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Root separation on generalized lemniscates[1]

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Abstract. We discuss several positivity type criteria for a polynomial to have all the roots inside, or outside certain planar semi-algebraic domains. The main examples of such domains are the quadrature domains for analytic functions. Compared to the classical separation results for the disk or half-plane, in this more general setting a number of free parameters enter into the positivity criteria. We also remark that the complements on the Riemann sphere of these semi-algebraic domains are appropriate for solving bounded analytic interpolation problems.

Key words: root separation, quadrature domain, positive definite matrix, bounded analytic interpolation.

1. Introduction

The prototype for the class of planar domains which make the object of this note is a quadrature domain for complex analytic functions. We first define the latter, more restrictive family of domains.

Let dA denote the planar Lebesgue measure. Following Aharonov and Shapiro [2] a bounded domain Ω of the complex plane is called a *quadrature domain* if there exists a finitely supported distribution u on the complex plane such that $\operatorname{supp}(u) \subset \Omega$ and the following Gaussian type quadrature formula:

$$\int_{\Omega} f dA = u(f), \qquad f \in L_a^1(\Omega),$$

holds for all integrable, complex analytic functions f in Ω. For instance a disk $D(a, r)$ satisfies such a formula with $u(f) = \pi r^2 f(a)$. This class of domains turns out to have many remarkable properties and connections to several areas of mathematics. For an account of their theory we refer to the monograph [16], the papers [2], [6], [7], [8] and the references cited there.

For the purposes of this note we need only know that the boundary of a quadrature domain Ω is real algebraic (and irreducible) given by an

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equation of the form:

$$|P(z)|^2 = \sum_{k=0}^{d-1} |Q_k(z)|^2, \tag{1}$$

where $P(z)$ is a monic polynomial of degree $d \geq 1$ and each $Q_k(z)$, $0 \leq k \leq d-1$ is a polynomial of degree k. This unique form of the boundary equation comes from operator theory (via the localized resolvent of a certain matrix) but we do not expand these details here, see [8]. However, in view of possible applications, it is worth recalling that every bounded planar domain can be approximated in the Hausdorff distance by a sequence of quadrature domains, see [6].

The aim of the present note is to show that the classical method of separating roots due to Hermite and later refined by Routh, Hurwitz, Schur, Cohn, Liénard and Chipard, and many other authors, can be combined with the form (1) of the equation of the boundary of a planar domain Ω, to obtain matricial criteria for the root location of an arbitrary polynomial, with respect to the domain Ω. The techical tools used below (Theorem 2.2) are elementary: the above expression of the defining equation of the domain will be combined with a simple Hilbert space remark; then Hermite's separation method (with respect to the half-space), or Schur's criterion (with respect to the disk) will be invoked.

It is interesting to remark that *simply connected* quadrature domains are in a sense "dual" to the sub-level domains ω of the real part of a rational function $R(z)$:

$$\omega = \{z \in \mathbf{C};\ \Re R(z) < 0\}, \tag{2}$$

treated by Hermite (for root separation purposes) in his original memoir [9]. Indeed, every simply connected quadrature domain Ω is of the form $\Omega = r\mathbf{C}_+$, where r is a rational function which is conformal on the upper half plane $\mathbf{C}_+$, see [6]. In this case a simple variable change via the map r will reduce the root separation problem with respect to Ω to the upper half plane, see Proposition 2.1 below.

Via the Bezoutiant of a pair of polynomials, almost all known root separation criteria can be put into matricial form and can be combined with classical inertia and stability results, see [5]. This line of research was amply developed from the perspective of matrix theory and linear control theory; for references see [4] and [14].

For an account of the history and vast literature devoted to root location see [10], and for an updated version [4], [3], and the references cited there.

The defining equation of type (1) of a bounded domain Ω brings immediately into discussion the recent investigations on complete Nevanlinna-Pick kernels [1]. We illustrate below, without expanding the subject, (see Theorem 2.5) how this leads to a simple interpolation statement on the complement of Ω in the Riemann sphere.

