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A modified Newton method for asymmetric variational inequality problems

Zhong-Zhi ZHANG and Yu-Fei YANG

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Abstract. Based on the regularized gap function introduced by Fukushima [5], we present a modification of Newton's method for solving the variational inequality problem $VI(X, F)$ by combining the trust region technique. Without the assumption that the mapping F is strongly monotone on the set X, we prove that the proposed algorithm converges globally to a solution of $VI(X, F)$ if the Jacobian matrix ∇F is positive definite on X. Under some additional assumptions, we deduce that the rate of convergence is quadratic.

Key words: variational inequality problem, Newton's method, trust region technique, global convergence, quadratic convergence.

1. Introduction

We consider the variational inequality problem (denoted by $VI(X, F)$): find an $x^{*}\in X$ such that

$$\langle F(x^{*}), x-x^{*}\rangle\geq 0, \quad \text{for all} \quad x\in X,$$

where $X\subseteq R^{n}$ is a nonempty closed convex set, $F : R^{n}\rightarrow R^{n}$ is a continuous mapping and $\langle\cdot, \cdot\rangle$ denotes the inner product in R^{n}. In the special case where $X=R_{+}^{n}$, the variational inequality problem reduces to the nonlinear complementarity problem.

The variational inequality problem has many applications in economics, engineering and various equilibrium models. In the last several years, there have been developed many numerical methods for solving the variational inequality problem. We refer to [7] for a comprehensive review about the early developments of such approaches. They are mainly divided into two classes: one is to reformulate the variational inequality problem as a system of equations and then to use methods and theory from the classical system of equations, such as projection methods, the nonlinear Jacobi method, SOR method, generalized gradient method, Newton's method and quasi-

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Newton method, see, e.g., [7, 15]; another is to reformulate the variational inequality problem as a equivalent optimization problem and then to use methods and theory from mathematical program, such as gap function, regularized gap function and merit functions derived from NCP functions, see, e.g., [3, 6, 9, 16, 19].

In recent years, reformulating the variational inequality problem as an equivalent optimization problem has become a very important class of iterative methods and has attracted much more attention since this approach enjoys some definite advantages, for example, the latter problem may be solved by descent algorithms which possess the global convergence property. In general, there are two ways to derive the global convergence of an algorithm: line search strategy and trust region technique. Trust region methods are an efficient class of iterative methods for solving optimization problems and have been received much attention, see, e.g., [4, 12, 13, 14]. They are often said to be more reliable and robust than the corresponding line search methods.

When $F(x)$ is the gradient of a differentiable function $\theta : R^n \to R$, $VI(X, F)$ can be reformulated as the following equivalent optimization problem:

$$\min_{\text{s.t. } x \in X} \theta(x).$$

It is well known that when the mapping F is differentiable, F satisfies the above condition if and only if the Jacobian matrix $\nabla F(x)$ is symmetric for all x. However, this symmetry condition does not hold in many practical equilibrium models.

For the general asymmetric variational inequality problem, Auslender [2] reformulated $VI(X, F)$ as the following equivalent optimization problem:

$$\min_{\text{s.t. } x \in X} g(x),$$

where

$$g(x) := \sup\{\langle F(x), x - y\rangle \mid y \in X\}$$

is called as gap function. This function was also introduced by Hearn [8] for convex programming problems. Based on the above reformulation, Marcotte [10] presented a descent algorithm for solving the monotone variational inequality problems. By using the gap function as the merit function, Marcotte and Dassault [11] proposed a globally convergent modification of Newton's method with exact line search. However, the gap function is in general nondifferentiable and the set X has to be assumed compact to ensure that the function g is well defined.

