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Examples of compact Toeplitz operators on the Bergman space

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Abstract. R. Yoneda studied compact Toeplitz operators on the Bergman space for special symbols and he posed several problems. In this paper, we give counterexamples for some of these problems.

Key words: Bergman space, Toeplitz operator, compact operator.

1. Introduction

Let D be the open unit disc in the complex plane $\mathbb{C}$. Let dA be the normalized area measure on D. The Bergman space on D, denoted by $L_a^2(D)$, is the space of analytic functions f on D such that

$$\|f\|^2 = \int_D |f(z)|^2 dA(z) < \infty.$$

Let P be the orthogonal projection from $L^2(D, dA)$ onto $L_a^2(D)$. For ϕ in $L^\infty(D)$ the Toeplitz operator $T_\phi : L_a^2(D) \to L_a^2(D)$ is defined by $T_\phi f = P(\phi f)$, $f \in L_a^2(D)$. Put

$$k_z(w) = \frac{1-|z|^2}{(1-\bar{z}w)^2} \quad \text{for} \quad z, w \in D,$$

and k_z is called the normalized reproducing kernel for z. For $z \in D$, define

$$\varphi_z(w) = \frac{z-w}{1-\bar{z}w}, \quad w \in D.$$

It is known several characterization for the compactness of T_ϕ. In [5, Theorem 4], Zheng proved the next theorem.

Theorem A *Let ϕ be in $L^\infty(D)$. Then the following are equivalent.*
(i) *T_ϕ is a compact operator on $L_a^2(D)$.*
(ii) *$\|T_\phi k_z\| \to 0$ as $|z| \to 1-$.*

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(iii) $\|P(\phi \circ \varphi_z)\| \to 0$ *as* $|z| \to 1-$.

In [1, Corollary 2.5], Axler and Zheng proved the next theorem.

Theorem B *Let* ϕ *be in* $L^\infty(D)$. *Then* T_ϕ *is a compact operator on* $L^2_a(D)$ *if and only if* $\tilde{\phi}(z) \to 0$ *as* $|z| \to 1-$, *where*

$$\tilde{\phi}(z) = \int_D (\phi \circ \varphi_z)(w) dA(w) \quad z \in D.$$

Theorem B supplies the most useful characterization of the compact Toeplitz operators in the sense that to check the condition $\tilde{\phi}(z) \to 0$ as $|z| \to 1-$ is easier than the conditions in Theorem A.

Let

$$S_z = \{w \in D : |z| < |w| < 1, \ |\arg z - \arg w| < 2\pi(1 - |z|)\}$$

be the Carleson square at z and $|S_z|$ be the dA-measure of S_z. The next theorem is an immediate consequence of Luecking's result [3, p.349].

Theorem C *Let* ϕ *be a nonnegative function on* D. *Then* T_ϕ *is a compact operator on* $L^2_a(D)$ *if and only if* $\hat{\phi}(z) \to 0$ *as* $|z| \to 1-$, *where*

$$\hat{\phi}(z) = \frac{1}{|S_z|} \int_{S_z} \phi(w) dA(w) \quad z \in D.$$

In [2], Korenblum and Zhu characterized the compactness of T_ϕ for a bounded radial function ϕ in D.

Theorem D *Let* ϕ *be a bounded radial function in* D. *Then* T_ϕ *is a compact operator on* $L^2_a(D)$ *if and only if*

$$\lim_{x \to 1-} \frac{1}{1-x} \int_x^1 \phi(r) dr = 0.$$

Recently, Yoneda generalized this theorem for some special symbols [4]. And he posed several problems. The purpose of this paper is to give counterexamples for some of his problems.

2. Examples

The following is one of Yoneda's problems.

Problem Let $\{a_n\}$ be a sequence in $[0, 1)$ such that $0 = a_0 < a_1 < \cdots < a_n$ and $a_n \to 1$ as $n \to \infty$. Let $E_n = [a_n, a_{n+1})$. Let $\phi(re^{i\theta}) =$

$\sum_{n=0}^{\infty} e^{in\theta}\chi_{E_n}(r)$. Whether T_ϕ is compact or not?

