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Real hypersurfaces in a complex projective space in which the reflections with respect to ξ-curves are isometric

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Abstract. We completely classify the real hypersurfaces M in a complex projective space $\mathbf{C}P_n$ in which the reflections with respect to integral curves of the structure vector ξ are isometric. We conclude that the reflections with respect to integral curves of the structure vector ξ are isometric in M if and only if M is locally congruent to one of homogeneous real hypersurfaces of type (A) and (B).

Key words: complex projective space, real hypersurface, reflection, ξ-curves.

1. Introduction

In Riemannian Geometry the group of isometric transformations of a manifold plays an important role. Riemannian symmetric space is defined by the condition that any geodesic symmetry gives an isometric transformation of it. Weakly symmetric space which is introduced by A. Selberg ([12]) is a space whose any two points can be interchanged by a suitable isometry of it. The concept of reflections of a manifold with respect to an embedded submanifold is a generalization of geodesic symmetries (cf. [2]). In the paper [1] J. Berndt and L. Vanhecke shows that homogeneous real hypersurfaces of type (A) in a non-flat complex space form (for definitions see Theorem T of §2) are weakly symmetric. In the paper [8] the author gives a new examples of weakly symmetric spaces. Their examples are not Riemannian symmetric. In both papers [1] and [8] they make use of reflections with respect to a totally geodesic submanifolds to construct suitable isometric transformations.

Real hypersurfaces in a complex space form present many interesting homogeneous Riemannian manifolds. So many differential geometers study these spaces. There are many charactrizations of some homogeneous real hypersurfaces in a complex space forms (cf. [11], [6], [4], [5], [3], [9]). Some

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of them are naturally reductive, some of them are weakly symmteric and none of them are Riemannian symmetric.

The purpose of this paper is to give a geometric characterization of homogeneous real hypersurfaces of type (A) and (B) in a complex projective space. We consider the reflections of a real hypersurface with respect to the integral curves of the structure vector field ξ, simply called ξ-curves in this paper (for definition see §2). The main theorem is the following:

Theorem 4.1 *Let M be a real hypersurface in $\mathbf{C}P_n$. The reflections with respect to ξ-curves are isometries of M if and only if M is locally congruent to one of homogeneous real hypersurfaces of type* (A) *and* (B).

2. Preliminaries

In this section we explain preliminary results concerning reflections with respect to a submanifold in a Riemannian manifold and real hypersurfaces of a complex projective space. In this paper all manifolds will be assumed to be connected and of class C^∞.

First, we explain some results concerning reflections. Let (M, g) be a Riemannian manifold and B a connected embedded submanifold. Then we define the following local diffeomorphism ϕ_B of M which is called the reflection with respect to the submanifold B:

$$\phi_B : p \mapsto \phi_B(p), \quad \exp_m(tu) \mapsto \exp_m(-tu)$$
$$\text{for } m \in B, \quad u \in T_m^\perp B, \quad \|u\| = 1, \tag{2.1}$$

where exp and $T^\perp B$ denote the expornential mapping of M and the normal bundle of B in M, respectively.

The necessary and sufficient conditions for ϕ_B to be isometric are:

Theorem C-V ([2]) *For analytic data the reflection ϕ_B is a local isometry if and only if*

(i) *B is totally geodesic;*

(ii) *$(\nabla^{2k}_{u\cdots u}R)(v,u)u$ is normal to B,*
$(\nabla^{2k+1}_{u\cdots u}R)(v,u)u$ is tangent to B, and
$(\nabla^{2k+1}_{u\cdots u}R)(x,u)u$ is normal to B,

for all normal vectors u, v of B, any tangent vector x of B and all $k \in \mathbf{N}$. Here ∇ and R denote the Levi Civita connecton and the Riemannian curvature tensor of M, respectively.

