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# Singularities for projections of contour lines of surfaces onto planes

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**Abstract.** We study semi-local patterns of the visions for contour lines of a surface when one looks at it from a distant view in any direction. The study of such a landscape (i.e. so-called "topography") is reduced to the study of a certain divergent diagram of smooth mappings $\mathbb{R} \leftarrow M \rightarrow \mathbb{R}^2$, where $M$ is a smooth surface. We give a generic semi-local classification of such divergent diagrams.

*Key words*: singularity, vision, divergent diagram, semi-local, web structure.

## 1. Introduction

A contour line of a surface $S$ in $\mathbb{R}^3 = \{(x, y, z)\}$ is a set $S \cap \{z = \text{constant}\}$. In this paper we study semi-local patterns of the viewing image (i.e. topography) when one looks at a surface with its contour lines from a distant point in $\mathbb{R}^3$. We give a generic semi-local classification for the singularities of orthogonal projections of contour lines of surfaces onto the plane.

There are many studies of certain visual images from the viewpoint of singularity theory started by Koenderink and Doorn [KD] (see also [K], [W], [B], [BG], [P], [A], [DT]). In particular Dufour and Tueno [DT] have investigated local and semi-local generic types of photographs (i.e. equal illumination curves) of lighted surfaces. In [DT] the generic classification of divergent diagrams $(\mathbb{R}, 0) \leftarrow (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ due to Arnol'd [A] and Dufour [D4] was applied. However normal forms of the semi-local case were not given.

In this paper, we give more detailed classification of topographies in the semi-local case. In particular, we give the normal forms of generic types, which contain "functional moduli". These results will be used to show the so-called "topological rigidity theorem" in the semi-local case of

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topographies in the forthcoming paper jointed with Dufour [DK], which is a generalization of Dufour's result in the local case [D4] (see Remark 1.4 below).

We now formulate our theorems. Throughout this paper we shall suppose that all mappings, map germs and manifolds are of class $C^\infty$.

Let $M$ be a surface in $\mathbb{R}^3 = \{(x,y,z)\}$. We denote by $Emb(M,\mathbb{R}^3)$ the space of all embeddings $M \to \mathbb{R}^3$ endowed with the Whitney $C^\infty$-topology. Let $E_d$ be a hyperplane in $\mathbb{R}^3$ with the normal direction $d$ such that $E_d \cap M = \phi$ and let $\pi_d : \mathbb{R}^3 \to E_d$ be the orthogonal projection along the direction $d$.

Let $i \in Emb(M,\mathbb{R}^3)$. If one looks at a contour line on $i(M)$ from a distant view in a direction $d$, then one will get $\pi_d(i(M) \cap \{z=c\})$ as the viewing image. Our subject is a semi-local classification of singularities for one parameter families $\{\pi_d(i(M) \cap \{z=c\})\}_{c \in \mathbb{R}}$, called *the topography* of $i(M)$ with respect to a direction $d$. Without loss of generality we can suppose that $d = (0, \cos\xi, \sin\xi) \in S^2 \cap \{x = 0\}$ where $0 < \xi \leqq \frac{\pi}{2}$. For a direction $d$, by the transformation of $\frac{\pi}{2} - \xi$ rotation around $x$-axis in $\mathbb{R}^3$, we choose new coordinates $(u,v,w)$. Then the direction $d$ becomes $(0,0,1)$ and the height function $z$ is expressed by $-v\sin(\frac{\pi}{2}-\xi) + w\cos(\frac{\pi}{2}-\xi)$ in the new coordinates. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection defined by $\pi(u,v,w) = (u,v)$. We call the following divergent diagram of mappings *a topographic diagram* of $i(M)$ :

$$\mathbb{R} \xleftarrow{\mu[i,\xi]} M \xrightarrow{g[i]} \mathbb{R}^2,$$

where $\mu[i,\xi] = -v \circ i \sin(\frac{\pi}{2}-\xi) + w \circ i \cos(\frac{\pi}{2}-\xi)$ and $g[i] = \pi \circ i$.