To overcome the above drawbacks, Fukushima [5] and Auchmuty [1], almost at the same time, presented the regularized gap function $f : R^n \to R$ defined by

$$f(x) := \max\left\{ \langle F(x), x-y\rangle - \frac{1}{2}\langle x-y, G(x-y)\rangle \mid y \in X \right\},$$

where G is an $n \times n$ symmetric positive definite matrix. Compared with the gap function, f is differentiable whenever so is the mapping F. And $VI(X,F)$ is equivalent to the following optimization problem:

$$\min_{\text{s.t. } x\in X} f(x).$$

Furthermore, Fukushima [5] proposed a descent algorithm for solving the variational inequality problem. Based on the use of the regularized gap function as the merit function, Taji et al. [17] presented a modification of Newton's method with global convergence and quadratic rate of convergence under mild assumptions. In particular, this method allows inexact line search and does not rely upon the compactness assumption on the set X. However, the mapping F has to be assumed strongly monotone so that a wide variety of practical applications may be treated outside this framework. Moreover, Wu et al. [18] presented an unified framework for descent algorithms that solve the monotone variational inequality problem, which includes, as special cases, some well known iterative methods and equivalent optimization formulations. A descent method was developed for an equivalent general optimization formulation and the global convergence was given. Their method may not require the strong monotonicity of the mapping F but still requires the compactness of the set X.

Motivated by the above points, in this paper we present a hybrid algorithm for solving the variational inequality problem based on the regularized gap function introduced by Fukushima [5]. The proposed algorithm incorporates Newton's method for the variational inequality problem with the trust region technique. We prove that, when the Jacobian matrix $\nabla F(x)$ is positive definite for all $x \in X$ instead of the assumption that F is strongly monotone on X, the proposed algorithm is globally convergent to a solution of $VI(X, F)$, and that the rate of convergence is quadratic under the same assumptions as those in [17]. In this paper we does not assume the compactness of the set X.

This paper is organized as follows. In the next section, we review the several basic concepts and some important properties on the regularized gap function f. In Section 3, we present a modified Newton method for solving the variational inequality problem and prove that it is well defined. In Section 4, we show that our algorithm is globally convergent, and, under some additional assumptions, the rate of convergence is quadratic. Some conclusive remarks are given in the last section.

2. Preliminaries

In this section, we recall several foundmental concepts and summarize some properties of the regularized gap function f.

Definition 2.1 The mapping $F : R^n \to R^n$ is said to be

(i) monotone on X if

$$\langle F(x) - F(y), x - y \rangle \geq 0 \quad \text{for all } \; x, y \in X,$$

(ii) strictly monotone on X if

$$\langle F(x) - F(y), x - y \rangle > 0 \quad \text{for all } \; x, y \in X, \quad x \neq y,$$

(iii) strongly monotone with modulus $\mu > 0$ on X if

$$\langle F(x) - F(y), x - y \rangle \geq \mu \|x - y\|^2 \quad \text{for all } \; x, y \in X,$$

where $\|\cdot\|$ denotes the Euclidean norm in R^n.

It is well known that, when the mapping F is differentiable, F is strictly monotone on X if the Jacobian matrix $\nabla F(x)$ is positive definite for all $x \in X$, and F is strongly monotone on X if and only if $\nabla F(x)$ satisfies

$$\langle x - y, \nabla F(x)(x - y) \rangle \geq \mu \|x - y\|^2 \quad \text{for all } \; x, y \in X.$$

It is obvious that any strongly monotone mapping is strictly monotone.

Definition 2.2 Let G be an $n \times n$ symmetric positive definite matrix. The projection of a point $x \in R^n$ onto the closed convex set X, denoted by

$Proj_{X,G}(x)$, is defined as the unique solution of the problem:

$$\min_{\text{s.t. } y \in X} \|y - x\|_G,$$

where $\|\cdot\|_G$ denotes the G-norm in R^n, which is defined by $\|x\|_G := \langle x, Gx \rangle^{1/2}$.