Secondly, we turn to some preliminaries concerning real hypersurfaces of a complex projective space. Let $\boldsymbol{C}P_n$ be an n-dimensional complex projective space with constant holomorphic sectional curvature 4 and let J and $\overline{g}$ be its complex structure and metric, respectively. Further, let M be a connected submanifold of $\boldsymbol{C}P_n$ with real codimension 1, simply called a real hypersurface in the following. We denote by g the induced Riemannian metric of M and by ν a local unit normal vector field of M in $\boldsymbol{C}P_n$.

The Gauss and Weingarten formulas are:

$$\overline{\nabla}_X Y = \nabla_X Y + g\,(AX, Y)\,\nu, \tag{2.2}$$

$$\overline{\nabla}_X \nu = -AX, \tag{2.3}$$

where $\overline{\nabla}$ and A denote the Levi Civita connection on $\boldsymbol{C}P_n$ and the shape operator of M in $\boldsymbol{C}P_n$, respectively.

We define an almost contact metric structure (ϕ, ξ, η, g) of M as usual. That is,

$$\xi = -J\nu, \quad \eta\,(X) = g\,(X, \xi)\,, \quad \phi X = (JX)^T\,, \quad \text{for} \;\; X \in TM,$$

where TM denotes the tangent bundle of M and $(\;\;)^T$ the tangential component of a vector. These structure tensors satisfy the following relations:

$$\begin{aligned} &\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta\,(\phi X) = 0, \quad \eta\,(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in TM, \end{aligned} \tag{2.4}$$

where I denotes the identity mapping of TM.

From (2.2) we easily have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.5}$$

$$\nabla_X \xi = \phi AX \tag{2.6}$$

for tangent vectors $X, Y \in TM$.

In our case the Gauss and the Codazzi equations of M become

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X &- g(X, Z)Y + g(\phi Y, Z)\phi X \\ &- g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.7}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi, \tag{2.8}$$

where $X, Y, Z \in TM$.

For the structure vector ξ we know the following lemmas:

Lemma 2.1 ([6] p.532 Lemma 2.1) *The structure vector ξ is principal if and only if the integral curves of ξ are geodesics.*

Lemma 2.2 ([6] p.533 Lemma 2.4) *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

Lemma 2.3 ([6] p.532 Lemma 3.2) *Assume that ξ is a principal curvature vector. If X is a principal curvature vector of M with $AX = \lambda X$, then the equation $(2\lambda - \alpha)A\phi X = (\alpha\lambda + 2)\phi X$ is satisfied.*

Typical examples of real hypersurfaces in $\boldsymbol{C}P_n$ are homogeneous ones which are orbits under analytic subgroups of $PU(n+1)$. The complete classification of them is obtained by R. Takagi ([13]) as follows:

Theorem T ([13]) *Let M be a homogeneous real hypersurface of $\boldsymbol{C}P_n$. Then M is locally congruent to one of the following spaces:*

(A) *a tube of radius r over a totally geodesic $\boldsymbol{C}P_k$ $(0 \leq k \leq n-1)$, $0 < r < \frac{\pi}{2}$;*

(B) *a tube of radius r over a complex quadric Q_{n-1}, $0 < r < \frac{\pi}{4}$;*

(C) *a tube of radius r over $\boldsymbol{C}P_1 \times \boldsymbol{C}P_{\frac{n-1}{2}}$, $n \geq 5$ is odd, $0 < r < \frac{\pi}{4}$;*

(D) *a tube of radius r over a complex Grassmann $G_{2,5}(\boldsymbol{C})$, $n = 9$, $0 < r < \frac{\pi}{4}$;*

(E) *a tube of radius r over a Hermitian symmetric space $SO(10)/U(5)$, $n = 15$, $0 < r < \frac{\pi}{4}$.*

Here $\boldsymbol{C}P_0$ means a single point.