Since our concern is to describe the discriminant set of $g[i]$ (the outline of $i(M)$) and the bifurcation of $g[i](\mu[i,\xi]^{-1}(c))$ along the parameter $c \in \mathbb{R}$ in the semi-local situation, we introduce the following definitions. Let $\{p_1, \ldots, p_r\}$ be a subset of $M$ whose elements are all distinct points in $M$ such that $\pi \circ i(p_1) = \cdots = \pi \circ i(p_r)$, where $r$ is a positive integer. Then the multigerm of a topographic diagram at $\{p_1, \ldots, p_r\}$ which is denoted by ${}_rT[i,\xi]$ (or briefly ${}_rT_i$) is called *a topographic multigerm of* $i$:

$$\begin{tikzcd}
(\mathbb{R},0) & (\mathbb{R}^2,0) \arrow[l, "\mu_1"'] \arrow[r, "g_1"] & (\mathbb{R}^2,0) \\
(\mathbb{R},0) & (\mathbb{R}^2,0) \arrow[l, "\mu_2"'] \arrow[ur, "g_2"] & \\
\vdots & & \\
(\mathbb{R},0) & (\mathbb{R}^2,0) \arrow[l, "\mu_r"'] \arrow[uuur, "g_r"] &
\end{tikzcd},$$

where $\mu_k$, $g_k$ are germs of $\mu[i,\xi]$, $g[i]$ at $p_k$ $(k = 1, \ldots, r)$ respectively. Let ${}_rT[i,\xi]$ and ${}_rT[i',\xi']$ be topographic multigerms, then they are said to be *equivalent* if there exist diffeomorphism germs $\lambda_k : (\mathbb{R},0) \to (\mathbb{R},0)$, $\psi_k : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ and $\Phi : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ such that $\lambda_k \circ \mu_k = {\mu'}_k \circ \psi_k$, $\Phi \circ g_k = {g'}_k \circ \psi_k$ for $k = 1, \ldots, r$.

We shall state our theorems of a genericity and their normal forms for topographic multigerms. Denote by $S_g$ the singular set of a mapping $g$. Denote by $\mathcal{E}_{x_1,\ldots,x_n}$ the ring of all smooth function germs on $\mathbb{R}^n$ at 0 with coordinates $(x_1, \ldots, x_n)$ and denote by $\mathcal{M}_{x_1,\ldots,x_n}$ the unique maximal ideal of $\mathcal{E}_{x_1,\ldots,x_n}$.

**Theorem A** *Let $r = 1, 2$. There exists a residual subset (hence dense) ${}_r\mathcal{O}$ in $Emb(M, \mathbb{R}^3)$ such that for any $i \in {}_r\mathcal{O}$ and any $\xi \in (0, \frac{\pi}{2}]$ the topographic multigerms ${}_rT[i,\xi]$ is one of the following types:*

*In the case of $r = 1$*

(I) *$\mu_1$ is a submersion and $g_1$ is regular.*

(II) *$\mu_1$ is of Morse type and $g_1$ is regular.*

(III) *$\mu_1$ is a submersion, $g_1$ is a fold, $\mu_1|_{S_{g_1}}$ is regular and $(\mu_1, g_1) : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ is regular.*

(IV) *$\mu_1$ is a submersion, $g_1$ is a fold, $\mu_1|_{S_{g_1}}$ is of Morse type and $(\mu_1, g_1) : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ is regular.*

(V) *$\mu_1$ is a submersion, $g_1$ is a fold, $(\mu_1, g_1) : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ is a Whitney umbrella whose line of double points is transversal at 0 to the direction $\{0\} \times \mathbb{R}^2$ in $\mathbb{R}^3$.*

(VI) *$\mu_1$ is a submersion, $g_1$ is a cusp and $(\mu_1, g_1) : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ is regular.*

*In the case of $r = 2$*

$(\mathrm{I},\mathrm{I})_0$, $(\mathrm{I},\mathrm{I})_1$, $(\mathrm{I},\mathrm{I})_2$ : $(\mu_1, g_1)$, $(\mu_2, g_2)$ *are both of type* (I) *and* $(\mu_1 \circ {g_1}^{-1}, \mu_2 \circ {g_2}^{-1}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *is regular, a fold, a cusp respectively.*

$(\mathrm{II}, \mathrm{I})$ : $(\mu_1, g_1)$ *is of type* (II), $(\mu_2, g_2)$ *is of type* (I)
*and* $(\mu_1 \circ {g_1}^{-1}, \mu_2 \circ {g_2}^{-1}) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ *is a fold.*

$(\mathrm{III}, \mathrm{I})^0$, $(\mathrm{III}, \mathrm{I})^1$ : $(\mu_1, g_1)$ *is of type* (III), $(\mu_2, g_2)$ *is of type* (I)
*and respectively* $g_1(S_{g_1}) \bar{\pitchfork} g_2(\mu_2^{-1}(0))$, $g_1(S_{g_1}) \not\bar{\pitchfork} g_2(\mu_2^{-1}(0))$
*with two point contact.*

$(\mathrm{IV}, \mathrm{I})$ : $(\mu_1, g_1)$ *is of type* (IV), $(\mu_2, g_2)$ *is of type* (I)
*and* $g_1(S_{g_1}) \bar{\pitchfork} g_2(\mu_2^{-1}(0))$.

