

On semisimple extensions of serial rings

Kazuhiko HIRATA and KOZO SUGANO

(Received October 17, 1997; Revised June 19, 1998)

Abstract. We prove that if B is a commutative local serial ring and A is a B -algebra which is a left semisimple extension of B , A is a uniserial ring. If in addition A is indecomposable as ring, the lengths of the composition serieses of Ae and B are same for each primitive idempotent e of A . We also give some necessary and sufficient conditions for A to be a left semisimple extension of a subring B of it, in the case where A and B are local serial rings or the case where B is a commutative local serial ring and A is a B -algebra which is serial.

Key words: serial ring, uniserial ring, composition series, semisimple extension.

Throughout this paper A will always be a ring with identity 1, and B a subring of A containing 1. In their previous paper [4] the authors introduced the notion of semisimple extensions of a ring. A ring A is said to be a left semisimple extension of B in the case where every left A -module M is (A, B) -projective, that is, the map π of $A \otimes_B M$ to M , defined by $\pi(a \otimes m) = am$ for any $a \in A$ and $m \in M$, splits as left A -homomorphism, or equivalently, for every left A -module M , every A -submodule which is a B -direct summand of M is always an A -direct summand. (See Theorem 1.1 [4]). The right semisimple extension is similarly defined, and the both left and right semisimple extension is called semisimple extension. Till now some typical examples of the semisimple extension are known, for example, each semisimple ring is a semisimple extension of each subring of it, and each separable extension is a semisimple extension. However, since the semisimplicity is a quite abstract condition, it is very difficult to research the structure of the semisimple extension or find proper examples of it.

In this paper we will give some structure theorem of semisimple extensions of (two-sided) uniserial local rings. A ring R is said to be left serial in the case where R is left artinian and Re has the unique composition series for each primitive idempotent e of R . In the case where R is a direct sum of finite primary left serial rings, R is said to be a left uniserial ring. It is

a well known fact that R is primary left uniserial if and only if R is a full matrix ring over a local left serial ring. Right (uni) serial ring is defined similarly, and a both left and right (uni) serial ring is called (uni) serial ring. It is also a well known fact that, if R is serial, R satisfies the following two conditions;

- (1) Each left R -module is a direct sum of indecomposable submodules
- (2) A left R -module is indecomposable if and only if it is a homomorphic image of some Re where e is a primitive idempotent of R .

In the case where R satisfies the condition (1), the indecomposable decomposition of each module compliments direct summands by Corollary 2 to Theorem A [7]. Therefore in this case we see that the indecomposable decomposition of each module is unique up to isomorphism by Theorem 12.4 [1], and that R is left artinian by Corollary 28.15 [1]. Consequently each left R -module has the projective cover. In addition it can be easily proved that, under the condition (2), for each primitive idempotent e of R Re has the unique maximal left subideal, and each epimorphism of Re to M is a projective cover of an indecomposable left R -module M . Under these preparations we have;

Theorem 1 *Let both A and B satisfy the above conditions (1) and (2), and suppose that A is a left semisimple extension of B . Then for each left ideal L of A and each primitive idempotent e of A , there exist a left ideal I of B and a primitive idempotent e' of A such that there is an A -isomorphism of Ae to Ae' whose restriction on Le is an isomorphism of Le to AJe' .*

Proof. Suppose that A is a left semisimple extension of B and let L and e be as in the theorem. Then Ae/Le is A -indecomposable and Ae/Le is an A -direct summand of $A \otimes_B Ae/Le$. On the other hand B satisfies the same condition. Therefore there exist classes $\{K_\alpha\}$ of left ideals of B and $\{f_\alpha\}$ of primitive idempotents of B such that $Ae/Le \cong \Sigma \oplus Bf_\alpha/K_\alpha f_\alpha$ as B -modules. Then we have $A \otimes_B Ae/Le \cong \Sigma \oplus A \otimes_B Bf_\alpha/K_\alpha f_\alpha \cong \Sigma \oplus Af_\alpha/AK_\alpha f_\alpha$. Write $f = f_\alpha$ and $K = K_\alpha$ for a fixed α , and let $Af/AKf = M_1 \oplus M_2$ be a decomposition of Af/AKf with M_1 indecomposable. As is stated above we have the projective covers $p_i : P_i \rightarrow M_i$ ($i = 1, 2$) and the following commutative diagram