M. Kimura obtains the following characterization of homogeneous real hypersurfaces:

Theorem K ([4]) *Let M be a real hypersurface in $\mathbf{C}P_n$. Then M has constant principal curvatures and ξ is a principal curvature vector if and only if M is locally congruent to a homogeneous real hypersurface.*

About the decomposition of the tangent space into the eigenspaces of the shape operator of a homogeneous real hypersurface, we know the following:

Theorem 2.4 ([14]) *The tangent space of the homogeneous real hypersurfaces can be decomposed as follows*:

for type (A) : $TM = \boldsymbol{R}\xi \oplus T_x \oplus T_{-\frac{1}{x}},\ A\xi = (x - \frac{1}{x})\xi,\ x > 0;$

for type (B) : $TM = \boldsymbol{R}\xi \oplus T_x \oplus T_{-\frac{1}{x}},\ A\xi = (\frac{-4x}{x^2-1})\xi,\ 0 < x < 1;$

for type (C), (D) *and* (E) : $\begin{cases} TM = \boldsymbol{R}\xi \oplus T_x \oplus T_{-\frac{1}{x}} \oplus T_{\frac{x+1}{1-x}} \oplus T_{\frac{x-1}{x+1}}, \\ A\xi = (\frac{-4x}{x^2-1})\xi,\ 0 < x < 1, \end{cases}$

where T_λ denotes the eigenspace of the shape operator with the principal curvature λ. Further, for type (B)–(E) *we have $\phi T_x = T_{-\frac{1}{x}}$* (cf. [6]).

There are some charactarizations of homogeneous real hypersurfaces of type (A). We know the following theorem:

Theorem 2.5 ([6], [11]) *Let M be a real hypersurface in $\mathbf{C}P_n$. Then the following three conditions are equivalent*:
(i) *M is locally congruent to a homogeneous real hypersurface of type* (A),
(ii) $\phi A = A\phi$,
(iii) $(\nabla_X A)Y = -\eta(Y)\phi X - g(\phi X, Y)\xi$.

For a homogeneous real hypersurface of type (B) we have the following:

Proposition 2.6 ([3]) *The shape operator of a homogeneous real hypersurface of type* (B) *satisfies the following*:

$$(\nabla_X A)Y = -\frac{\alpha}{4}\{2\eta(X)(A\phi - \phi A)Y + \eta(Y)(A\phi - 3\phi A)X + g((A\phi - 3\phi A)X, Y)\xi\}. \quad (2.9)$$

Both homogeneous real hypersurfaces of type (A) and (B) are characterized by the following:

Theorem K-M ([5]) *Let M be a real hypersurface in $\mathbf{C}P_n$. Then the second fundamental form of M is η-parallel and ξ is a principal curvature vector if and only if M is locally congruent to one of homogeneous real hypersurfaces of type* (A) *and* (B).

Here the second fundamental form of M is said to be η-parallel if the equation $g((\nabla_X A)Y, Z) = 0$ is satisfied for any $X, Y, Z \in T^0M = \{X \in TM : X \perp \xi\}$.

3. Lemmas

In this section we prove lemmas needed later.

First, we prove the following:

Lemma 3.1 *In homogeneous real hypersurfaces of type* (A) *and* (B) *the following relations are satisfied*:

$$\begin{cases} (\nabla^{2n+1}_{u\cdots u}\phi)v \in \mathrm{Span}\,\{\xi\}\,, & (\nabla^{2n}_{u\cdots u}\phi)v \in T^0M, \\ (\nabla^{2n+1}_{u\cdots u}\phi)\xi \in T^0M, & (\nabla^{2n}_{u\cdots u}\phi)\xi \in \mathrm{Span}\,\{\xi\}\,, \\ (\nabla^{2n+1}_{u\cdots u}A)v \in \mathrm{Span}\,\{\xi\}\,, & (\nabla^{2n}_{u\cdots u}A)v \in T^0M, \\ (\nabla^{2n+1}_{u\cdots u}A)\xi \in T^0M, & (\nabla^{2n}_{u\cdots u}A)\xi \in \mathrm{Span}\,\{\xi\}\,, \\ \nabla^{2n+1}_{u\cdots u}\xi \in T^0M, & \nabla^{2n}_{u\cdots u}\xi \in \mathrm{Span}\,\{\xi\}\,, \end{cases} \tag{3.1}$$

for $u, v \in T^0M = \{X \in TM : X \perp \xi\}$ *and* $n \in \mathbf{N}$, *where* $\mathrm{Span}\,\{\xi\}$ *is the subspace of* TM *spanned by* ξ *over* $\mathbf{R}$.