$(\mathrm{V}, \mathrm{I})$ : $(\mu_1, g_1)$ *is of type* (V), $(\mu_2, g_2)$ *is of type* (I)
*and* $g_1(S_{g_1}) \bar{\pitchfork} g_2(\mu_2^{-1}(0))$.

$(\mathrm{VI}, \mathrm{I})$ : $(\mu_1, g_1)$ *is of type* (VI), $(\mu_2, g_2)$ *is of type* (I)
*and the tangent cone of* $g_1(S_{g_1}) \bar{\pitchfork} g_2(\mu_2^{-1}(0))$.

$(\mathrm{III}, \mathrm{III})$ : $(\mu_1, g_1), (\mu_2, g_2)$ *are both of type* (III) *and* $g_1(S_{g_1}) \bar{\pitchfork} g_2(S_{g_2})$.

**Remark 1.1** In the case of $r = 2$, the generic condition of $(\mathrm{I}, \mathrm{I})_1$ (resp. $(\mathrm{I}, \mathrm{I})_2$) means that $g_1({\mu_1}^{-1}(0))$ and $g_2({\mu_2}^{-1}(0))$ have second (resp. third) order contact.

**Remark 1.2** In the case of $r = 3$, the generic types except the following three types are essentially the same as the case of $r = 2$:
$(\mathrm{I}, \mathrm{I}, \mathrm{I})_{1,1}$ : $(\mu_j, g_j; \mu_k, g_k)$ is of type $(\mathrm{I}, \mathrm{I})_1$ for $1 \leq j < k \leq 3$.

$(\mathrm{III}, \mathrm{I}, \mathrm{I})_1^{0,0}$ : $(\mu_1, g_1; \mu_2, g_2)$ is of type $(\mathrm{III}, \mathrm{I})^0$ and $(\mu_2, g_2; \mu_3, g_3)$
is of type $(\mathrm{I}, \mathrm{I})_1$.

$(\mathrm{III}, \mathrm{III}, \mathrm{I})^{0,0}$ : $(\mu_1, g_1; \mu_2, g_2)$ is of type (III, III) and $(\mu_j, g_j; \mu_3, g_3)$
is of type $(\mathrm{III}, \mathrm{I})^0$ for $j = 1, 2$.

That is, except the above three types, we can add only type (I) to the case of $r = 2$ such that $(\mu_j, g_j; \mu_3, g_3)$, $j = 1, 2$, are non-degenerate in the case of $r = 2$, namely are not $(\mathrm{I}, \mathrm{I})_1$, $(\mathrm{I}, \mathrm{I})_2$ and $(\mathrm{III}, \mathrm{I})^1$. In the same sense, for the case $r \geq 4$ all of the generic types are essentially the same as the case of $r = 3$.

**Theorem B** *Normal forms of the topographic multigerms of the each generic type are the following*:

*In the case of* $r = 1$.
(I) $\mu_1 = y_1$, $g_1 = (x_1, y_1)$.

(II) $\mu_1 = x_1^2 \pm y_1^2, \; g_1 = (x_1, y_1)$.

(III) $\mu_1 = x_1 + y_1, \; g_1 = (x_1, y_1^2)$.

(IV) $\mu_1 = x_1^2 + y_1, \; g_1 = (x_1, y_1^2)$.

(V) $\mu_1 = x_1 + x_1y_1 + y_1^3, \; g_1 = (x_1, y_1^2)$.

(VI) $\mu_1 = y_1 + \alpha \circ g_1, \; g_1 = (x_1, y_1^3 + x_1y_1)$, *where* $\alpha \in \mathcal{M}_{u,v}$.

*In the case of* $r = 2$.

$(\mathrm{I}, \mathrm{I})_0$ $\mu_1 = y_1, \; g_1 = (x_1, y_1)$;
$\mu_2 = x_2, \; g_2 = (x_2, y_2)$.

$(\mathrm{I}, \mathrm{I})_1$ $\mu_1 = y_1, \; g_1 = (x_1, y_1)$;
$\mu_2 = x_2^2 + y_2, \; g_2 = (x_2, y_2)$.

$(\mathrm{I}, \mathrm{I})_2$ $\mu_1 = y_1, \; g_1 = (x_1, y_1)$;
$\mu_2 = x_2^3 + x_2y_2 + y_2, \; g_2 = (x_2, y_2)$.

(II, I) $\mu_1 = x_1^2 \pm y_1^2, \; g_1 = (x_1, y_1)$;
$\mu_2 = x_2, \; g_2 = (x_2, y_2)$.