$$\begin{array}{ccc}
 Af & \xrightarrow{\mu} & M_1 \oplus M_2 \\
 \rho \downarrow & & \parallel \\
 P_1 \oplus P_2 & \xrightarrow{p} & M_1 \oplus M_2
 \end{array}$$

where $p = p_1 + p_2$ is the projective cover of $M_1 \oplus M_2$ and μ is the canonical epimorphism, ρ is an epimorphism such that $p\rho = \mu$. Then there exists a monomorphism λ of P of Af such that $\rho\lambda = \text{identity on } P$ where $P = P_1 \oplus P_2$. Write $\bar{P}_i = \lambda(P_i)$ for each i . Clearly we have $p = \mu\lambda$ and $\mu(\bar{P}_i) = p(P_i) = M_i$ for each i . We have also $Af = \bar{P}_1 \oplus \bar{P}_2 \oplus \text{Ker } \rho$, and $\mu(\text{Ker } \rho) = 0$. Hence we have $\mu(\bar{P}_2 \oplus \text{Ker } \rho) = M_2$. Now there exist mutually orthogonal idempotents e' and e'' of A such that $f = e' + e''$, $\bar{P}_2 \oplus \text{Ker } \rho = Ae''$ and $\bar{P}_1 = Ae'$. Here e' is primitive, since Ae' is the projective cover of the indecomposable A -module M_1 . Then by the above arguments we have the following commutative diagram, where all rows and columns are exact

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & Ae'' & \xrightarrow{\mu''} & M_2 & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & AKf & \longrightarrow & Af & \xrightarrow{\mu} & M_1 \oplus M_2 \longrightarrow 0 \\
 & & & & \pi \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Ker } \mu' & \longrightarrow & Ae' & \xrightarrow{\mu'} & M_1 \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

In the above diagram μ' and μ'' are the restrictions of μ on Ae' and Ae'' respectively, and π is the projection of $Af (= Ae' \oplus Ae'')$ to Ae' , which is given by the right multiplication of e' . Then the exactness of the above commutative diagram yields the epimorphism of AKf to $\text{Ker } \mu'$, which is the restriction of π to AKf . Hence we have $\text{Ker } \mu' = AKfe'$, and $M_1 \cong Ae'/AKfe'$. Thus we have shown that each indecomposable direct summand of Af/AKf is of the form $Ae'/Ae'I$, with $I (= Kf)$ a left ideal of B and e' a primitive idempotent of A . This fact together with the condition (1) shows that $Af_\alpha/AK_\alpha f_\alpha \cong \Sigma \oplus Ae_{\alpha i}/Ae_{\alpha i}I_{\alpha i}$ for some left ideals $I_{\alpha i}$ of B and primitive

idempotents $e_{\alpha i}$ of A , and consequently

$$\begin{aligned} Ae/Le &\llcorner \bigoplus A \otimes_B Ae/Le \cong \Sigma \oplus A \otimes_B Bf_{\alpha}/K_{\alpha}f_{\alpha} \\ &\cong \Sigma \oplus Af_{\alpha}/AK_{\alpha}f_{\alpha} \cong \Sigma \oplus Ae_{\alpha i}/AI_{\alpha i}e_{\alpha i} \end{aligned}$$

Then by the uniqueness of the decomposition, we have $Ae/Le \cong Ae_{\alpha i}/AI_{\alpha i}e_{\alpha i}$ for some α and i . Since as is stated above the canonical maps

$$Ae \rightarrow Ae/Le \quad \text{and} \quad Ae_{\alpha i} \rightarrow Ae_{\alpha i}/AI_{\alpha i}e_{\alpha i}$$

are projective covers respectively, there exists an isomorphism ϕ of Ae to $Ae_{\alpha i}$ such that

$$\begin{array}{ccc} Ae & \xrightarrow{\phi} & Ae_{\alpha i} \\ \downarrow & & \downarrow \\ Ae/Le & \cong & Ae_{\alpha i}/AI_{\alpha i}e_{\alpha i} \end{array}$$

is commutative. Obviously we have $\phi(Le) = AI_{\alpha i}e_{\alpha i}$. Thus we have proved the theorem. \square

Now we will apply Theorem 1 to two cases. One is the case where B is a commutative local serial ring and A is a B -algebra, the other is the case where both A and B are local serial rings. In either case the converse of Theorem 1 is true.