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2. Main results

Let $d \geq 1$ be a fixed integer and let $R : \mathbf{C} \longrightarrow \mathbf{C}^d$ be a rational, vector valued map with the property that $\lim_{z\to\infty} R(z) = 0$. Then the open set, which will be called a *generalized lemniscate*,

$$\Omega_R = \{z \in \mathbf{C};\ \|R(z)\| > 1\}$$

is relatively compact in $\mathbf{C}$, and it contains all the poles of R. Without loss of generality we can assume that the vector space $\mathbf{C}^d$ is spanned by the vectors $R(z)$, $z \in \mathbf{C}$.

Note that this is a slightly more general set than a quadrature domain, in which case we can take, with the notation in formula (1) above:

$$R(z) = \left(\frac{Q_0(z)}{P(z)}, \frac{Q_1(z)}{P(z)}, \dots, \frac{Q_{d-1}(z)}{P(z)}\right).$$

Suppose first that the domain $\Omega = \Omega_R$ above is a simply connected quadrature domain obtained as the image of the upper half plane by a rational conformal map r of degree d. There is no accident that the degree of r and the dimension of the Euclidean space where the map R takes values coincide, see for instance [8]. Let

$$f(z) = \prod_{k=1}^{n}(z - \alpha_k) = z^n + c_1 z^{n-1} + \cdots + c_n, \tag{3}$$

be a monic polynomial of degree n. The zero set of f will henceforth be denoted $V(f)$, while the number of zeroes of f contained in a certain set A

will be denoted by $|V(f) \cap A|$.

The map $f(r(u))$ is rational, of degree dn, and its coefficients are polynomial functions of the coefficients $c_1, \ldots, c_n$. The counting of the zeroes of $f \circ r$ in the upper half-plane can be identified, following Hermite, with the inertia data (number of zeroes, and positive, respectively negative squares) of an associated quadratic form. See [10] for several equivalent statements of this fact.

Due to the injectivity of r on the upper half plane, remark that the number of zeroes of f in the domain Ω is equal to the number of zeroes of $f \circ r$ in $\mathbf{C}_+$.

The number of zeroes of f on the boundary of Ω depends on the double tangency points of the domain Ω. Indeed, recall [6] that the conformal map r may not be injective on the boundary of the upper half plane. Specifically, there are finitely many pairs (σ_j, λ_j), $1 \leq j \leq m$, of distinct points of the extended real axis $\hat{\mathbf{R}} = \mathbf{R} \cup \infty$, (σ_j, λ_j), $1 \leq j \leq m$, with the property that $r(\sigma_j) = r(\lambda_j)$, $1 \leq j \leq m$. Except for this set, the function r is injective on $\mathbf{R}$. If one of the roots of the polynomial f coincides with one of the points $r(\sigma_j)$, then the composed function $f \circ r$ will have both points σ_j, λ_j in its zero set.

In conclusion, we have proved the following proposition.

Proposition 2.1 *Let r be a rational conformal map of the upper half-plane onto the quadrature domain Ω, and assume that $\partial\Omega$ has m double points ($m \geq 0$). Then*

$$|V(f) \cap \Omega| = |V(f \circ r) \cap \mathbf{C}_+|,$$

and

$$|V(f) \cap \partial\Omega| \leq |V(f \circ r) \cap \hat{\mathbf{R}}| \leq |V(f) \cap \partial\Omega| + m.$$

Returning now to the general class of generalized lemniscates Ω_R associated as before to a vector valued rational map R, we remark that a point α belongs to $\mathbf{C} \setminus \overline{\Omega_R}$ if and only if, by definition, $\|R(\alpha)\| < 1$. In its turn, the latter condition is equivalent to $|\langle R(\alpha), v\rangle| < 1$ for all unit vectors $v \in \mathbf{C}^d$, or at least for the vectors of the form $v = R(\beta)/\|R(\beta)\|$, where β is not a pole of at least one, or a common zero of all, entries of R. Note that in the last formula $R(\alpha)$ depends rationally on the root α. Schur's criterion of separation with respect to the unit disk can then be applied, [15]. In order to state the main result, let us introduce the rational function

$$Q(z,\overline{w}) = 1 - \langle R(z), R(w)\rangle,$$

so that the domain Ω_R has the defining equation

$$\Omega_R = \{z \in \mathbf{C};\ Q(z,\overline{z}) < 0\}.$$

Theorem 2.2 *Let $R : \mathbf{C} \longrightarrow \mathbf{C}^d$ be a rational function satisfying $\lim_{z\to\infty} R(z) = 0$, and let $\Pi \subset \mathbf{C}$ be the set of all poles and common zeroes of R.*