$(\mathrm{III}, \mathrm{I})^0$ $\mu_1 = x_1 + y_1, \; g_1 = (x_1, y_1^2)$;
$\mu_2 = x_2 + \theta(x_2, y_2), \; g_2 = (x_2, y_2)$,
*where* $\theta \in \mathcal{M}_{x_2,y_2}$ *with* $\theta(x_2, 0) = 0$.

$(\mathrm{III}, \mathrm{I})^1$ $\mu_1 = x_1 + y_1, \; g_1 = (x_1, y_1^2)$;
$\mu_2 = x_2^2 + \theta(x_2, y_2), \; g_2 = (x_2, y_2)$,
*where* $\theta \in \mathcal{M}_{x_2,y_2}$ *with* $\theta(x_2, 0) = 0, \; \frac{\partial \theta}{\partial y_2}(0) \neq 0$.

(IV, I) $\mu_1 = x_1^2 + y_1, \; g_1 = (x_1, y_1^2)$;
$\mu_2 = x_2 + \theta(x_2, y_2), \; g_2 = (x_2, y_2)$,
*where* $\theta \in \mathcal{M}_{x_2,y_2}$ *with* $\theta(x_2, 0) = 0$.

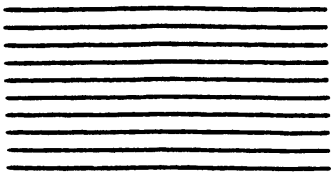
(V, I) $\mu_1 = x_1 + x_1y_1 + y_1^3, \; g_1 = (x_1, y_1^2)$;
$\mu_2 = x_2 + \theta(x_2, y_2), \; g_2 = (x_2, y_2)$,
*where* $\theta \in \mathcal{M}_{x_2,y_2}$ *with* $\theta(x_2, 0) = 0$.

(VI, I) $\mu_1 = y_1 + \alpha \circ g_1, \; g_1 = (x_1, y_1^3 + x_1y_1)$;
$\mu_2 = x_2 + \theta(x_2, y_2), \; g_2 = (x_2, y_2)$,
*where* $\alpha \in \mathcal{M}_{u,v}$, $\theta \in \mathcal{M}_{x_2,y_2}$ *with* $\theta(x_2, 0) = 0$.

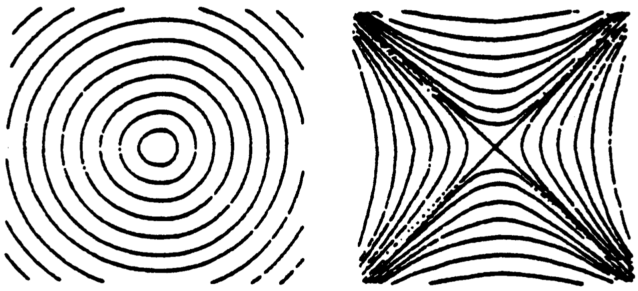
(III, III) $\mu_1 = y_1 + \alpha_1 \circ g_1, \; g_1 = (x_1, y_1^2)$;
$\mu_2 = x_2 + \alpha_2 \circ g_2, \; g_2 = (x_2^2, y_2)$,
*where* $\alpha_1, \alpha_2 \in \mathcal{M}_{u,v}$ *with* $\frac{\partial \alpha_1}{\partial u}(0) \neq 0, \; \frac{\partial \alpha_2}{\partial v}(0) \neq 0$.

**Remark 1.3** In order to understand our classification of topographies geometrically, let us describe the level curves $\{g_k({\mu_k}^{-1}(c))\}$ and the discriminant set of $g_k, 1 \leq k \leq r$ $(r = 1, 2)$ for each type.

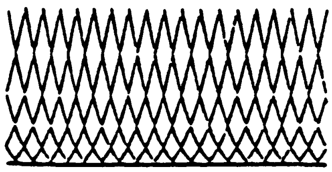
(I)



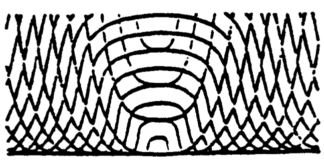
(II)



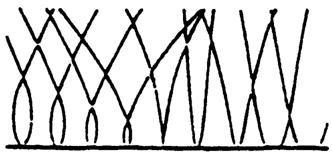
(III)



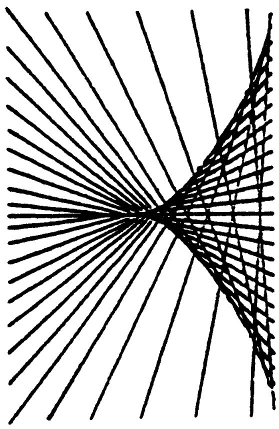
(IV)



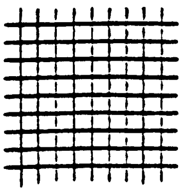
(V)



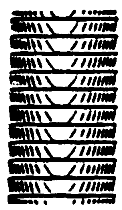
(VI)



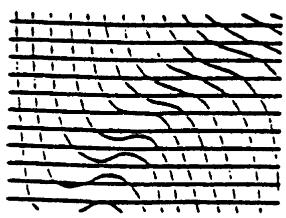
$(\mathrm{I}, \mathrm{I})_0$



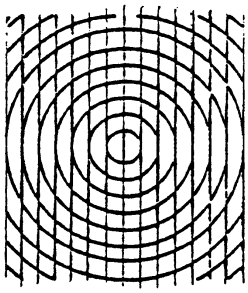
$(\mathrm{I}, \mathrm{I})_1$



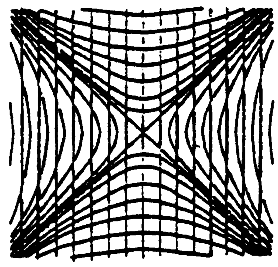
$(\mathrm{I}, \mathrm{I})_2$



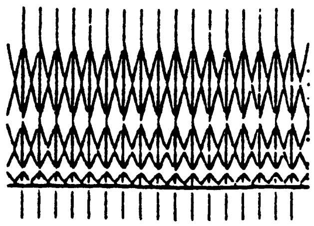
(II, I)



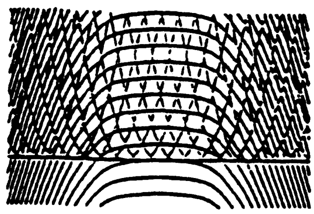
(II, I)



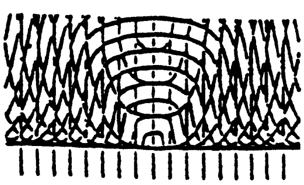
(III, I)$^0$



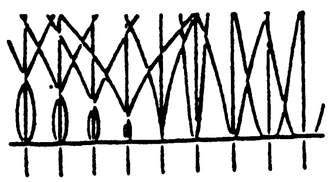
(III, I)$^1$



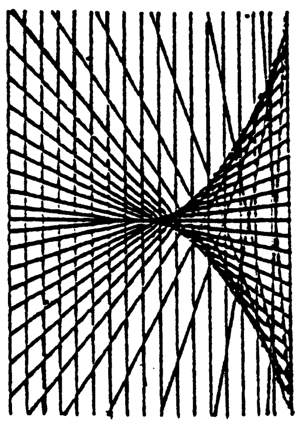
(IV, I)



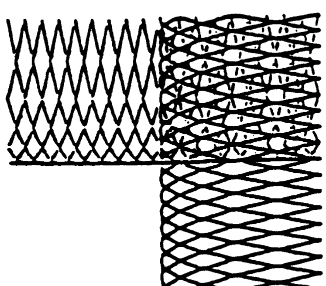
(V, I)



(VI, I)



(III, III)



**Remark 1.4** (Functional moduli) The normal forms in Theorem B depend on arbitrary functions with some conditions, that is so-called "functional moduli". For the type (VI) the functional moduli have been characterized and the complete invariant has been detected ([D5], [IK]). In a semi-local case of $r = 2$ without functional moduli, that is types $(\mathrm{I}, \mathrm{I})_k$ ($k = 0, 1, 2$), $(\mathrm{II}, \mathrm{I})$ have been studied by Dufour [D1, 2, 3] from the viewpoint of "bi-stability" and their normal forms have been obtained.

**Remark 1.5** (Web structures) In topographies a "d-web structure" (a configuration of d foliations) appears. In the local case ($r = 1$) Dufour [D4] showed that the topological classification and the $C^\infty$ classification of (topographic) germs $(\mathbb{R}, 0) \leftarrow (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ are generically the same by investigating the web structures. Such property is called topological rigidity. Also in [DK] it has been shown that for the case of $r = 2$ topological rigidity generically holds.

## 2. Proof of Theorem A

### 2.1. Preliminary

In order to describe the genericity we shall prepare a transversality lemma. Let $M^{(r)} = \{(p_1, \ldots, p_r) \in M^r \mid p_j \neq p_k \text{ for } 1 \leq j < k \leq r\}$, where $M^r$ is the $r$-fold product of $M$ and let ${}_rJ^l(M, \mathbb{R}^3)$ denote the multi jet space from $M$ to $\mathbb{R}^3$ of multiplicity $r$ and order $l$. For a map $f : M \rightarrow \mathbb{R}^3$, denote by ${}_rj^l f : M^{(r)} \rightarrow {}_rJ^l(M, \mathbb{R}^3)$ the multi $l$-extension of $f$. Denote by ${}_rJ^l_{emb}(M, \mathbb{R}^3)$ the set of all multi jets of $i \in Emb(M, \mathbb{R}^3)$. Note that ${}_rJ^l_{emb}(M, \mathbb{R}^3)$ is open in ${}_rJ^l(M, \mathbb{R}^3)$. We suppose that $l$ is a positive integer with $l \geqq 3$.

Consider the map ${}_r\Phi : {}_rJ^l_{emb}(M, \mathbb{R}^3) \rightarrow {}_rJ^l_{emb}(M, \mathbb{R}^3)$ defined by

$$
{}_r\Phi(j^l i(p_1), \ldots, j^l i(p_r)) = (j^l T_i(p_1), \ldots, j^l T_i(p_r)),
$$

where $T_i = (\mu_i, \pi \circ i) \in Emb(M, \mathbb{R}^3)$ defined in Section 1. We can easily see that the map ${}_r\Phi$ is a diffeomorphism. Then by the multi jet transversality theorem we easily have

**Lemma 2.1** *Let $\mathcal{S}$ be a submanifold of ${}_rJ^l_{emb}(M, \mathbb{R}^3)$. If* $\operatorname{codim} \mathcal{S} > 2r$,

$$
\mathcal{O}_{\mathcal{S}} := \{i \in Emb(M, \mathbb{R}^3) \mid {}_rj^l T_i(M^{(r)}) \cap \mathcal{S} = \phi\}
$$

*is residual in $Emb(M, \mathbb{R}^3)$.*

According to Dufour [D4], we now define subvarieties of $J^l(2,3)$ as follows.

$\Sigma_a =$ *Closure*$\{j^3(\mu, g)(0) \mid g$ is a fold map at 0 and $\mu|_{S_g}$ is degenerate at $0\}$,

$\Sigma_b =$ *Closure*$\{j^3(\mu, g)(0) \mid g$ is a fold map at 0 and $(\mu, g)$ is a Whitney umbrella whose line of double points is non-transversal at 0 to the direction $\{0\} \times \mathbb{R}^2$ in $\mathbb{R}^3\}$,

$\Sigma_c =$ *Closure*$\{j^3(\mu, g)(0) \mid g$ is a cusp at 0 and $(\mu, g)$ is singular at $0\}$,

$\Sigma_d =$ *Closure*$\{j^3(\mu, g)(0) \mid \mu$ is singular and degenerate at $0\}$,

where $\mu : \mathbb{R}^2,\ 0 \to \mathbb{R},\ 0,\ g : \mathbb{R}^2,\ 0 \to \mathbb{R}^2,\ 0$.

By its definition, $\Sigma_d$ is a subvariety of codimension 3 in $J^l(2,3)$.

**Lemma 2.2** ([D4]) *$\Sigma_a$, $\Sigma_b$, $\Sigma_c$ are subvarieties of codimension 3 in $J^l(2,3)$.*

## 2.2. Proof of Theorem A.

In the case of $r = 1$, as was shown in [D4], by Lemma 2.1 and Lemma 2.2 we have Theorem A.

In the case of $r = 2$. Denote by ${}_r\Delta$ the diagonal set in the $r$-fold product of $\mathbb{R}^2$. Then ${}_r\Delta$ is a codimension $2r - 2$ submanifold of $\mathbb{R}^2 \times \cdots \times \mathbb{R}^2$ ($r$-times). Let

$$\mathcal{S}_* = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times \Sigma_* \times J^l(2,3),$$

these being algebraic sets in ${}_2J^l(\mathbb{R}^2, \mathbb{R}^3) = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times (\mathbb{R}^2 \times \mathbb{R}^2) \times J^l(2,3) \times J^l(2,3)$, where $* = a, b, c, d$ and $\Sigma_*$ is as above. By Lemma 2.2, $\operatorname{codim} \mathcal{S}_* > 4$. Hence, by Lemma 2.1 a residual subset $\mathcal{O}_0 = \mathcal{O}_{\mathcal{S}_a} \cup \mathcal{O}_{\mathcal{S}_b} \cup \mathcal{O}_{\mathcal{S}_c} \cup \mathcal{O}_{\mathcal{S}_d}$ in $Emb(M, \mathbb{R}^3)$ has the following property: for any $i \in \mathcal{O}_0$, $(\mu_1, g_1)$ is one of the types (I), ..., (VI), $(\mu_2, g_2)$ is of type (I).

We need the following to see relations between type (I), ..., (VI) and type (I).

**Proposition 2.3** *There exist residual subsets $\mathcal{O}_{(\nu,\mathrm{I})}$ in $Emb(M, \mathbb{R}^3)$ such that for any $i \in \mathcal{O}_{(\nu,\mathrm{I})}$, ${}_2T_i$ is of type $(\nu, \mathrm{I})$, where $\nu = \mathrm{I}, \ldots, \mathrm{VI}$. Here $(\mathrm{I},\mathrm{I})$ consists of three types $(\mathrm{I},\mathrm{I})_0$, $(\mathrm{I},\mathrm{I})_1$ and $(\mathrm{I},\mathrm{I})_2$, and similarly $(\mathrm{III},\mathrm{I})$ consists of $(\mathrm{III},\mathrm{I})^0$ and $(\mathrm{III},\mathrm{I})^1$.*

*Proof.* We shall give the proof only the case $\nu = \mathrm{III}$. Define $\Sigma_{fold} \subset J^l(2,2)$ by $\Sigma_{fold} = \{j^l g(0) \mid g$ is a fold map at $0\}$. It is a well-known fact that $\Sigma_{fold}$

is a codimension 1 subvariety in $J^l(2,2)$. Let $g : M \to \mathbb{R}^2$ and let $p_1, p_2 \in M$ with $g(p_1) = g(p_2)$. Suppose that $p_1$ is a fold point of $g$. Take local coordinates $(x_1, y_1)$ at $p_1$ and $(u, v)$ at $g(p_1)$ such that $g(x_1, y_1) = (x_1, y_1^2)$. Then, to describe the contact of the discriminant set $g_1(S_{g_1})$ and the curve $g_2(\mu_2^{-1}(0))$, we define a function germ $K : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ by $K(u) = \mu_2 \circ g_2^{-1}(u, 0)$.

Define an algebraic set $\mathcal{S}_{(\mathrm{III,I})}$ in ${}_2J^l(\mathbb{R}^2, \mathbb{R}^3)$ by

$$\mathcal{S}_{(\mathrm{III,I})} = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times (J^l(2,1) \times \Sigma_{fold}) \times \left\{ \frac{dK}{du}(0) = \frac{d^2K}{du^2}(0) = 0 \right\}.$$

Since obviously $\operatorname{codim} \mathcal{S}_{(\mathrm{III,I})} = 5$, by Lemma 2.1 ${}_2T_i$ is of type $(\mathrm{III}, \mathrm{I})$ for any $i$ in the residual set $\mathcal{O}_{\mathcal{S}_{(\mathrm{III,I})}}$. This completes the proof for the case $\nu = \mathrm{III}$.

By the same argument we can prove the other case. $\square$

Next we shall prove the genericity of type $(\mathrm{III}, \mathrm{III})$. The following lemma is obtained by a theorem of Whitney and Lemma 2.1.

**Lemma 2.4** *There exists a residual set $\mathcal{O}_1$ in $Emb(M, \mathbb{R}^3)$ such that for any $i \in \mathcal{O}_1$, the singularity of the topographic multigerm ${}_2T_i$ is one of the following types: $g_1$ and $g_2$ are both fold map germs such that the discriminant sets of $g_1$, $g_2$ are transversal at $0$ each other; $g_1$ is a cusp map germs and $g_2$ is non-singular.*

**Proposition 2.5** *There exists a residual set $\mathcal{O}_{(\mathrm{III,III})}$ in $Emb(M, \mathbb{R}^3)$ such that for any $i \in \mathcal{O}_{(\mathrm{III,III})}$, ${}_2T_i$ is of type* $(\mathrm{III}, \mathrm{III})$.

*Proof.* We set $\Sigma_{sing} := \{j^l(\mu, g)(0) \mid \mu|_{S_g}$ is singular at $0$ or $(\mu, g)$ is singular at $0$, where $g$ is a fold map at $0\}$. Note that in the case when $g = (x_1, x_2^2)$, these conditions are written as $\frac{\partial \mu}{\partial x_1}(0) = 0$ or $\frac{\partial \mu}{\partial x_2}(0) = 0$. So we can define an algebraic set $\mathcal{S}$ in ${}_2J^l(\mathbb{R}^2, \mathbb{R}^3)$ by

$$\mathcal{S} = \mathbb{R}^{2(2)} \times \mathbb{R} \times \mathbb{R} \times {}_2\Delta \times \Sigma_{sing} \times (J^l(2,1) \times \Sigma_{fold}).$$