By using the projection operator $Proj_{X,G}(\cdot)$, we can rewrite $f(x)$ as

$$f(x) = -\langle F(x), H(x) - x \rangle - \frac{1}{2}\langle H(x) - x, G(H(x) - x) \rangle,$$

where

$$H(x) := Proj_{X,G}(x - G^{-1}F(x)).$$

The projection operator $Proj_{X,G}(\cdot)$ is nonexpansive, i.e.,

$$\|Proj_{X,G}(x) - Proj_{X,G}(y)\|_G \le \|x - y\|_G \quad \text{for all} \quad x, y \in R^n.$$

Hence when the mapping F is continuous, so is the mapping H. Furthermore, $f(x)$ possesses the following properties, see [5, 17] for details.

Lemma 2.1 *x solves $VI(X, F)$ if and only if x is a fixed point of the mapping H, i.e., $x = H(x)$.*

Lemma 2.2 *$f(x) \ge 0$ for all $x \in X$, and $f(x) = 0$ if and only if x solves $VI(X, F)$. Hence x solves $VI(X, F)$ if and only if it solves the following optimization problem with the minimum value equal to zero:*

$$\min_{s.t.\ x \in X} f(x). \tag{1}$$

Lemma 2.3 *If the mapping F is continuous, then the function f is also continuous. Furthermore, if F is continuously differentiable, then f is also continuously differentiable and its gradient is given by*

$$\nabla f(x) = F(x) - [\nabla F(x) - G](H(x) - x).$$

Lemma 2.2 indicates that solving $VI(X, F)$ amounts to finding a global optimal solution of problem (1). However, most of the existing optimization algorithms are only able to find a stationary point of (1), which is not necessary a global minimizer of (1). To ensure that every stationary point of (1) is a solution of $VI(X, F)$, additional conditions are assumed. This is done in the next lemma.

Lemma 2.4 *Assume that the mapping F is continuously differentiable and its Jacobian matrix $\nabla F(x)$ is positive definite for all $x \in X$. If $x \in X$ is a stationary point of problem (1), i.e.,*

$$\langle \nabla f(x), y - x \rangle \geq 0 \quad \text{for all} \;\; y \in X,$$

then x solves $VI(X, F)$, and hence x is a global optimal solution of problem (1).

Lemma 2.5 *If the mapping F is continuously differentiable and its Jacobian matrix $\nabla F(x)$ is positive definite for all $x \in X$. Then for each $x \in X$ the vector $d := H(x) - x$ satisfies the descent condition*

$$\langle \nabla f(x), d \rangle < 0,$$

whenever $d \neq 0$.

In Newton's method for solving $VI(X, F)$, for some fixed $x \in X$, we have to solve a linearized variational inequality problem (denoted by $LVIP(x)$): Find $z \in X$ such that for all $y \in X$,

$$\langle F(x) + \nabla F(x)^T (z - x), y - z \rangle \geq 0.$$

The above problem does not always have a solution. But when $\nabla F(x)$ is positive definite, it has a unique solution z, which is denoted by $Z(x)$. The linearized problem $LVIP(x)$ is usually easier to solve than the original problem $VI(X, F)$.

3. Algorithm

In this section, we first propose a modification of Newton's method for solving $VI(X, F)$ based on the regularized gap function and the trust region technique, and then we prove that the proposed algorithm is well defined.

Algorithm 3.1

Step 0. Choose an initial feasible point $x^0 \in X$, constants $\alpha, \beta, \sigma \in (0, 1)$ and $M \geq 0$.

Set $k := 0$.

Step 1. If $f(x^k) = 0$, stop.

Step 2. Find $Z(x^k) \in X$ that solves the subproblem $LVIP(x^k)$.

Let $d^k := Z(x^k) - x^k$.

Step 3. If

$$f(x^k + d^k) \le \alpha f(x^k), \tag{2}$$

then set $x^{k+1} := x^k + d^k$, $k := k+1$, go to Step 1; otherwise, set $\Delta := \|d^k\|$.

Step 4. Solve the trust region subproblem:

$$\begin{aligned} \min \quad & Q_k(s) = \frac{1}{2}M\|s\|^2 + \nabla f(x^k)^T s, \\ \text{s.t.} \quad & x^k + s \in X \quad \text{and} \quad \|s\| \le \Delta. \end{aligned} \tag{3}$$

Let $s(\Delta)$ be the solution of (3).