Then a monic polynomial f has all its roots $\alpha_1, \ldots, \alpha_n$ in the open set $\mathbf{C} \setminus \overline{\Omega}$ if and only if, for every $\beta \in \mathbf{C} \setminus \Pi$, the polynomial

$$F_\beta(X) = \prod_{j=1}^{n} \left(X - \frac{1 - Q(\alpha_j, \overline{\beta})}{\sqrt{1 - Q(\beta, \overline{\beta})}} \right) \tag{4}$$

has all its roots in the unit disk.

Proof. Let f be a polynomial with all roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in the set $\mathbf{C} \setminus \overline{\Omega}$. Then $\|R(\alpha_i)\| < 1$ for all $i, 1 \le i \le n$. Consequently, if $\beta \in \mathbf{C} \setminus \Pi$ we obtain:

$$|\langle R(\alpha_i), R(\beta)\rangle| < \|R(\beta)\|,$$

which is exactly condition (4) in the statement.

Conversely, if (4) holds for all $\beta \in \mathbf{C}\setminus\Pi$, then by reversing the preceding argument we find that $\|R(\alpha_i)\| < 1$, $1 \le i \le n$. □

Note that the polynomial $F_\beta(X)$ is a symmetric function of the roots $\alpha_j, 1 \le j \le n$, hence its coefficients are rational functions of $c_1, \ldots, c_n$. Therefore Schur's criterion will involve only rational combinations of the coefficients $c_1, \ldots, c_n$.

Specifically, if $F(z)$ is a polynomial with complex coefficients of degree d, we define the associated polynomials:

$$\overline{F}(z) = \overline{F(\overline{z})}, \quad F^*(z) = z^d \overline{F}\left(\frac{1}{z}\right);$$

then the inertia of the bilinear form:

$$G_F(X,Y) = \frac{F^*(X)\overline{F}^*(Y) - F(X)\overline{F}(Y)}{1 - XY},$$

gives full information about the root location of F with respect to the unit disk. That is, if G_F has d_+ positive squares and d_- negative squares, then the polynomial F has exactly d_+ roots in the unit disk, d_- roots outside the closed disk, and $d-d_+-d_-$ roots lie on the unit circle, see [10], Proposition XVa.

Variations of Theorem 2.2 are readily available: for instance one can replace the rational map $R(z)$ by a polynomial map, or instead of F_β one can consider the polynomial involving the squares of the roots of F_β, and so on.

If we want to have more information about the root location of the polynomial $f(z) = (z-\alpha_1)\cdots(z-\alpha_n)$, then the scalar products $\langle R(\alpha_j), v\rangle$, with v a fixed unit vector, can be replaced by an expression such as $\langle R(\alpha_j), S(\overline{\alpha_j})\rangle$, where $S(z)$ is a vector valued rational function, of norm less than one in a large disk, where the roots are first estimated to be. Then, by counting parameters, the degree of S can be chosen to be dependent on n, the degree of f.

In order to state such a result, we make the following notation: for $S : \mathbf{C} \longrightarrow \mathbf{C}^d$ a vector valued rational map, let

$$F_S(X) = \prod_{j=1}^{n}(X - \langle R(\alpha_j), S(\overline{\alpha_j})\rangle). \tag{5}$$

Note that this polynomial in X depends rationally on the entries α_j and is symmetrical in them. We have then

Corollary 2.3 *In the conditions of Theorem 2.2, let $U = t\mathbf{D}$ be a disk centered at the origin, that contains all the roots of the polynomial $f(z)$.*

Let $S : \mathbf{C} \longrightarrow \mathbf{C}^d$ be a rational map of degree less than or equal to s on each entry, satisfying $\|S(z)\| \le 1$, $z \in U$, where we assume: $(2s+1)^d > dn$.