We shall show that both cases occur. An example for which T_ϕ is not compact is given in Example 1 and an example for which T_ϕ is compact is given in Example 2.

Example 1 We choose a sequence $\{R_n\} \subset (\frac{1}{2}, 1)$ such that R_n increases to 1. By induction, we can choose sequences $\{a_n\}$ and $\{r_n\}$ which satisfy the following;

$$\left|\frac{1}{(a_n)^n} - 1\right| < \frac{1}{n} \quad \text{for } n \geq 1, \tag{1}$$

$$0 = a_0 < a_n < r_n < a_{n+1} < 1 \quad \text{for } n \geq 1, \tag{2}$$

and

$$\varphi_{r_n}(R_n) = a_n, \quad \varphi_{r_n}(-R_n) < a_{n+1}. \tag{3}$$

First, put $r_0 = R_0$. Then $\varphi_{r_0}(R_0) = a_0 = 0$ and $a_0 < r_0$. We find a_1 such that $\left|\frac{1}{a_1} - 1\right| < 1$ and $\varphi_{r_0}(-R_0) < a_1$. Then $a_0 < r_0 < \varphi_{r_0}(-R_0) < a_1$. Suppose that $r_0, \ldots, r_{k-1}$ and $a_0, \ldots, a_k$ are chosen satisfying (1), (2) and (3). There exists r_k such that $\varphi_{r_k}(R_k) = a_k$. Then $a_k < r_k$. Choose a_{k+1} such that $\left|\frac{1}{(a_{k+1})^{k+1}} - 1\right| < \frac{1}{k+1}$ and $\varphi_{r_k}(-R_k) < a_{k+1}$. Then $a_k < r_k < a_{k+1}$. This completes the induction.

Put $E_n = [a_n, a_{n+1})$ and $\phi(re^{i\theta}) = \sum_{n=0}^{\infty} e^{in\theta}\chi_{E_n}(r)$. Then

$$\left|\int_D \phi \circ \varphi_{r_n} dA - \int_{D_{R_n}} \phi \circ \varphi_{r_n} dA\right| \to 0 \quad \text{as } n \to \infty \tag{4}$$

and

$$\left|\int_D z^n \circ \varphi_{r_n} dA - \int_{D_{R_n}} z^n \circ \varphi_{r_n} dA\right| \to 0 \quad \text{as } n \to \infty, \tag{5}$$

where $D_{R_n} = \{z \in \mathbb{C} : |z| < R_n\}$. We have

$$\varphi_{r_n}(R_n) \leq \left|\frac{r_n - w}{1 - r_n w}\right| \leq \varphi_{r_n}(-R_n), \quad w \in D_{R_n}.$$

Then by (3),

$$\varphi_{r_n}(D_{R_n}) \subset \{re^{i\theta} : a_n \leq r < a_{n+1}\}. \tag{6}$$

Therefore

$$\int_{D_{R_n}} \phi \circ \varphi_{r_n} dA = \int_{D_{R_n}} e^{in\theta} \circ \varphi_{r_n} dA = \int_{D_{R_n}} \frac{z^n \circ \varphi_{r_n}}{|z^n \circ \varphi_{r_n}|} dA. \tag{7}$$

By (6) and (1),

$$\left| \int_{D_{R_n}} \left(\frac{z^n \circ \varphi_{r_n}}{|z^n \circ \varphi_{r_n}|} - z^n \circ \varphi_{r_n} \right) dA \right| \leq \int_{D_{R_n}} \left| \frac{1}{(a_n)^n} - 1 \right| dA$$
$$\leq \frac{1}{n} dA(D_{R_n}).$$

Then by (7),

$$\left| \int_{D_{R_n}} \phi \circ \varphi_{r_n} dA - \int_{D_{R_n}} z^n \circ \varphi_{r_n} dA \right| \to 0 \quad \text{as } n \to \infty.$$

Hence by (4) and (5),

$$\int_D \phi \circ \varphi_{r_n} dA - \int_D z^n \circ \varphi_{r_n} dA \to 0 \quad \text{as } n \to \infty.$$

