 *Let M be a rational exterior compact manifold and $f : M \to M$ be a C^1 essential map. Let m be an odd prime number such that:*

(i) *neither of quotient eigenvalues is an m-th primitive root of unity*

(ii) *the period of any periodic point is not a multiple of m.*

Then f has a δ_m-periodic point.

Proof. Let us notice that $n \in T_A$ iff $\det(I - A^n) \neq 0$. On the other hand, by Theorem 3.3 we have: $\det(I - A^n) = \prod_{i=1}^{k}(1 - \lambda_i^n) = L(f^n)$. If among λ_i, $(i = 1, \ldots, k)$ there is no m-th primitive root of unity then $L(f^{ml})$ is different from zero for infinitely many l. Thus, by Theorem 3.6 for sufficiently large l we obtain $i_{ml}(f) \neq 0$, which proves the statement due to Theorem 2.7. □

Remark 3.8 By Proposition 3.5, Theorem 3.7 refers only to maps with infinitely many periodic points. Moreover, for a given self-map of a rational exterior compact manifold M there is such number N_f (although usually very large) that for all prime $m > N_f$ there is always a point with minimal period m (cf. [G]). As a result Theorem 3.7 acts effectively only for $m < N_f$.

For every odd prime m we may formulate the following alternative.

Theorem 3.9 *Let M be a rational exterior compact manifold. Then there exists a number s_M, such that for every essential C^1 self-map f of M and all natural $s > s_M$, $m^s \in T_A$ either there is a δ_m-periodic point or there are points of minimal period m^s.*

Proof. Let us take s_M such that $m^{s_M} > n_M$, where $n_M = n_X$ is taken from Theorem 3.6. Then for every $s > s_M$ we have:

$$i_{m^s}(f) = \sum_{x \in P_1(f)} c_{m^s}(x) + \sum_{x \in P_m(f)} c_{m^{s-1}}(x) + \cdots + \sum_{x \in P_{m^s}(f)} c_1(x) \neq 0$$

If there is no δ_m-periodic point then from the convention of Theorem 2.5 and Lemma 2.6 we conclude that for $1 \leq r \leq s$ we have $c_{m^r}(x) = 0$. As a result $\sum_{x \in P_{m^s}(f)} c_1(x) \neq 0$ which gives the thesis. □

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