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Local equivalence of Sacksteder and Bourgain hypersurfaces

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Abstract. Finding examples of tangentially degenerate submanifolds (submanifolds with degenerate Gauss mappings) in an Euclidean space R^4 that are noncylindrical and without singularities is an important problem of differential geometry. The first example of such a hypersurface was constructed by Sacksteder in 1960. In 1995 Wu published an example of a noncylindrical tangentially degenerate algebraic hypersurface in R^4 whose Gauss mapping is of rank 2 and which is also without singularities. This example was constructed (but not published) by Bourgain.

In this paper, the authors analyze Bourgain's example, prove that, as was the case for the Sacksteder hypersurface, singular points of the Bourgain hypersurface are located in the hyperplane at infinity of the space R^4, and these two hypersurfaces are locally equivalent.

Key words: Gauss mapping, varieties with degenerate Gauss mappings, hypercubic, Sacksteder, Bourgain.

1. It is important to find examples of tangentially degenerate submanifolds in order to understand the theory of such manifolds. These examples prove the existence of tangentially degenerate submanifolds and help to illustrate the theory. The first known example of a tangentially degenerate hypersurface of rank 2 without singularities in R^4 was constructed by Sacksteder [S60]. This example was examined from the differential geometry point of view by Akivis in [A87]. In particular, Akivis proved that the Sacksteder hypersurface has no singularities since they "went to infinity". In the same paper, Akivis presented a series of examples generalizing Sacksteder's example in R^4, constructed a new series of examples of three-dimensional submanifolds $V^3 \subset P^n(\mathbb{R})$, $n \geq 4$, of rank 2, whose focal surfaces are imaginary, and proved existence of submanifolds of this kind. Note that more examples of tangentially degenerate submanifolds without singularities can be found in [I98, I99a, I99b]. The examples are essentially based on classical Cartan's hypersurfaces (see [C39]).

Mori [M94] claims that he constructed "a one-parameter family of com-

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plete nonruled deformable hypersurfaces in R^4 with rank $r = 2$ almost everywhere". However, it follows immediately from his formulas that the hypersurfaces of his family are ruled hypersurfaces. Moreover, they are cylinders.

Also much progress on the study of tangentially degenerate submanifolds over the complex numbers has been made in [GH79], [L99], and [AGL]. In these papers and in the papers [FW95], [W95], and [WZ01], one can find more examples of tangentially degenerate submanifolds over the complex numbers.

Recently Wu [W95] published an example of a noncylindrical tangentially degenerate algebraic hypersurface in an Euclidean space R^4 which has a degenerate Gauss mapping but does not have singularities. This example was constructed (but not published) by Bourgain (see also [I98, I99a, I99b]). In the present paper, we investigate Bourgain's example from the point of view of the paper [A87] (see also Section 4.7 of our book [AG93]). In particular, we prove that, as was the case for the Sacksteder hypersurface, the Bourgain hypersurface has no singularities since they "went to infinity". Namely this analysis suggested an idea that Bourgain's and Sacksteder's examples must be equivalent. Moreover, this analysis showed that a hypersurface constructed in these examples is torsal, i.e., it is stratified into a one-parameter family of plane pencils of straight lines.

In addition, at the end of our paper we prove that the examples of Bourgain and Sacksteder are locally equivalent.

2. In Cartesian coordinates x_1, x_2, x_3, x_4 of the Euclidean space R^4, the equation of the Bourgain hypersurface B is

$$x_1 x_4^2 + x_2(x_4 - 1) + x_3(x_4 - 2) = 0 \tag{1}$$

(see [W95] or [I98, I99a, I99b]). Equation (1) can be written in the form

$$x_1 x_4^2 + (x_2 + x_3)x_4 - (x_2 + 2x_3) = 0. \tag{2}$$

Make in (2) the following admissible change of Cartesian coordinates:

$$x_2 + x_3 \to x_2, \quad x_2 + 2x_3 \to x_3.$$

Then equation (2) becomes

$$x_1 x_4^2 + x_2 x_4 - x_3 = 0. \tag{3}$$

Introduce homogeneous coordinates in R^4 by setting $x_i = \frac{z_i}{z_0}$, $i = 1, 2, 3, 4$. Then equation (3) takes the form

$$f = z_1 z_4^2 + z_0 z_2 z_4 - z_0^2 z_3 = 0. \tag{4}$$

Equation (4) defines a cubic hypersurface F in the space $\overline{R}^4 = R^4 \cup P^3_\infty$ which is an enlarged space R^4, i.e., it is the space R^4 enlarged by the hyperplane at infinity P^3_∞ (whose equation is $z_0 = 0$).