Then, with the above notations, we have:

$$|V(f) \cap \Omega_R| = \max_S |V(F_S) \setminus \overline{\mathbf{D}}|,$$

and

$$|V(f) \setminus \overline{\Omega_R}| = \min_S |V(F_S) \cap \mathbf{D}|.$$

Proof. Let $d_+ = |V(f) \setminus \overline{\Omega_R}|$ and $d_- = |V(f) \cap \Omega_R|$.

Since $\|S(\overline{\alpha_j})\| \le 1$ for all j, $1 \le j \le n$, we have $\langle R(\alpha_j), S(\overline{\alpha_j})\rangle \le \|R(\alpha_j)\|$. Therefore, the polynomial F_S has at least d_+ zeroes in the unit disk and at most d_- zeroes ouside its closure.

To see that these bounds are attained, we remark that, due to the degree assumption, the map $S(z)$ can be chosen to have prescribed values at every point $\overline{\alpha_i}$, $1 \leq j \leq n$. Thus we can choose the values $S(\overline{\alpha_j})$ so that $\langle R(\alpha_j), S(\overline{\alpha_j})\rangle = \|R(\alpha_j)\|$. □

Going into another direction, it is easy to establish *sufficient* criteria for the roots of the polynomial f to be all contained in the exterior of $\overline{\Omega}$. Let us denote the defining rational map by $R(z) = (R_1(z), \dots, R_d(z))$.

Corollary 2.4 *In the conditions of Theorem* 2.2, *let* a_i, $1 \leq i \leq d$, *be positive numbers satisfying* $a_1^2 + a_2^2 + \cdots + a_d^2 = 1$.

Define the polynomials:

$$F_i(X) = \prod_{j=1}^{n} \left(X - \frac{R_i(\alpha_j)}{a_i} \right), \quad 1 \leq i \leq d. \tag{6}$$

If the roots of each $F_i, 1 \leq i \leq d$, *are contained in the unit disk, then the roots of* f *are contained in* $\mathbf{C} \setminus \overline{\Omega}$.

Proof. It is sufficient to remark that, under the assumption for the roots of F_j, for each fixed j, $1 \leq j \leq n$, we have

$$\|R(\alpha_j)\|^2 \leq \sum_{i=1}^{d} \|R_i(\alpha_j)\|^2 < \sum_{i=1}^{d} a_i^2 = 1.$$

□

Henceforth we denote by $\tilde{\mathbf{C}} = \mathbf{C} \cup \infty$ the Riemann sphere. Without aiming at full generality, we consider below a bounded interpolation problem on the complement $\tilde{\mathbf{C}} \setminus \Omega$ of a quadrature domain Ω with smooth boundary, defined by the equation:

$$Q(z, \overline{z}) = |P(z)|^2 - \sum_{k=0}^{d-1} |Q_k(z)|^2 < 0. \tag{7}$$

Recall, cf. formula (1), that $P(z)$ is a degree d polynomial with all roots contained in Ω, while $Q_k(z)$ are polynomials of degree exactly equal to k, $0 \leq k \leq d-1$.

Theorem 2.5 *Let* Ω *be a simply-connected, smooth quadrature domain of equation* (7). *Let* $a_i \in \Omega^c = \tilde{\mathbf{C}} \setminus \Omega$, $1 \leq i \leq n$, *be distinct points and let*

$b_i \in \mathbf{D}$, $1 \le i \le n$.

Then there exists an analytic function $f : \Omega^c \longrightarrow \mathbf{D}$ satisfying the interpolation constraints $f(a_i) = b_i$, $1 \le i \le n$, if and only if the matrix

$$\left(\frac{1 - b_i \overline{b_j}}{Q(a_i, \overline{a_j})} \right)_{i,j=1}^{n}$$

is non-negative definite.