Step 5. If

$$f(x^k + s(\Delta)) \le f(x^k) + \sigma \nabla f(x^k)^T s(\Delta), \tag{4}$$

then set $\Delta_k := \Delta$, $s^k := s(\Delta_k)$, $x^{k+1} := x^k + s^k$, $k := k+1$, go to Step 1; otherwise, set $\Delta := \beta\Delta$, go to Step 4.

In subproblem (3), instead of using the quadratic term $\frac{1}{2}M\|s\|^2$, we may use the more general quadratic term $\frac{1}{2}s^T B_k s$, where the positive semidefinite matrix B_k may be updated by some quasi-Newton formula. The convergence results will remain valid as long as $\{B_k\}$ is bounded.

The above algorithm also allows an inexact solution of the trust region subproblem (3). In fact, we can replace Step 4 by the following step:

Step 4*. Choose a symmetric matrix $B_k \in R^{n\times n}$ and $M_k \in [0, M]$ such that $\|B_k\| \le M_k$.

Let $\hat{s}(\triangle)$ be the solution of the problem:

$$\begin{aligned} \min \quad & \hat{Q}_k(s) := \frac{1}{2}M_k\|s\|^2 + \nabla f(x^k)^T s, \\ \text{s.t.} \quad & x^k + s \in X \quad \text{and} \quad \|s\| \le \triangle. \end{aligned}$$

Compute $s(\triangle)$ such that

$$\begin{aligned} & \phi_k(s(\triangle)) \le \delta \hat{Q}_k(\hat{s}(\triangle)), \\ & x^k + s(\triangle) \in X \quad \text{and} \quad \|s(\triangle)\| \le \triangle, \end{aligned}$$

where $\phi_k(s) := \frac{1}{2}s^T B_k s + \nabla f(x^k)^T s$ and $\delta \in (0,1)$ is a given constant.

It is not difficult to deduce that all results are still valid if Step 4 is replaced by Step 4*. The above idea has also been used in recent literature [4, 12, 13].

The next proposition shows that Algorithm 3.1 is well defined.

Proposition 3.1 *Assume that the mapping F is continuously differentiable and its Jacobian matrix $\nabla F(x)$ is positive definite for all $x \in X$. Then Algorithm* 3.1 *is well defined; i.e., x^{k+1} can be obtained either by Step* 3 *or by repeating Step* 4 *and Step* 5 *a finite number of times whenever $f(x^k) \neq 0$.*

Proof. We only need to prove that Algorithm 3.1 can not cycle infinitely between Step 4 and Step 5 if x^{k+1} is not obtained by Step 3.

In fact, by Lemma 2.2, $f(x^k) \neq 0$ implies that x^k is not a solution of $VI(X, F)$ and hence it follows from Lemma 2.4 that x^k is not also a stationary point of (1). So, there exists a feasible descent direction $d \in R^n \backslash \{0\}$ such that for some $\lambda_0 > 0$,

$$\nabla f(x^k)^T d < 0 \quad \text{and} \quad x^k + \lambda d \in X, \quad \forall \lambda \in (0, \lambda_0).$$

For $\Delta > 0$ sufficiently small, $\frac{\Delta}{\|d\|} < \lambda_0$ and

$$\begin{aligned}\nabla f(x^k)^T s(\Delta) \leq Q_k(s(\Delta)) &\leq Q_k\left(\frac{\Delta}{\|d\|} d\right) \\ &= \frac{M}{2}\Delta^2 + \frac{\Delta}{\|d\|}\nabla f(x^k)^T d,\end{aligned}$$

which shows

$$\limsup_{\Delta \to 0} \frac{\nabla f(x^k)^T s(\Delta)}{\Delta} \leq \frac{1}{\|d\|}\nabla f(x^k)^T d \triangleq 2c_1 < 0.$$

Thus, there exists $\bar{\Delta} > 0$ such that

$$\frac{\nabla f(x^k)^T s(\Delta)}{\Delta} \leq c_1 < 0, \quad \forall \Delta \in (0, \bar{\Delta}).$$