Proposition 1 *Let B be a commutative ring and A a B -algebra, and suppose that both A and B satisfy the conditions (1) and (2). Then A is a left semisimple extension of B if and only if, for each left ideal L and primitive idempotent e of A , there exist an ideal I of B and a primitive idempotent e' of A such that there is an A -isomorphism of Ae to Ae' whose restriction on Le is an isomorphism of Le to AIe' .*

Proof. The ‘only if’ part is due to Theorem 1. In order to prove the converse we need only to prove that each indecomposable left A -module is (A, B) -projective. But this is almost clear, since each indecomposable left A -module is of the form Ae/AIe for some primitive idempotent e of A and an ideal I of B , and is isomorphic to $Ae \otimes_B B/I$. The latter is (A, B) -projective, since it is an A -direct summand of $A \otimes_B B/I$, which is (A, B) -projective. \square

In what follows we will always denote the radicals of A and B by N

and J , respectively.

Theorem 2 *Let B be a commutative local serial ring, and A a B -algebra. Then if A is a left semisimple extension of B , A is a uniserial ring. In addition if A is indecomposable as a ring, the length of the composition series of Ae coincides with that of B for each primitive idempotent e of A .*

Proof. AJ is a nilpotent ideal of A , and consequently contained in N . Since B is local, we have $AJ \cap B = J$. Then A/AJ is a left semisimple extension of a field B/J , and A/AJ is a semisimple ring (See Proposition 1.2 and Corollary 1.7 [4]). Therefore we have $N = AJ$, which is nilpotent. Thus we see that A is semiprimary. We have also $N^i = AJ^i$ for each number i . Let e be a primitive idempotent of A , and r the natural number such that $J^r e = 0$ and $J^{r-1} e \neq 0$. Since A is semiprimary, Ae/Ne is simple. It is obvious that $Ae/Ne = Ae/AJe \cong B/J \otimes_B Ae$. On the other hand we have $B/J \cong J^i/J^{i+1}$ for each $1 \leq i \leq r-1$. Therefore each $J^i/J^{i+1} \otimes_B Ae$ is also simple. Now consider the following commutative diagram

$$\begin{array}{ccccccc}
 J^{i+1} \otimes_B Ae & \longrightarrow & J^i \otimes_B Ae & \longrightarrow & J^i/J^{i+1} \otimes_B Ae & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & & & \\
 J^{i+1} Ae & \longrightarrow & J^i Ae & \longrightarrow & J^i Ae/J^{i+1} Ae & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & & & \\
 0 & & 0 & & & &
 \end{array}$$

where all rows and columns are exact. Then there exists an epimorphism

$$J^i/J^{i+1} \otimes_B Ae \rightarrow J^i Ae/J^{i+1} Ae \rightarrow 0$$

which means that each $J^i Ae/J^{i+1} Ae = N^i e/N^{i+1} e$ is simple. Note that if $N^i e = N^{i+1} e$ for $i < r$, we have $N^i e = N^{i+1} e = N^{i+2} e = \dots = N^r e = 0$, which contradicts the assumption on r . Thus we see that

$$0 = N^r e < N^{r-1} e < \dots < Ne < N^0 e = Ae$$

is a composition series. Now assume that there exists a left A -submodule I of Ae which is different from any $N^i e$ ($0 \leq i \leq r$). Then there exists the largest integer k such that $I \subset N^k e$. Then $J \not\subset N^{k+1} e$, and we have $I + N^{k+1} e = N^k e$, since $N^{k+1} e$ is a maximal submodule of $N^k e$. But this is a contradiction, since we have $I = N^k e$ by Nakayama's lemma. Note that $N^k e = AJ^k e$ is a principal left ideal of A , since J^k is principal.