Proof. According to the main result of [1], we only have to prove that the bounded multiplier algebra of the Hilbert space H_1 of analytic functions with reproducing kernel

$$\frac{P(z)\overline{P(w)}}{Q(z, \overline{w})}, \quad z, w \in \Omega^c,$$

coincides with $H^\infty(\Omega^c)$. Indeed, due to formula (7), this kernel has the complete Nevanlinna-Pick property (in the terminology of [1]).

The identification of the multiplier space can be done directly, by an inspection of the singularity of $\frac{1}{Q(z,\overline{w})}$ along the diagonal, at boundary points of Ω^c, supposed by assumption to be smooth, or via operator theory, as sketched below.

Let T be the irreducible hyponormal operator with rank one self-commutator $[T^*, T] = \xi \otimes \xi$ and principal function equal to the characteristic function of Ω. According to [13], the defining equation of the domain Ω is related to the operator T by the equation:

$$\frac{P(z)\overline{P(w)}}{Q(z, \overline{w})} = 1 + \langle (T - z)^{-1}\xi, (T - w)^{-1}\xi \rangle, \quad z, w \in \Omega^c. \tag{8}$$

Following an idea of D. Xia, cf. the references in [12], we can represent the operator T as multiplication by the complex variable z on a Hilbert space H_2 with reproducing kernel $\frac{P(z)\overline{P(w)}}{Q(z,\overline{w})}$ supported by the boundary $\partial\Omega$. This analytic model provides in particular a similarity between T and the standard Hardy space shift $M_z \in L(H^2(\Omega))$, for details see [12].

On the other hand the spaces H_1 and H_2 are in Grothendieck-Köthe duality, hence H_1 is isomorphic to $H^2(\Omega^c)$, and this implies that the multiplier algebra of H_1 is precisely $H^\infty(\Omega^c)$.

Indeed, let us consider the direct sum Hilbert space $K = \mathbf{C} \oplus H_2$. Then formula (8) yields:

$$\frac{P(z)\overline{P(w)}}{Q(z,\overline{w})} = \langle 1 \oplus (T-z)^{-1}\xi, 1 \oplus (T-w)^{-1}\xi \rangle.$$

A generic element in the reproducing Hilbert space H_1 is of the form

$$f(z) = \sum_{j=1}^{m} c_j \frac{P(z)\overline{P(w_j)}}{Q(z,\overline{w_j})} = \langle 1 \oplus (T-z)^{-1}\xi, x \rangle,$$

where $x = \sum_{j=1}^{m} \overline{c_j}(1 \oplus (T-w_j)^{-1}\xi)$, and w_j are distinct points of Ω^c. Without loss of generality we can change the constant c_j corresponding to the point at infinity (or add it), so that $\sum_{j=1}^{m} c_j = 0$. In this way the first component of x is zero and moreover $\|f\|_{H_1} = \|x\|_{H_2}$.

Let us denote $\Gamma = \partial\Omega$ and consider the Lebesgue space $L^2(\Gamma)$ with respect to the arc length element. Let $g(z)$ be an analytic function defined in a neighbourhood of $\overline{\Omega}$. The main result of [12] asserts that the norms $\|g\|_{2,\Gamma}$ and $\|g(T)\xi\|$ are equivalent.

Let f be an analytic function outside Ω, vanishing at infinity and let g be an analytic function defined in a neighbourhood of $\overline{\Omega}$. Then:

$$\int_{\Gamma} f(z)g(z)dz = -2\pi i \langle g(T)\xi, x \rangle.$$

The first bilinear form above is what we called the Grothendieck-Köthe pairing and it is well known to be non-degenerated. Since the second duality pair above is non-degenerated, too and the norms $\|g\|_{2,\Gamma}$ and $\|g(T)\xi\|$ are equivalent, we infer that there are positive constants c, C with the property that:

$$c\|f\|_{2,\Gamma} \leq \|f\|_{H_1} = \|x\| \leq C\|f\|_{2,\Gamma}.$$

In particular this implies that the multiplier algebra of the Hilbert space H_1 of analytic functions on Ω^c is exactly $H^\infty(\Omega^c)$. □

Note that in Theorem 2.5 above the necessary and sufficient condition in the statement requires only the defining function of the boundary, and not more involved kernels, such as the Green function, the Szegö or Bergman kernels of Ω^c. We will resume elsewhere this interpolation subject on quadrature domains, in more generality and full detail.