Now, by [6, p.52],

$$\int_D z^n \circ \varphi_{r_n} dA = \langle z^n k_{r_n}, k_{r_n} \rangle = (r_n)^n.$$

Therefore

$$\int_D \phi \circ \varphi_{r_n} dA - (r_n)^n \to 0 \quad \text{as } n \to \infty.$$

By (1), $(a_n)^n \to 1$. Then by (2), $(r_n)^n \to 1$. Hence

$$\int_D \phi \circ \varphi_{r_n} dA \to 1 \quad \text{as } n \to \infty.$$

By Theorem B, T_ϕ is not compact.

Example 2 Let $0 \leq t < 1$. Then we have

$$\sup_{0 \leq r \leq t} \left| \int_0^{2\pi} \frac{e^{in\theta}}{|1 - re^{i\theta}|^4} d\theta / 2\pi \right| \to 0 \quad \text{as } n \to \infty.$$

Let N_t be the smallest positive integer satisfying

$$\sup_{0 \leq r \leq t} \left| \int_0^{2\pi} \frac{e^{in\theta}}{|1 - re^{i\theta}|^4} d\theta / 2\pi \right| \leq \frac{1}{2} \quad \text{for all } n \geq N_t. \tag{8}$$

Then it is easy to see that $N_0 = 1$, N_t increase with respect to t, $N_t \to \infty$ as $t \to 1$, N_t is left continuous, and $N_t = 1$ for sufficient small t. Put

$$\{n_j\}_{j=0}^{\infty} = \{N_t : 0 \leq t < 1\}, \quad \text{where } n_j < n_{j+1} \text{ for any } j.$$

Then $n_0 = 1$. For each positive integer j, we define $c_j = \inf\{t : N_t = n_j\}$. Then we get

$$0 = c_0 < c_1 < \cdots < 1,$$
$$\{t : N_t = n_0\} = [0, c_1],$$

and

$$\{t : N_t = n_j\} = (c_j, c_{j+1}] \quad j \geq 1.$$

Next we divide the interval $[0, c_1]$ into n_1 equal intervals. And we divide the interval $(c_j, c_{j+1}]$ into n_{j+1} equal intervals. Then we get divided points $\{a_k\}$ such that

$$0 = a_0 < a_1 < \cdots < a_k < 1 \quad \text{and} \quad a_k \to 1 \text{ as } k \to \infty.$$

For a sufficiently large k, there exist a unique $j_k \geq 1$ such that $[a_k, a_{k+1}) \subset [c_{j_k}, c_{j_k+1}]$. We put $E_k = [a_k, a_{k+1})$. Then by the above, we have

$$N_t \leq n_{j_k} \quad \text{for all } t \in E_k \quad \text{and} \quad n_{j_k} \leq k. \tag{9}$$

Put $\phi(re^{i\theta}) = \sum_{k=0}^{\infty} e^{ik\theta} \chi_{E_k}(r)$. Let $r \in E_k$. By (9), $N_r \leq n_{j_k} \leq k$. Since N_t is left continuous, $N_{a_{k+1}} \leq k$. By (8),

$$\sup_{0 \leq r \leq a_{k+1}} \left| \int_0^{2\pi} \frac{e^{ik\theta}}{|1 - re^{i\theta}|^4} d\theta / 2\pi \right| \leq \frac{1}{2}.$$

Therefore

$$\left| \int_0^{2\pi} \frac{e^{ik\theta}}{|1 - |z|re^{i\theta}|^4} d\theta / 2\pi \right| \leq \frac{1}{2}$$

for $r \in E_k$ and $z \in D$. Thus

$$\left| \int_D \phi \circ \varphi_z dA \right| = \left| \int_D \phi |k_z|^2 dA \right| = (1 - |z|^2)^2 \left| \int_D \frac{\phi(w)}{|1 - \bar{z}w|^4} dA(w) \right|$$
$$= (1 - |z|^2)^2 \left| \int_0^{2\pi} \int_0^1 \frac{\sum_{k=0}^{\infty} e^{ik\theta} \chi_{E_k}(r)}{|1 - \bar{z}re^{i\theta}|^4} 2r dr d\theta / 2\pi \right|$$