Therefore $\{N^i e\}$ are the only submodules of Ae , and the above composition series is the unique composition series of Ae . On the other hand since A is semiprimary, we can write $A = Ae_1 \oplus Ae_1 \oplus \cdots \oplus Ae_n$, where $\{e_i\}$ is a system of orthogonal primitive idempotents of A . Then since each Ae_i has a composition series, so does $\Sigma \oplus Ae_i$. Hence A is a left artinian ring. Thus we have shown that A is a left (and right) serial ring. By the way, in order to prove only that A is uniserial, we do not have to require the entire argument given above. For instance, we can apply Theorem 2.51 [2] which asserts that, if A is semiprimary and if each prime ideal of A is left principal, then A is left uniserial. However, when the last assertion of the theorem is proved, it will become necessary. Therefore, we give an explicit decomposition of A and so on here. Let P be a prime ideal of A . Since A is semiprimary, P is maximal as a two-sided ideal and contains N . Hence for each i there exists the natural epimorphism of Ae_i/Ne_i to Ae_i/Pe_i . Suppose $Ae_i \neq Pe_i$. Then since Ae_i/Ne_i is simple, this epimorphism is an isomorphism, and we have $Pe_i = Ne_i$. Therefore, after renumbering e_i 's if necessary, we can write

$$\begin{aligned} P &= Pe_1 \oplus Pe_2 \oplus \cdots \oplus Pe_n \\ &= Ae_1 \oplus Ae_2 \oplus \cdots \oplus Ae_m \oplus Ne_{m+1} \oplus Ne_{m+2} \oplus \cdots \oplus Ne_n \end{aligned}$$

Then since $Ne_j = AJe_j = A\pi e_j$ for some $\pi \in J \subset B$, we see that P is generated by $e_1 + e_2 + \cdots + e_m + \pi e_{m+1} + \pi e_{m+2} + \cdots + \pi e_n$ as a left ideal. Thus all prime ideals are left principal, and any two distinct prime ideals P and Q are mutually prime, i.e., $P + Q = A$, because they are maximal. Hence we have $PQ = QP$. Then since the multiplication of any two prime ideals of a semiprimary ring A is commutative, A is a direct sum of finitely many primary rings. (See e.g., [2] Theorem 2.43 and Lemma 6 §4 Chap. 2). Therefore A is uniserial. We now prove the last assertion of the theorem. Let e be a primitive idempotent of A and r the natural number such that $J^r = 0$ and $J^{r-1} \neq 0$. Then since $N = AJ$, we have $N^i = AJ^i$, $N^r = 0$ and $N^{r-1} \neq 0$. Since A is uniserial and indecomposable, A is a matrix ring of a local serial ring. Hence we have $Ae_i \cong Ae$ for any primitive idempotents e_i . Therefore we have $N^{r-1}e = AJ^{r-1}e \neq 0$. Then by the same argument as the proof of the first part of the theorem we have that

$$0 < N^{r-1}e < \cdots < N^2e < Ne < Ae$$

is the unique composition series of Ae . □

Note that in the proof of Theorem 2 we used only the condition that A is a semiprimary B -algebra such that $N = AJ$. Therefore Theorem 2 holds under a weaker condition that A is a semisimple B -algebra in the sense of Hattori [3]. Moreover the above theorem can be described more generally as follows;

Theorem 3 *Let B be a commutative local serial ring and A a semiprimary B -algebra such that $N = AJ$. Then A is a uniserial ring. If furthermore A is indecomposable as a ring, the length of Ae is equal to the length of B for each primitive idempotent e of A .*

By Theorem 1 and a part of the proof of Theorem 2 we have

Theorem 4 *Let B be a commutative local serial ring and assume that A is a serial B -algebra. Then A is a left semisimple extension of B if and only if $N = JA$.*