3. Examples

We consider below two examples, in low degrees, which illustrate the advantages and the limits of Proposition 2.1, respectively Theorem 2.2.

3.1. Let $z = \phi(w) = w^2 + rw$, $r > 2$, be the conformal map of the unit disk $\mathbf{D}$ onto a quadrature domain Ω of equation:

$$|z|^4 - (2 + r^2)|z|^2 - r^2 z - r^2 \overline{z} + 1 - r^2 < 0,$$

see [8]. Let $f(z) = z^2 + c_1 z + c_2$ be a degree two polynomial whose roots we want to locate with respect to Ω.

According to Proposition 2.1, we have to compute the composed map:

$$\begin{aligned} f \circ \phi(w) &= (w^2 + rw)^2 + c_1(w^2 + rw) + c_2 \\ &= w^4 + 2rw^3 + (r^2 + c_1)w^2 + rc_1 w + c_2, \end{aligned}$$

and to the latter to apply Schur's criterion.

Note that in this case there are no multiple points on the boundary, so that:

$$|V(f) \cap \Omega| = |V(f \circ \phi) \cap \mathbf{D}|,$$

and

$$|V(f) \cap \partial\Omega| = |V(f \circ \phi) \cap \partial\mathbf{D}|.$$

3.2. Suppose now that we have a quadrature domain Ω of equation

$$|z|^4 - z^2 - \overline{z}^2 - r^2|z|^2 < 0, \quad r > \sqrt{2},$$

see again [8]. But this time suppose that we do not know the conformal map of the disk onto Ω.

First we write the equation of Ω in canonical form:

$$|z|^4 - z^2 - \overline{z}^2 - r^2|z|^2 = (z^2 - 1)(\overline{z}^2 - 1) - r|z|^2 - 1,$$

so that an associated rational map is:

$$R(z) = \left(\frac{rz}{z^2 - 1}, \frac{1}{z^2 - 1} \right),$$

and $\Omega = \{z;\ \|R(z)\| > 1\}$.

Let $f(z) = z^2 + c_1 z + c_2 = (z - \alpha_1)(z - \alpha_2)$ be the polynomial whose roots we want to locate (with respect to Ω) and let $w = (\overline{u}, \overline{v})$ be an arbitrary unit vector in $\mathbf{C}^2$.

According to (a weaker version of) Theorem 2.2, we have to form the polynomial:

$$\begin{aligned} F_w &= (X - \langle R(\alpha_1), w\rangle)(X - \langle R(\alpha_2), w\rangle) \\ &= \left(X - \frac{r\alpha_1 u + v}{\alpha_1^2 - 1}\right)\left(X - \frac{r\alpha_2 u + v}{\alpha_2^2 - 1}\right) = X^2 - d_1 X + d_2, \end{aligned}$$

where:

$$d_1 = -\frac{r(c_1 - c_2 c_1)u + (c_1^2 - 2c_2 - 2)v}{c_2^2 - c_1^2 + 2c_2 + 1},$$

and

$$d_2 = \frac{r^2 c_2 u^2 - r c_1 u v + v^2}{c_2^2 - c_1^2 + 2c_2 + 1}.$$

Thus, by applying Schur's criterion, both roots of f lie outside $\overline{\Omega}$ if and only if $|d_2|^2 + |d_1 - \overline{d_1} d_2| < 1$, for all unit vectors $w = (\overline{u}, \overline{v})$.

Our root separation problem is thus reduced to a standard extremum problem, that of finding the minimum:

$$\min_{|u|^2+|v|^2=1} (1 - |d_2|^2 - |d_1 - \overline{d_1} d_2|) > 0.$$

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