Denote by A_α, $\alpha = 0, 1, 2, 3, 4$, fixed basis points of the space $\overline{R}^4$. Suppose that these points have constant normalizations, i.e., that $dA_\alpha = 0$. An arbitrary point $z \in \overline{R}^4$ can be written in the form $z = \sum_\alpha z_\alpha A_\alpha$. We will take a proper point of the space $\overline{R}^4$ as the point A_0, and take points at infinity as the points A_1, A_2, A_3, A_4.

Equation (4) shows that the proper straight line $A_0 \wedge A_4$ defined by the equations $z_1 = z_2 = z_3 = 0$ and the plane at infinity defined by the equations $z_0 = z_4 = 0$ belong to the hypersurface F defined by equation (4).

We write the equations of the hypersurface F in a parametric form. To this end, we set

$$z_0 = 1, \quad z_4 = p, \quad z_1 = u, \quad z_3 = pv.$$

Then it follows from (4) that

$$z_2 = v - pu.$$

This implies that an arbitrary point $z \in F$ can be written as

$$z = A_0 + uA_1 + vA_2 + p(A_4 - uA_2 + vA_3). \tag{5}$$

The parameters p, u, v are independent nonhomogeneous parameters on the hypersurface F.

3. Let us find singular points of the hypersurface F. Such points are defined by the equations $\frac{\partial f}{\partial z_\alpha} = 0$. It follows from (4) that

$$\begin{cases} \dfrac{\partial f}{\partial z_0} = z_2 z_4 - 2 z_0 z_3, \\ \dfrac{\partial f}{\partial z_1} = z_4^2, \quad \dfrac{\partial f}{\partial z_2} = z_0 z_4, \quad \dfrac{\partial f}{\partial z_3} = -z_0^2, \\ \dfrac{\partial f}{\partial z_4} = 2 z_1 z_4 + z_0 z_2. \end{cases} \tag{6}$$

All these derivatives vanish simultaneously if and only if $z_0 = z_4 = 0$. Thus the 2-plane at infinity $\sigma = A_1 \wedge A_2 \wedge A_3$ is the locus of singular points of the hypersurface F.

Consider a point $B_0 = A_0 + pA_4$ on the straight line $A_0 \wedge A_4$. By (4), to the point B_0 there corresponds the straight line $a(p)$ in the 2-plane at infinity σ, and the equation of this straight line is

$$p^2 z_1 + p z_2 - z_3 = 0. \tag{7}$$

The family of straight lines $a(p)$ depends of the parameter p, and its envelope is the conic C defined by the equation

$$z_2^2 + 4 z_1 z_3 = 0. \tag{8}$$

The straight line $a(p)$ is tangent to the conic C at the point

$$B_1(p) = A_1 - 2pA_2 - p^2 A_3. \tag{9}$$

Equation (9) is a parametric equation of the conic C. The point

$$\frac{dB_1}{dp} = -2(A_2 + pA_3) \tag{10}$$

belongs to the tangent line to the conic C at the point $B_1(p)$.

Consider the 2-planes $\tau = B_0 \wedge B_1 \wedge \frac{dB_1}{dp}$. Such 2-planes are completely determined by the location of the point B_0 on the straight line $A_0 \wedge A_4$, and they form a one-parameter family. All these 2-planes belong to the hypersurface F. In fact, represent an arbitrary point z of the 2-plane τ in the form

$$\begin{aligned} z &= \alpha B_0 + \beta B_1 - \frac{1}{2}\gamma \frac{dB_1}{dp} \\ &= \alpha A_0 + \beta A_1 + (-2p\beta + \gamma) A_2 + (-p^2 \beta + p\gamma) A_3 + p\alpha A_4. \end{aligned} \tag{11}$$

The coordinates of the point z are

$$z_0 = \alpha, \;\; z_1 = \beta, \;\; z_2 = \gamma - 2p\beta, \;\; z_3 = p(\gamma - p\beta), \;\; z_4 = p\alpha. \tag{12}$$

Substituting these values of the coordinates into equation (4), one can see that equation (4) is identically satisfied. Thus the hypersurface F is foliated into a one-parameter family of 2-planes $\tau(p) = B_0 \wedge B_1 \wedge \frac{dB_1}{dp}$.