Proof. We prove the lemma by induction on n.

For $n = 0$ the lemma can easily be verified by (2.4), (2.5), (2.6), (2.9), Theorem 2.4 and Theorem 2.5.

Now we assume that the lemma holds for all natural numbers $n \leq k-1$. Then we are going to prove the lemma for $n = k$.

First, we prove the relation $(\nabla^{2k}_{u\cdots u}\phi)v \in T^0M$. From (2.5) we have

$$\begin{aligned} (\nabla^{2k}_{u\cdots u}\phi)v = & \sum_{\nu=0}^{2k-1} \binom{2k-1}{\nu} g(v, \nabla^{\nu}_{u\cdots u}\xi)(\nabla^{2k-1-\nu}_{u\cdots u}A)u \\ & - \sum_{\nu=0}^{2k-1} \binom{2k-1}{\nu} g((\nabla^{\nu}_{u\cdots u}A)u, v)\nabla^{2k-1-\nu}_{u\cdots u}\xi. \end{aligned} \tag{3.2}$$

By the induction hypothesis we can easily verify that the right-hand side of (3.2) belongs to T^0M.

Secondly, we prove the relation $(\nabla^{2k}_{u\cdots u}\phi)\xi \in \mathrm{Span}\,\{\xi\}$. From (2.6) we have

$$\begin{aligned} (\nabla^{2k}_{u\cdots u}\phi)\xi = & \sum_{\nu=0}^{2k-1} \binom{2k-1}{\nu} g(\xi, \nabla^{\nu}_{u\cdots u}\xi)(\nabla^{2k-1-\nu}_{u\cdots u}A)u \\ & - \sum_{\nu=0}^{2k-1} \binom{2k-1}{\nu} g((\nabla^{\nu}_{u\cdots u}A)u, \xi)\nabla^{2k-1-\nu}_{u\cdots u}\xi. \end{aligned} \tag{3.3}$$

By the induction hypothesis we can also check that the right-hand side of (3.3) belongs to Span $\{\xi\}$.

Thirdly, we prove the relation $(\nabla^{2k}_{u\cdots u}A)v \in T^0M$. For a homogeneous real hypersurface of type (A), using Theorem 2.5 (iii), we have

$$(\nabla^{2k}_{u\cdots u}A)v = -\sum_{\nu=0}^{2k-1}\binom{2k-1}{\nu} g(v, \nabla^{\nu}_{u\cdots u}\xi)(\nabla^{2k-1-\nu}_{u\cdots u}\phi)u - \sum_{\nu=0}^{2k-1}\binom{2k-1}{\nu} g((\nabla^{\nu}_{u\cdots u}\phi)u, v)\nabla^{2k-1-\nu}_{u\cdots u}\xi. \tag{3.4}$$

By induction hypothesis the right-hand side of (3.4) belongs to T^0M.

For a real hypersurface of type (B), using (2.9), we have

$$(\nabla^{2k}_{u\cdots u}A)v = -\frac{\alpha}{4}\Big\{2\sum_{\nu=0}^{2k-1}\binom{2k-1}{\nu} g(u, \nabla^{\nu}_{u\cdots u}\xi)(\nabla_{u\cdots u}B)v + \sum_{\nu=0}^{2k-1}\binom{2k-1}{\nu} g(v, \nabla^{\nu}_{u\cdots u}\xi)(\nabla^{2k-1-\nu}_{u\cdots u}C)u + \sum_{\nu=0}^{2k-1}\binom{2k-1}{\nu} g((\nabla^{\nu}_{u\cdots u}C)u, v)\nabla^{2k-1-\nu}_{u\cdots u}\xi\Big\}, \tag{3.5}$$

where $B = A\phi - \phi A$ and $C = A\phi - 3\phi A$.