Set

$$\rho(\Delta) = \frac{f(x^k + s(\Delta)) - f(x^k)}{\nabla f(x^k)^T s(\Delta)}.$$

Then

$$|\rho(\Delta) - 1| = \left| \frac{f(x^k + s(\Delta)) - f(x^k) - \nabla f(x^k)^T s(\Delta)}{\nabla f(x^k)^T s(\Delta)} \right|$$
$$= \frac{o(\|s(\Delta)\|)}{|\nabla f(x^k)^T s(\Delta)|} \leq \frac{o(\Delta)}{|c_1|\Delta},$$

which implies

$$\lim_{\Delta \to 0} \rho(\Delta) = 1.$$

This indicates that after a finite number of reduction of Δ,

$$\rho(\Delta) \geq \sigma,$$

which implies that (4) holds. □

4. Global and superlinear convergence

In this section, we shall prove that Algorithm 3.1 is globally convergent to a solution of $VI(X, F)$ if ∇F is positive definite on X and that the rate of convergence is quadratic if some additional assumptions hold.

First, from the proof of Proposition 2.2 in [17], we can deduce the following result, which will play an important role in our global convergence analysis.

Proposition 4.1 *Assume that the mapping F is continuously differentiable and its Jacobian matrix $\nabla F(x)$ is positive definite for all $x \in X$. Then x solves $VI(X, F)$ if and only if x satisfies $x = Z(x)$.*

Note that we do not deduce that the mapping $Z : X \to X$ is continuous, but this does not affect the application of the above proposition. In fact, it is not difficult to prove that if $x^k \to x^*$ and $d^k := Z(x^k) - x^k \to 0$, then x^* solves $VI(X, F)$.

We are now in a position to prove the global convergence for the proposed algorithm.

Theorem 4.1 *Assume that the mapping F is continuously differentiable and its Jacobian matrix $\nabla F(x)$ is positive definite for all $x \in X$. If the level set $L_0 := \{x \in X \mid f(x) \leq f(x^0)\}$ is bounded, then the whole sequence $\{x^k\}$ generated by Algorithm* 3.1 *converges to the unique solution of $VI(X, F)$.*

Proof. It follows from the construction of Algorithm 3.1 that $\{f(x^k)\}$ is monotonously descent and bounded from below. Thus

$$\{f(x^{k+1}) - f(x^k)\} \to 0, \quad \text{as} \ \ k \to \infty. \tag{5}$$

The compactness of the level set L_0 shows that the sequence $\{x^k\}$ has at least one accumulation point. Assume that x^* is an accumulation point of $\{x^k\}$. Then there exists a subsequence $\{x^k\}_{k\in K}$ such that

$$\lim_{\substack{k\to\infty \\ k\in K}} x^k = x^*.$$

Suppose that there exists infinitely many indices $k \in K$ satisfying (2). Without loss of generality, it can be assumed that

$$f(x^{k+1}) \le \alpha f(x^k), \quad \forall k \in K.$$

Then $\lim_{k\to\infty} f(x^k) = 0$, which implies that x^* is a solution of $VI(X, F)$.

Suppose that there exists only finitely many indices $k \in K$ satisfying (2). It can be assumed that

$$f(x^{k+1}) \le f(x^k) + \sigma \nabla f(x^k)^T s^k, \quad \forall k \in K. \tag{6}$$

Set

$$\tilde{d}^k := H(x^k) - x^k, \quad \tilde{d}^* := H(x^*) - x^*$$

and let

$$\inf_{k\in K} \Delta_k = \Delta^*.$$

Then

$$\lim_{\substack{k\to\infty \\ k\in K}} \tilde{d}^k = \tilde{d}^*$$

and there exists $K_1 \subseteq K$ such that

$$\lim_{\substack{k\to\infty \\ k\in K_1}} \Delta_k = \Delta^*.$$

Subsequencing if necessary, it can be assumed that for some d^*,

$$\lim_{\substack{k\to\infty \\ k\in K_1}} d^k = d^*.$$

By Lemma 2.1 and Proposition 4.1, to prove that x^* is a solution of $VI(X, F)$, we only need to show that $\tilde{d}^* = 0$ or $d^* = 0$. Assume the contrary, then it follows from Lemma 2.2 and Lemma 2.5 that

$$f(x^*) \neq 0 \quad \text{and} \quad \nabla f(x^*)^T \tilde{d}^* < 0.$$

There are two possibilities and we shall get a contradiction in each case.