In a 2-plane $\tau(p)$ consider a pencil of straight lines with center at B_1. The straight lines of this pencil are defined by the point B_1 and the point

$B_2 = A_2 + pA_3 + q(A_0 + pA_4)$. The straight lines $B_1 \wedge B_2$ depend on two parameters p and q. These lines belong to the 2-plane $\tau(p)$, and along with this 2-plane they belong to the hypersurface F. Thus they form a foliation on the hypersurface F.

We prove that this foliation is a Monge-Ampère foliation. In the space $\overline{R}^4$, we introduce the moving frame formed by the points

$$\begin{cases} B_0 = A_0 + pA_4, \\ B_1 = A_1 - 2pA_2 - p^2 A_3, \\ B_2 = A_2 + pA_3 + qA_0 + pqA_4, \\ B_3 = A_3, \\ B_4 = A_4. \end{cases} \tag{13}$$

It is easy to prove that these points are linearly independent, and the points A_α can be expressed in terms of the points B_α as follows

$$\begin{cases} A_0 = B_0 - pB_4, \\ A_1 = B_1 + 2pB_2 - p^2 B_3 - 2pqB_0, \\ A_2 = B_2 - pB_3 - qB_0, \\ A_3 = B_3, \\ A_4 = B_4. \end{cases} \tag{14}$$

Consider a displacement of the straight lines $B_1 \wedge B_2$ along the hypersurface F. Suppose that Z is an arbitrary point of this straight line,

$$Z = B_1 + \lambda B_2. \tag{15}$$

Differentiating (15) and taking into account (14) and $dA_\alpha = 0$, we find that

$$dZ \equiv (2qdp + \lambda dq)B_0 + \lambda dp(B_3 + qB_4) \pmod{B_1, B_2}. \tag{16}$$

It follows from relation (16) that

1. A tangent hyperplane to the hypersurface F is spanned by the points B_1, B_2, B_0, and $B_3 + qB_4$. This hyperplane is fixed when the point Z moves along the straight line $B_1 \wedge B_2$. Thus the hypersurface F is tangentially degenerate of rank 2, and the straight lines $B_1 \wedge B_2$ form a Monge-Ampère foliation on F.

2. The system of equations

$$\begin{cases} 2q dp + \lambda dq = 0, \\ \lambda dp = 0 \end{cases} \tag{17}$$

defines singular points on the straight line $B_1 \wedge B_2$, and on the hypersurface F it defines torses. The system of equations (17) has a nontrivial solution with respect to dp and dq if and only if its determinant vanishes: $\lambda^2 = 0$. Hence by (15), a singular point on the straight line $B_1 \wedge B_2$ coincides with the point B_1. For $\lambda = 0$, system (17) implies that $dp = 0$, i.e., $p = \text{const}$. Thus it follows from (9) that the point $B_1 \in C$ is fixed, and as a result, the torse corresponding to this constant parameter p is a pencil of straight lines with center at B_1 located in the 2-plane $\tau(p) = B_0 \wedge B_1 \wedge B_2$.

3. All singular points of the hypersurface F belong to the conic $C \subset P^\infty$ defined by equation (8). Thus if we consider the hypersurface F in an Euclidean space R^4, then on F there are no singular points in a proper part of this space.

4. The hypersurface F considered in the proper part of an Euclidean space is not a cylinder since its rectilinear generators do not belong to a bundle of parallel straight lines. A two-parameter family of rectilinear generators of F decomposes into a one-parameter family of plane pencils of parallel lines.

4. No one of properties 1–4 characterizes Bourgain's hypersurfaces completely: they are necessary but not sufficient for these hypersurfaces. The following theorem gives a necessary and sufficient condition for a hypersurface to be of Bourgain's type.