On the other hand we have

$$(\nabla^{2n+1}_{u\cdots u}B)v = \sum_{\nu=0}^{2n+1}\binom{2n+1}{\nu}\left\{(\nabla^{\nu}_{u\cdots u}A)(\nabla^{2n+1-\nu}_{u\cdots u}\phi)v - (\nabla^{\nu}_{u\cdots u}\phi)(\nabla^{2n+1-\nu}_{u\cdots u}A)v\right\}. \tag{3.6}$$

According to the induction hypothesis, the right-hand side of (3.6) belongs to Span $\{\xi\}$ for $n \leq k-1$. We can deduce the relations $(\nabla^{2n}_{u\cdots u}B)v \in T^0M$, $(\nabla^{2n+1}_{u\cdots u}C)v \in$ Span $\{\xi\}$ and $(\nabla^{2n}_{u\cdots u}C)v \in T^0M$ for $n \leq k-1$ analogously. Using these facts and the induction hypothesis, we deduce that the right-hand side of (3.5) belongs to T^0M.

Next we prove the relation $\nabla^{2k}_{u\cdots u}\xi \in$ Span $\{\xi\}$. According to the relation (2.6), we have

$$\nabla^{2k}_{u\cdots u}\xi = \sum_{\nu=0}^{2k-1} \binom{2k-1}{\nu} (\nabla^{\nu}_{u\cdots u}\phi)(\nabla^{2k-1-\nu}_{u\cdots u}A)u. \tag{3.7}$$

By the hypothesis the right-hand side of (3.7) belongs to Span $\{\xi\}$.

The remainders can be proved analogously. The lemma is now proved by all the above arguments. □

Lemma 3.2 *Let M be a real hypersurface in $\mathbf{C}P_n$. If ξ is a principal curvature vector with principal curvature* 0, *then for any principal curvature function λ on M is constant along the integral curves of ξ.*

Proof. Let $u \in T^0M$ be a unit principal curvature vector and λ the corresponding principal curvature. By our assumption $A\xi = 0$ and the Codazzi equation (2.8), we have

$$\begin{aligned} 0 = \nabla_u(A\xi) &= (\nabla_u A)\xi + A\phi A u \\ &= (\nabla_\xi A)u - \phi u + A\phi A u \\ &= (\xi\lambda)u + (\lambda I - A)\nabla_\xi u - \phi u + A\phi A u. \end{aligned} \tag{3.8}$$

Taking inner product both sides of (3.8) with u, we have

$$\xi\lambda = 0. \tag{3.9}$$

Combining (3.9) with Lemma 2.2, we prove the lemma. □

4. Proof of the theorem

Now we prove the theorem.

Theorem 4.1 *Let M be a real hypersurface in $\mathbf{C}P_n$. The reflections with respect to ξ-curves are isometries of M if and only if M is locally congruent to one of homogeneous real hypersurfaces of type* (A) *and* (B).

Proof. First, we prove that the reflections with respect to ξ-curves are isometric in homogeneous real hypersurfaces of type (A) and (B).

By (2.7) we have the following relations:

$$\begin{aligned} (\nabla^n_{u\cdots u}R)(v,u)u = {} & 3\sum_{\nu=0}^{n} \binom{n}{\nu} g((\nabla^{\nu}_{u\cdots u}\phi)u, v)(\nabla^{n-\nu}_{u\cdots u}\phi)u \\ & - \sum_{\nu=0}^{n} \binom{n}{\nu} g((\nabla^{\nu}_{u\cdots u}A)v, u)(\nabla^{n-\nu}_{u\cdots u}A)u \end{aligned}$$

$$+\sum_{\nu=0}^{n}\binom{n}{\nu} g((\nabla^{\nu}_{u\cdots u}A)u,u)(\nabla^{n-\nu}_{u\cdots u}A)v, \qquad (4.1)$$