Proof. Assume $N = AJ$, and let L be a left ideal of A and e a primitive idempotent of A . Since A is left serial, $0 = N^r e < N^{r-1} e < \dots < Ne < N^0 e = Ae$ is the unique composition series of Ae . Hence we have $Le = N^i e$ for some i , and $Le = AJ^i e$. Then by Proposition 1 A is a left semisimple extension of B . The converse can be proved by the same way as Theorem 2. \square

Now we will consider the case where B is not necessarily commutative. Assume again that A and B satisfy the conditions (1) and (2), and furthermore that A has no nonzero idempotent except for 1. In the case where A and B are local serial rings, A and B satisfy these conditions. Under these conditions we see that each indecomposable left A -module (resp. B -module) is isomorphic to A/L (resp. B/I) for some left ideal L of A (resp. I of B). Then the same methods as the proofs of Theorem 1 and Proposition 1 can be applied to A and B . In addition each left A -isomorphism of A to A is given by the right multiplication of some unit element of A . Therefore we have;

Theorem 5 *Let A and B satisfy the conditions (1) and (2), and assume that A has no idempotent except for 1 and 0. Then A is a left semisimple extension of B if and only if for each left ideal L of A there exist a left ideal I of B and a unit u of A such that $Lu = AI$.*

Applying the above theorem we see that the same results as Theorems 2 and 4 hold in the case of local serial rings as follows;

Theorem 6 *Let A and B be local serial rings. Then the following conditions are equivalent;*

- (i) A is a left semisimple extension of B
- (ii) $N = AJ$
- (iii) The lengths of the composition serieses of the left A -module A and the left B -module B are same.
- (iv) A is a right semisimple extension of B

Proof. First we will show $J \subseteq N$. Suppose $J \not\subseteq N$. Then $JA \not\subseteq N$, and we have $JA = A$, since A is local. Then $A = JA = J^2A = \cdots = J^rA = 0$, which is a contradiction. Thus we have $J \subseteq N$. Now assume (i). Then by Theorem 5 there exist a number $i \geq 1$ and a unit u of A such that $N = Nu = AJ^i$, while we have $AJ^i \subseteq AJ \subseteq N$, which imply (ii). Conversely assume (ii). Then we have $N^i = AJ^i$ for each i . Then again by Theorem 5 we have (i) since $\{N^i\}$ is the set of all left ideals of A . We have also (iii), since $N^r = 0$ if and only if $J^r = 0$. Lastly assume (iii). Then $N^r = J^r = 0$ and $N^{r-1} \neq 0 \neq J^{r-1}$. If $N^k = AJ$ for $r > k > 1$, $N^{k(r-1)} = AJ^{r-1} \neq 0$. But in this case we have $k(r-1) = rk - k \geq r + r - k > r$, and $N^{k(r-1)} = 0$, a contradiction. Thus we have (ii). Since the condition (iii) is left and right symmetry for local (two-sided) serial rings A and B , the conditions (i) to (iii) are equivalent to (vi). \square

Theorem 7 *Let B be a local serial ring and A a local ring, and assume that A is finitely generated as a left B -module. Then if A is a left semisimple extension of B , A is a left serial ring.*

Proof. By the assumption A is left artinian and A/N is an artinian left B -module. Hence A/N is a finite direct sum of indecomposable modules, and we can write $A/N = \Sigma \oplus B/J^\alpha$ (finite), where each α is a natural number. Then since A is left semisimple over B , we have $A/N \llcorner \bigoplus A \otimes_B A/N \cong \Sigma \oplus A \otimes_B B/J^\alpha \cong \Sigma \oplus A/AJ^\alpha$ as left A -modules. On the other hand as is shown above we have $J \subseteq N$. Then each A/AJ^α is A -indecomposable and artinian, since $AJ^\alpha \subseteq N$ and A is local left artinian. Then we can apply the theorem of Krull-Remak-Schmidt to obtain $A/N \cong A/AJ^\alpha$ for some α . Now by comparing the lengths of composition serieses we have $N = AJ^\alpha$. But B is a both left and right principal ideal ring. Hence N is principal

as a left ideal. But N is the unique prime ideal of A , since N is nilpotent and the unique maximal ideal of A . Then A is a left serial ring. (See for example Theorem 2.51 [2]). \square