Theorem 1 *Let l be a proper straight line of an Euclidean space R^4 enlarged by the plane at infinity P^3_∞, and let C be a conic in the 2-plane σ. Suppose that the straight line l and the conic C are in a projective correspondence. Let $B_0(p)$ and $B_1(p)$ be two corresponding points of l and C, and let τ be the 2-plane passing through the point B_0 and tangent to the conic C at the point B_1. Then*

(a) *when the point B_0 is moving along the straight line l, the plane τ describes a Bourgain hypersurface, and*

(b) *any Bourgain hypersurface satisfies the above construction.*

Proof. The necessity (b) of the theorem hypotheses follows from our previous considerations. We prove the sufficiency (a) of these hypotheses. Take a fixed frame $\{A_u\}$, $u = 0, 1, 2, 3, 4$, in the space R^4 enlarged by the plane at infinity P^3_∞ as follows: its point A_0 belongs to l, the point A_4 is the point at infinity of l, and the points A_1, A_2, and A_3 are located at the 2-plane at infinity σ in such a way that a parametric equation of the straight line l is $B_0 = A_0 + pA_4$, and the equation of C has the form (9). The plane τ is defined by the points B_0, B_1, and $\frac{dB_1}{dp}$. The parametric equations of this plane have the form (12). Excluding the parameters α, β, γ, and p from these equations, we will return to the cubic equation (4) defining the Bourgain hypersurface B in homogeneous coordinates. □

The method of construction of the Bourgain hypersurface used in the proof of Theorem 1 goes back to the classical methods of projective geometry developed by Steiner [St32] and Reye [R68].

5. In conclusion we prove the following theorem.

Theorem 2 *The Sacksteder hypersurface S and the Bourgain hypersurface B are locally equivalent, and the former is the standard covering of the latter.*

Proof. In an Euclidean space R^4, in Cartesian coordinates x_1, x_2, x_3, x_4, the equation of the Sacksteder hypersurface S (see [S60]) has the form

$$x_4 = x_1 \cos x_3 + x_2 \sin x_3. \tag{18}$$

The right-hand side of this equation is a function on the manifold $M^3 = \mathbb{R}^2 \times S^1$ since the variable x_3 is cyclic. Equation (18) defines a hypersurface on the manifold $M^3 \times \mathbb{R}$. The circumference $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ has a natural projective structure of P^1. In the homogeneous coordinates $x_3 = \frac{u}{v}$, the mapping $S^1 \to P^1$, can be written as $x^3 \to (u, v)$. By removing the point $\{v = 0\}$ from S^1, we obtain a 1-to-1 correspondence

$$S^1 - \{v = 0\} \longleftrightarrow \mathbb{R}^1. \tag{19}$$

Now we can consider the Sacksteder hypersurface S in R^4 or, if we enlarge R^4 by the plane at infinity P^3_∞, in the space P^4.

Next we show how by applying the mapping $S^1 \to P^1$, we can transform equation (18) of the Sacksteder hypersurface S into equation(4) of the

Bourgain hypersurface B. We write this mapping in the form

$$x_3 = 2 \arctan \frac{u}{v}, \quad \frac{u}{v} \in R, \quad |x_3| < \pi. \tag{20}$$

It follows from (20) that

$$\left\{ \begin{array}{l} \dfrac{u}{v} = \tan \dfrac{x_3}{2}, \\ \cos x_3 = \dfrac{1 - \tan^2 \dfrac{x_3}{2}}{1 + \tan^2 \dfrac{x_3}{2}} = \dfrac{v^2 - u^2}{v^2 + u^2}, \\ \sin x_3 = \dfrac{2 \tan \dfrac{x_3}{2}}{1 + \tan^2 \dfrac{x_3}{2}} = \dfrac{2uv}{v^2 + u^2}. \end{array} \right. \tag{21}$$

Substituting these expressions into equation (18), we find that

$$x_4(u^2 + v^2) = x_1(v^2 - u^2) + 2x_2uv,$$

i.e.,

$$(x_4 + x_1)u^2 + (x_4 - x_1)v^2 - 2x_2uv = 0. \tag{22}$$

Make a change of variables

$$z_1 = x_4 - x_1, \quad z_2 = -2x_2, \quad z_3 = x_1 + x_4, \quad z_0 = u, \quad z_4 = v.$$

As a result, we reduce equation (22) to equation (4). It follows that the Sacksteder hypersurface S defined by equation (18) is locally equivalent to the Bourgain hypersurface defined by equation (4).

Note also that if the cyclic parameter x_3 changes on the entire real axis $\mathbb{R}$, then we obtain the standard covering of the Bourgain hypersurface B by means of the Sacksteder hypersurface S. □

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