Case 1: $\Delta^* = 0$. Then there exists $\bar{k} \in K_1$ such that for all $k \in K_1$ with $k \geq \bar{k}$,

$$\Delta_k < \|d^*\|/2 \quad \text{and} \quad \|d^k\| \geq \|d^*\|/2.$$

On the other hand, we obtain Δ_k by reducing $\|d^k\|$. Thus, for all $k \in K_1$ with $k \geq \bar{k}$, there exists $\bar{\Delta}_k := \Delta_k/\beta$ such that

$$f(x^k + s(\bar{\Delta}_k)) > f(x^k) + \sigma \nabla f(x^k)^T s(\bar{\Delta}_k). \tag{7}$$

Set

$$v^k := \frac{\|s(\bar{\Delta}_k)\|}{\|\tilde{d}^k\|} \tilde{d}^k.$$

Noting that

$$\|v^k\| = \|s(\bar{\Delta}_k)\| \leq \bar{\Delta}_k \to 0, \qquad \tilde{d}^k \to \tilde{d}^* \neq 0$$

and

$$x^k, x^k + \tilde{d}^k \in X.$$

Hence, for $k \in K_1$ sufficiently large,

$$x^k + v^k \in X$$

and

$$\begin{aligned} \nabla f(x^k)^T s(\bar{\Delta}_k) &\leq Q_k(s(\bar{\Delta}_k)) \leq Q_k(v^k) \\ &= \frac{M}{2}\|s(\bar{\Delta}_k)\|^2 + \frac{\|s(\bar{\Delta}_k)\|}{\|\tilde{d}^k\|} \nabla f(x^k)^T \tilde{d}^k. \end{aligned}$$

So, we get

$$\limsup_{k \in K_1} \frac{\nabla f(x^k)^T s(\bar{\Delta}_k)}{\|s(\bar{\Delta}_k)\|} \leq \frac{1}{\|\tilde{d}^*\|} \nabla f(x^*)^T \tilde{d}^* \triangleq 2c_2 < 0,$$

which shows that for $k \in K_1$ sufficiently large,

$$\frac{\nabla f(x^k)^T s(\bar{\Delta}_k)}{\|s(\bar{\Delta}_k)\|} \le c_2 < 0.$$

Set

$$\rho_k := \frac{f(x^k + s(\bar{\Delta}_k)) - f(x^k)}{\nabla f(x^k)^T s(\bar{\Delta}_k)}.$$

Then for $k \in K_1$ sufficiently large,

$$\begin{aligned}|\rho_k - 1| &= \left|\frac{f(x^k + s(\bar{\Delta}_k)) - f(x^k) - \nabla f(x^k)^T s(\bar{\Delta}_k)}{\nabla f(x^k)^T s(\bar{\Delta}_k)}\right| \\ &= \frac{o(\|s(\bar{\Delta}_k)\|)}{|\nabla f(x^k)^T s(\bar{\Delta}_k)|} \le \frac{o(\|s(\bar{\Delta}_k)\|)}{|c_2|\|s(\bar{\Delta}_k)\|},\end{aligned}$$

which implies

$$\lim_{\substack{k\to\infty \\ k\in K_1}} \rho_k = 1.$$

This contradicts (7).