$$\begin{aligned}(\nabla^{2k+1}_{u\cdots u}R)(\xi,u)u = 3\sum_{\nu=0}^{2k+1}\binom{2k+1}{\nu} g((\nabla^{\nu}_{u\cdots u}\phi)u,\xi)(\nabla^{2k+1-\nu}_{u\cdots u}\phi)u \\ -\sum_{\nu=0}^{2k+1}\binom{2k+1}{\nu} g((\nabla^{\nu}_{u\cdots u}A)\xi,u)(\nabla^{2k+1-\nu}_{u\cdots u}A)u \\ +\sum_{\nu=0}^{2k+1}\binom{2k+1}{\nu} g((\nabla^{\nu}_{u\cdots u}A)u,u)(\nabla^{2k+1-\nu}_{u\cdots u}A)\xi,\end{aligned} \qquad (4.2)$$

where $u, v \in T^0M$.

Using Lemma 3.1 and the right-hand side of (4.1) and (4.2), we have

$$(\nabla^{2k+1}_{u\cdots u}R)(v,u)u \in \mathrm{Span}\,\{\xi\}\,, \quad (\nabla^{2k}_{u\cdots u}R)(v,u)u \in T^0M,$$
$$(\nabla^{2k+1}_{u\cdots u}R)(\xi,u)u \in T^0M.$$

So by Lemma 2.1 and Theorem C-V, we have our assertion.

Secondly, we prove that only homogeneous real hypersurfaces of type (A) and (B) satisfy the condition of the theorem. So in the following we assume that the reflections of M with respect to ξ-curves are isometric. Under the assumption, ξ is always principal by Theorem C-V (i) and Lemma 2.1.

We discuss dividing into the following two cases:
Case 1 $\alpha \neq 0$; Case 2 $\alpha = 0$.

Case 1: Using (2.5) and (2.7), we have the following relation for $u \in T^0M$:

$$\begin{aligned}(\nabla_u R)(\xi,u)u = -3g(Au,u)\phi u - g(\phi Au,(\alpha I - A)u)Au \\ + g(Au,u)(\alpha I - A)\phi Au + \alpha g((\nabla_u A)u,u)\xi.\end{aligned} \qquad (4.3)$$

By Theorem C-V (ii) the right-hand side of (4.3) must be belonging to T^0M. So we have

$$g((\nabla_u A)u,u) = 0, \quad u \in T^0M. \qquad (4.4)$$

Here we use the condition $\alpha \neq 0$.

Using polarization, we can lead that (4.4) reduces to the following:

$$g((\nabla_u A)v,w) = 0, \quad u,v,w \in T^0M. \qquad (4.5)$$

This means that M has an η-parallel second fundamental form. So by Theorem K-M a real hypersurface M is locally congruent to one of homogeneous real hypersurfaces of type (A) and (B).

Case 2: First, we prove that M is homogeneous. On account of Theorem K, Lemma 2.2 and Lemma 3.2, we only need to prove that any principal curvature function in the direction of a principal curvature vector in T^0M is constant.

Using (2.5) and (2.7), we have

$$\begin{aligned}(\nabla_u R)(v,u)u = {}& 3g(\phi v,u)g(Au,u)\xi - g((\nabla_u A)v,u)Au \\ & - g(Av,u)(\nabla_u A)u + g((\nabla_u A)u,u)Av \\ & + g(Au,u)(\nabla_u A)v,\end{aligned} \tag{4.6}$$

where $u, v \in T^0M$.

By our assumption and Theorem C-V (ii) the right-hand side of (4.6) must belong to Span $\{\xi\}$. So we have

$$\begin{aligned}& g((\nabla_u A)v,u)Au + g(Av,u)(\nabla_u A)u \\ & \quad - g((\nabla_u A)u,u)Av - g(Au,u)(\nabla_u A)v \in \operatorname{Span}\{\xi\}.\end{aligned} \tag{4.7}$$

On the other hand we have the following relation by (2.7):

$$\begin{aligned}(\nabla^2_{u,u}R)(v,u)u = {}& 3\{g(\phi u,v)(\nabla^2_{u,u}\phi)u + 2g((\nabla_u\phi)u,v)(\nabla_u\phi)u \\ & + g((\nabla^2_{u,u}\phi)u,v)\phi u\} - g(Av,u)(\nabla^2_{u,u}A)u \\ & - 2g((\nabla_u A)v,u)(\nabla_u A)u - g((\nabla^2_{u,u}A)v,u)Au \\ & + g(Au,u)(\nabla^2_{u,u}A)v + 2g((\nabla_u A)u,u)(\nabla_u A)v \\ & + g((\nabla^2_{u,u}A)u,u)Av,\end{aligned} \tag{4.8}$$

for $u, v \in T^0M$.

Using our assumption $\alpha = 0$ and (2.5), we have

$$(\nabla_u\phi)v = -g(Au,v)\xi, \tag{4.9}$$

$$(\nabla^2_{u,u}\phi)v = g(\phi Au,v)Au - g((\nabla_u A)u,v)\xi - g(Au,v)\phi Au, \tag{4.10}$$

$$(\nabla_u A)\xi = -A\phi Au, \tag{4.11}$$

$$(\nabla^2_{u,u}A)\xi = -2(\nabla_u A)\phi Au - A\phi(\nabla_u A)u. \tag{4.12}$$

Substituting the right-hand sides of (4.9)–(4.12) into the right-hand side of

(4.8) and using Theorem C-V (ii), we have the following equation:

$$\begin{aligned} 0 &= g((\nabla^2_{u,u}R)(v,u)u,\xi) \\ &= 3g((\nabla_u A)u,u)g(\phi v,u) + g(Av,u)g(A\phi(\nabla_u A)u,u) \\ &\quad - g(Au,u)g(A\phi(\nabla_u A)u,v) + 2g(Av,u)g((\nabla_u A)\phi Au,u) \\ &\quad - 2g(Au,u)g((\nabla_u A)\phi Au,v) + 2g((\nabla_u A)v,u)g(A\phi Au,u) \\ &\quad - 2g((\nabla_u A)u,u)g(A\phi Au,v). \end{aligned} \tag{4.13}$$

Owing to (4.7), we have

$$\begin{aligned} &g(Av,u)(\nabla_u A)u - g(Au,u)(\nabla_u A)v \\ &\quad = -g((\nabla_u A)v,u)Au + g((\nabla_u A)u,u)Av + (\xi\text{-component}). \end{aligned} \tag{4.14}$$

Due to (4.14) the equation (4.13) reduces to

$$\begin{aligned} &3g((\nabla_u A)u,u)g(\phi v,u) + g(Av,u)g(A\phi(\nabla_u A)u,u) \\ &\quad - g(Au,u)g(A\phi(\nabla_u A)u,v) = 0. \end{aligned} \tag{4.15}$$

Let $u \in T^0M$ be a unit principal curvature vector with $Au = \lambda u$ and $v = \phi u$, then (4.15) becomes

$$4g((\nabla_u A)u,u) = 0. \tag{4.16}$$

So we have

$$u\lambda = 0. \tag{4.17}$$

Let v be a principal curvature vector with $Av = \mu v$ which is perpendicular to both u and ϕu. Further we define $w = \phi v$. Then (4.15) becomes

$$\lambda\mu g((\nabla_u A)u,w) = 0. \tag{4.18}$$

Owing to Lemma 2.3, $\lambda\mu \neq 0$ is satisfied in our case. So using (4.18) and the Codazzi equation (2.8), we have

$$w\lambda = g(u,(\nabla_w A)u) = g(u,(\nabla_u A)w) = 0. \tag{4.19}$$

Further, from Lemma 2.3 and (4.17) we have

$$0 = \phi u\left(\frac{1}{\lambda}\right) = -\frac{(\phi u)\lambda}{\lambda^2}. \tag{4.20}$$

By (4.17), (4.19) and (4.20) we conclude that λ is constant.

In homogeneous real hypersurfaces only the real hypersurface of type (A) with $r = \frac{\pi}{4}$ attains $\alpha = 0$. So in Case 2 M must be congruent to the homogeneous real hypersurface of type (A). The theorem is now proved by all the above arguments. □

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