Finally we will give examples of ring extensions which satisfy the conditions of Theorem 6. Let D be a division ring with a discrete valuation v . Proposition 17.6 [6] shows that such division rings really exist. An uniformizer at v is an element z of D such that $v(z) < 1$ and $v(z)$ generates the cyclic group $V(D - \{0\})$. As usual we write

$$\begin{aligned} O(D) &= O(D, v) = \{x \in D \mid v(x) \leq 1\}, \\ P(D) &= P(D, v) = \{x \in D \mid v(x) < 1\} \end{aligned}$$

It is well known that $O(D)$ is a local ring with the radical $P(D)$, and $P(D) = O(D)z = zO(D)$ for each uniformizer z at v . It is also obvious that $O(D)/P(D)^n$ is a local serial ring with the length of the composition series n for each natural number n . Now the next proposition gives examples one of which satisfies the conditions of Theorem 6 and some other do not.

Proposition 2 *Let D be a division ring with a discrete valuation v and E a division subring of D . Then $O(D)/P(D)^n$ is a semisimple extension of $O(E)/P(E)^n$ for each natural number n if and only if E contains a uniformizer at v , that is, if and only if $v(D) = v(E)$.*

Proof. First suppose that E does not contain any uniformizer at v , and let z and y be uniformizers at v and $v|_E$, respectively, where $v|_E$ is the restriction of v on E , which is clearly discrete. Then $v(y) = v(z)^r$ for some natural number r (≥ 2), and we have $v(z^{-r}y) = 1$ and see that $z^{-r}y = a$ is a unit of $O(D)$. Then $P(E) = yO(E) = z^r a O(E) \subseteq z^r O(D)$, and $P(E) \subseteq z^r O(D) \cap O(E)$. Let x be any element of $z^r O(D) \cap O(E)$. Then $x = z^r b$ for some b in $O(D)$, and $x = z^r a a^{-1} b = y c$, where $c = a^{-1} b \in O(D)$. Hence we have $c = y^{-1} x \in E \cap O(D) = O(E)$, and $x = y c \in y O(E) = P(E)$. Thus we have $P(E) = z^r O(D) \cap O(E) = P(D)^r \cap O(E)$. Therefore $O(E)/P(E)$ is a subring of $O(D)/P(D)^r$. Both are local serial, but do not satisfy the condition of Theorem 6. Next suppose that E contains a uniformizer z at v . Then z is also a uniformizer at $v|_E$, and we have $P(E) = O(E)z = zO(E)$, and consequently, $P(E)^n = O(E)z^n = z^n O(E) = P(D)^n \cap O(E)$ for each natural number n . Here the last equality can be shown by the same argument as in the proof of ‘only if’ part. Then $O(E)/P(E)^n$ is a

subring of $O(D)/P(D)^n$, and these local serial rings satisfy the condition of Theorem 6. \square

References

- [1] Anderson F. and Fuller K., *Rings and Categories of Modules*. Springer G.T.M. **13** (1992).
- [2] Asano K., *Theory of Rings and Ideals* (in Japanese). Kyoritsu Press, Tokyo (1949).
- [3] Hattori A., *Semisimple algebras over a commutative ring*. J. Math. Soc. Japan **15** (1963), 404–419.
- [4] Hirata K. and Sugano K., *On semisimple extensions and separable extensions over non commutative rings*. J. Math. Soc. Japan **18** (1966), 360–373.
- [5] Knörr R., *Blocks, Vertices and Normal Subgroups*. Math. Z. **148** (1976), 53–60.
- [6] Pierce R. *Associative Algebras*. Springer G.T.M. **88** (1980).
- [7] Zimmermann-Huisgen B., *Rings whose right modules are direct sum of indecomposable modules*. Proc. Amer. Math. Soc. **77** (1979), 191–197.

Kazuhiko Hirata
Department of Mathematics
Chiba University
Chiba 263-8522, Japan

Kozo Sugano
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan