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A remark on a theorem of Y. Kurata

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Abstract. In [K] Y. Kurata proved that the Goldie torsion theory splits centrally for dual rings. Here we extend his result to semilocal rings with left essential socle such that $\mathrm{Soc}({}_RR)^2 \subseteq \mathrm{Soc}(R_R)$. An example will demonstrate that our observation extends Kurata's result.

Key words: Goldie torsion theory, central splitting, semilocal rings, essential socle.

All rings are associative rings with unit, all left (or right) R-modules are unital and all torsion theories are considered to be hereditary. The singular submodule of a left R-module M is denoted by $Z({}_RM)$. We abbreviate $S := \mathrm{Soc}({}_RR)$ and $J := \mathrm{Jac}(R)$ for the left socle resp. the Jacobson radical. We denote the left Goldie torsion theory, that is the torsion theory whose torsion free modules are exactly the nonsingular left R-modules, by τ_G (see [G, 1.14] or [AD]) and we denote the torsion submodule of a module M by $\tau_G(M)$. A torsion theory τ is called *jansian* (or TTF) if the class of τ-torsion modules is closed under taking products. Moreover a jansian torsion theory τ is called *centrally splitting* if $\tau(R)$ is a direct summand of R and τ-torsion free modules are closed under homomorphic images. (see [Be, Theorem 1]). A classical result of Alin and Dickson [AD, Theorem 3.1] states that τ_G is centrally splitting for a ring R if and only if R is a direct product of a semisimple ring and a ring with essential left singular ideal. (Alin and Dickson use the term *global dimension zero* instead of *centrally splitting*, meaning that all torsionfree modules are injective. We have that τ_G is centrally splitting if and only if all nonsingular left R-modules are injective. The sufficiency is clear (see also [G, 5.10]). The necessity follows since if nonsingular modules are closed under homomorphic images and τ_G-torsion submodules split off, then each nonsingular module must equal its injective hull. By the remark on page 201 in [AD] τ_G is also jansian.)

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I would like to thank Patrick Smith for having pointed out the above example while I was visiting Glasgow and Carl Faith for all his valuable comments and his interest.

Let us start with an easy Lemma. Note that (1) also follows from a more general statement in [Ba, Proposition 1.10 (d)].

Lemma 1 *Let R be a ring with essential left socle S. Then*

(1) $S = S^2 \oplus (S \cap Z(_RR))$, *where S^2 is projective and R/S^2 is τ_G-torsion.*

(2) J *is τ_G-torsion if and only if $S^2J = 0$.*

Proof. The socle can be decomposed as $S = S_0 \oplus S_1$ where $S_1 := S \cap Z(_RR)$ and S_0 is a projective left R-module. $S^2 \subseteq S_0$, because for $x, y \in S$ with $y = y_0 + y_1$ where $y_0 \in S_0$ and $y_1 \in S_1$. The product $xy = xy_0 \in S_0$ as $xy_1 \in SZ(_RR) = 0$. Thus $S^2 \subseteq S_0$ holds and there exists a left module $\tilde{S}$ such that $S_0 = S^2 \oplus \tilde{S}$. We have $S\tilde{S} \subseteq S^2 \cap \tilde{S} = 0$. If $_RS$ is essential in $_RR$, then $\tilde{S}$ becomes singular (as it is annihilated by S) and must be zero as it is also projective. Thus $S_0 = S^2$. Also R/S^2 becomes τ_G-torsion as $S_1 \simeq S/S^2$ and R/S are singular. This proves (1). Assume $S^2J = 0$, then J is an R/S^2-module and hence τ_G-torsion by (1). On the contrary, if J is τ_G-torsion, then $\mathrm{Hom}_R(S^2, J) = 0$ and therefore $S^2J = 0$. □

For a semilocal ring R we have $\mathrm{Soc}(R_R) = l.ann(\mathrm{Jac}(R))$, therefore the condition $S^2J = 0$ is equivalent to $\mathrm{Soc}(_RR)^2 \subseteq \mathrm{Soc}(R_R)$.

Every semilocal ring R has a decomposition $R = R_0 \oplus R_1$ of left R-modules R_0 and R_1, where R_0 is semisimple artinian and J is essential in R_1 (see [L, Theorem 3.5]).

Lemma 2 *Let R be a semilocal ring. Then τ_G is centrally splitting if and only if J contains an essential singular left R-submodule. In this case* $\mathrm{Soc}(_RR)^2 \subseteq \mathrm{Soc}(R_R)$ *holds.*

Proof. "⇒" By Alin and Dickson's theorem $R = S \times T$ where S is semisimple artinian and T has essential left singular ideal. Thus $J = \mathrm{Jac}(T) \subseteq T$. Obviously T is also semilocal and has a decomposition $T = T_0 \oplus T_1$ with J essential in T_1. Since $Z(_TT)$ is essential in T we get that $T_0 = 0$, hence $Z(_TT) \cap J$ is an essential submodule of J. "⇐" Recall the above decomposition of semilocal rings $R = R_0 \oplus R_1$. As J is essential in R_1, R_1/J is singular. By hypothesis J is τ_G-torsion and so is R_1. Thus $\mathrm{Hom}_R(R_0, R_1) = 0 = \mathrm{Hom}_R(R_1, R_0)$ as R_0 is semisimple projective. Hence $R = R_0 \times R_1$ is a direct product of a semisimple ring and a ring with essential singular ideal. By [AD, Theorem 3.1] τ_G is centrally splitting.

As R_1 is τ_G-torsion it does not contain any projective simple submodule.

Hence $S^2 = R_0$ and we have $S^2 J = 0$ as $\mathrm{Hom}_R(R_0, R_1) = 0$. □

As a special case we get the following criterion for the splitting of τ_G for semilocal rings with essential left socle that extends Kurata's result for dual rings.

Theorem 3 *Let R be a semilocal ring with essential left socle. Then τ_G is centrally splitting if and only if* $\mathrm{Soc}({}_RR)^2 \subseteq \mathrm{Soc}(R_R)$.

Proof. The necessity is clear by Lemma 2. Assume that R is semilocal with essential left socle and $S^2 J = 0$, then by Lemma 1 (2) J is τ_G-torsion and by Lemma 2 the result follows. □

In order to verify that our result extends Kurata's result, we give an example of a commutative semilocal ring with essential simple socle that is not semiperfect and hence not a dual ring. I am very grateful to Patrick F. Smith for the following example.

Example (P.F. Smith) *For any number n, there exists a commutative semilocal subdirectly irreducible non-local (and hence not semiperfect) ring with exactly n maximal ideals.* Take n different prime numbers $p_1, \dots, p_n$. Then $R := \{\frac{a}{b} \in \mathbb{Q} \mid p_i \nmid b \ \forall i = 1, \dots, n\}$ is a semilocal integral domain with n maximal ideals, which is not local. Let $M = \mathbb{Z}_{p_1^\infty}$ be the p_1-Prüfer group then M is a faithful R-module with essential simple socle isomorphic to $\mathbb{Z}/p_1\mathbb{Z}$. Form the trivial extension

$$S := R \propto M := \left\{ \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \mid a \in R, \ m \in M \right\}.$$

Then S is a commutative semilocal subdirectly irreducible non-local ring with exactly n maximal ideals.

Patrick Smith's results follows from the following lemma:

Lemma 4 *Let R be a commutative semilocal ring, which is not local and assume there is a faithful subdirectly irreducible (SDI) R-module M. Then $S := R \propto M$ is a commutative semilocal SDI ring which is not local.*

Proof. Let M be faithful with essential simple submodule N. Take an element $s = a \propto m \in S$. If $a \neq 0$, then $(a \propto m) \cdot (0 \propto M) = 0 \propto aM \neq 0$ since M is faithful. As $N \subseteq aM$ as R-modules we get $(0 \propto N) \subseteq (0 \propto aM) \subseteq sS$. If $a = 0$ and $m \neq 0$, then $sS = (0 \propto m)S = (0 \propto mR) \supseteq (0 \propto$

N) as $mR \supseteq N$. Thus S has an essential simple S-submodule $0 \propto N$. As $\text{Jac}(S) = \text{Jac}(R) \propto M$, S is semilocal, but not local as R is not local and S is indecomposable. □

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