Case 2: $\Delta^* > 0$. Since $\{s^k\}_{k\in K_1}$ are bounded, there exist $K_2 \subseteq K_1$ and $s^* \in R^n$ such that

$$\lim_{\substack{k\to\infty \\ k\in K_2}} s^k = s^*.$$

It is easy to prove that s^* is a solution of the problem

$$\begin{aligned}&\min \quad Q(s) = \frac{1}{2}M\|s\|^2 + \nabla f(x^*)^T s, \\ &\text{s.t.} \quad x^* + s \in X \quad \text{and} \quad \|s\| \le \Delta^*\end{aligned} \tag{8}$$

and hence $Q(s^*) \le 0$.

On the other hand, it follows from (5) and (6) that

$$0 \le \lim_{\substack{k\to\infty \\ k\in K_2}} \nabla f(x^k)^T s^k \le \lim_{\substack{k\to\infty \\ k\in K_2}} Q_k(s^k) = Q(s^*),$$

which implies

$$Q(s^*) = 0.$$

From (8), we can deduce that $Q(s^*) = 0$ if and only if x^* is a stationary point of the problem (1), i.e.,

$$\nabla f(x^*)^T(x - x^*) \geq 0, \quad \forall x \in X.$$

This contradicts $\nabla f(x^*)^T \tilde{d}^* < 0$.

The above analysis shows that any accumulation point of the sequence $\{x^k\}$ is a solution of $VI(X, F)$. Since $VI(X, F)$ has at most a solution, it follows that x^* is the unique solution of $VI(X, F)$ and hence the whole sequence $\{x^k\}$ converges to x^*. This completes the proof. □

To obtain a rate of convergence result, we need to strengthen assumptions on the mapping F. The following result comes from [17].

Lemma 4.1 *Let x^* be a solution of $VI(X, F)$. If F is strongly monotone with modulus μ on X, then f satisfies the inequality*

$$f(x) \geq \left(\mu - \frac{1}{2}\|G\|\right) \|x - x^*\|^2 \quad \textit{for all} \;\; x \in X.$$

In particular, if the matrix G is chosen sufficiently small to satisfy $\|G\| < 2\mu$, then

$$\lim_{\substack{\|x\| \to \infty \\ x \in X}} f(x) = +\infty.$$

Moreover, we need the following strict complementarity condition, see [11].

Definition 4.1 Suppose that X is a polyhedral and that problem $VI(X, F)$ has a unique solution x^*. Let T^* denote the minimal face of X containing x^*. Then we say that strict complementarity holds at x^* if $x \in X$ and $\langle F(x^*), x - x^* \rangle = 0$ imply $x \in T^*$.

Applying Lemma 4.1 and similar to the proof of Theorem 5.1 in [17], we can deduce the following rate of convergence result.

Theorem 4.2 *Assume that the set X is polyhedral convex, F is continuously differentiable and strongly monotone with modulus μ on X, ∇F is Lipschitz continuous on a neighborhood N of the unique solution x^* of problem $VI(X, F)$ and the strict complementarity condition holds at x^*. If the matrix G is chosen sufficiently small to satisfy $\|G\| < 2\mu$ and the level set $L_0 := \{x \in X \mid f(x) \leq f(x^0)\}$ is bounded. Then the sequence $\{x^k\}$*

generated by Algorithm 3.1 *converges quadratically to the solution* x^*.

5. Concluding remarks

In this paper we developed a hybrid method for solving the variational inequality problem. Based on the regularized gap function introduced by Fukushima [5], our algorithm combines Newton's method for the variational inequality problem and the trust region technique. Without the assumption that the mapping F is strongly monotone on X, we establish the global convergence of the proposed algorithm if ∇F is positive definite on X. Moreover, we get the quadratic rate of convergence if some additional assumptions hold.

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Zhong-Zhi Zhang
Hunan Computing Centre
Changsha, Hunan 410012
P. R. China

Yu-Fei Yang
Department of Applied Mathematics
Hunan University
Changsha, Hunan 410082
P. R. China

Present address:
School of Mathematics
The University of New South Wales
Sydney, NSW 2052, Australia
E-mail: yang@maths.unsw.edu.au