$$\leq (1-|z|^2)^2 \sum_{k=0}^{\infty} \int_{a_k}^{a_{k+1}} 2rdr \left| \int_0^{2\pi} \frac{e^{ik\theta}}{|1-\bar{z}re^{i\theta}|^4} d\theta/2\pi \right|$$

$$\leq (1-|z|^2)^2 \sum_{k=0}^{\infty} \int_{a_k}^{a_{k+1}} rdr$$

$$= \frac{1}{2}(1-|z|^2)^2 \to 0 \quad \text{as} \quad |z| \to 1.$$

Hence by Theorem B, T_ϕ is compact.

For any ψ in $L^\infty(D)$, we put

$$\psi_j(r) = \int_0^{2\pi} \psi(re^{i\theta})e^{-ij\theta} d\theta/2\pi \quad (j \in Z),$$

where Z is the set of all integers. Yoneda asked whether the following conditions are equivalent or not;

(i) T_ψ is compact,

(ii) $\lim_{x\to 1-} \frac{1}{1-x} \int_x^1 \psi_j(r)dr = 0 \quad (j \in Z)$.

In [4, Theorem 1], Yoneda proved that condition (i) implies condition (ii). But condition (ii) does not imply condition (i). For, let $\triangle$ be a triangle with vertices $e^{i\alpha}$, $e^{i\beta}$, $e^{i\gamma}$, and ψ be the characteristic function of $\triangle$. By Theorem C, it is easy to see that T_ψ is not compact. Since

$$|\psi_j(r)| \leq \int_0^{2\pi} \psi(re^{i\theta}) d\theta/2\pi \to 0 \quad \text{as} \quad r \to 1,$$

then we have

$$\lim_{x\to 1-} \frac{1}{1-x} \int_x^1 \psi_j(r)dr = 0 \quad (j \in Z).$$

For any ϕ in $L^\infty(D)$, we put

$$\Phi(xe^{i\theta}) = \frac{1}{1-x} \int_x^1 \phi(re^{i\theta})dr$$

and if the limit exists as $x \to 1-$, we put

$$\Phi(e^{i\theta}) = \lim_{x\to 1-} \Phi(xe^{i\theta}).$$

Then Yoneda showed the existence of ϕ such that $\Phi(e^{i\theta}) = 0$ $a.e.$ θ and T_ϕ is not compact. And Yoneda asked whether the following assertion holds or not; if $\phi(re^{i\theta})$ is a θ-continuous function for each $r \in [0,1]$ and $\Phi = 0$ *a.e.* θ, then T_ϕ is compact. Let $\triangle$ be a triangle with vertices $e^{i\alpha}$, $e^{i\beta}$, $e^{i\gamma}$, and $\chi_\triangle$ be the characteristic function of $\triangle$. There exists a sequence of continuous functions $\{\phi_n\}_n$ such that $0 \leq \phi_{n+1} \leq \phi_n \leq 1$ on D, $\phi_n(re^{i\theta}) \to 0$ as $r \to 1-$ for $e^{i\theta} \notin \{e^{i\alpha}, e^{i\beta}, e^{i\gamma}\}$, and $\phi_n \to \chi_\triangle$ pointwisely. Then $\Phi_n(e^{i\theta}) = 0$ *a.e.*, and by Theorem C, it is not difficult to see that T_{ϕ_n} is not compact for a large n.

Also Yoneda asked [4, p.573] that there is an example of ϕ such that T_ϕ is compact and $\Phi(e^{i\theta}) \not\equiv 0$. An example of such ϕ is the following. Let E be a concave triangle with vertices 1, $\frac{1}{2}i$, and $-\frac{1}{2}i$ such that the angle of E at 1 is zero. Let ϕ be the characteristic function of E. Then by Theorem C, T_ϕ is compact and $\Phi(1) = 1$.

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