Since the codimensions of $\mathcal{S}$ is obviously five, from Lemma 2.1 and Lemma 2.4 $\mathcal{O}_{(\mathrm{III,III})} := \mathcal{O}_{\mathcal{S}} \cap \mathcal{O}_1$ has the required properties. $\square$

*Proof of Theorem* A *in the case of $r = 2$.* Set $\mathcal{O} = \bigcup_{\nu=\mathrm{I}}^{\mathrm{VI}} (\mathcal{O}_{(\nu,\mathrm{I})} \cap \mathcal{O}_0) \cup \mathcal{O}_{(\mathrm{III,III})}$. Then by Proposition 2.3 and 2.5, $\mathcal{O}$ has the required properties in

Theorem A. □

## 3. Proof of Theorem B

In the case of $r = 1$, the normal forms except type (II) have been obtained in [D4]. For type (II) it is clear by Morse's lemma.

In the case of $r = 2$, we shall detect normal forms as follows.

*Case of type* $(\mathrm{III}, \mathrm{I})^0$, $(\mathrm{III}, \mathrm{I})^1$. By the result of the case $r = 1$ we can suppose that

$$\mu_1 = x_1 + y_1, \quad g_1 = (x_1, y_1^2); \ (\mu_2, g_2) \text{ is of type (I)}.$$

By the coordinate change $g_2^{-1}$ in the source space of $g_2$ and $\mu_2$, we may suppose that $g_2 = \mathrm{id}_{\mathbb{R}^2}$.

For the case $(\mathrm{III}, \mathrm{I})^0$ the generic condition about transversality means that $\frac{\partial \mu_2}{\partial x_2}(0) \neq 0$. In the target space of $\mu_2$, by the coordinate change $\mu_2(x_2, 0)^{-1}$ which is the inverse function germ of $\mu_2(x_2, 0)$, we have $\mu_2(x_2, 0) = x_2$. Thus we obtain the normal form of $(\mathrm{III}, \mathrm{I})^0$.

For the case $(\mathrm{III}, \mathrm{I})^1$ the generic condition means that $\frac{\partial \mu_2}{\partial x_2}(0) = 0$, $\frac{\partial \mu_2}{\partial y_2}(0) \neq 0$ and $\frac{\partial^2 \mu_2}{\partial x_2^2}(0) \neq 0$. Hence $\mu_2(x, 0) = \pm h(x)^2$ for some diffeomorphism germ $h$ on $(\mathbb{R}, 0)$.

We now need the following obvious lemma.

**Lemma 3.1** *Let $g = (x, y^2) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a fold map germ and let $H : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, $K : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be diffeomorphism germs defined by*

$$H = (\alpha \circ g, y\beta \circ g), \quad K = (\alpha, v\beta^2),$$

*where $\alpha, \beta \in \mathcal{E}_{u,v}$ with $\alpha(0) = 0$, $\frac{\partial \alpha}{\partial u}(0) \neq 0$, $\beta(0) \neq 0$. Then $g \circ H = K \circ g$ holds.*

**Remark** The converse of Lemma 3.1 is little bit weak. That is, the converse is true if $K = (\widetilde{\alpha}, v\widetilde{\beta^2})$, where $\widetilde{\alpha} = \alpha$ and $\widetilde{\beta} = \pm\beta$ on $\{v \geq 0\}$.

We can write $h(x_1 + y_1) = \alpha(x_1, y_1^2) + y_1\beta(x_1, y_1^2)$ for some $\alpha, \beta \in \mathcal{E}_{u,v}$. Then we can define diffeomorphism germs $H$, $K$ as in Lemma 3.1. Applying the following coordinate changes we obtain the normal form of $(\mathrm{III}, \mathrm{I})^1$: $h$, $H$, $K$, $K$, $\pm\,\mathrm{id}_{(\mathbb{R},0)}$, respectively in the target of $\mu_1$, in the source of $g_1$, in the target of $g_1$, in the source of $\mu_2$, in the target of $\mu_2$.

*Case of type* (IV, I), (V, I), (VI, I). By the same argument as the case $(\mathrm{III},\mathrm{I})^0$, we obtain the normal forms in Theorem B.

*Case of type* (III, III). We need the following two lemmas. The first lemma is well-known and the second one is obvious, so their proofs are omitted.

**Lemma 3.2** *Let* $(\mathbb{R}^2,0) \xrightarrow{g_1} (\mathbb{R}^2,0) \xleftarrow{g_2} (\mathbb{R}^2,0)$ *be a bi-germ such that* $g_1$, $g_2$ *are both fold map germs and that the discriminant sets of* $g_1$, $g_2$ *are transversal at* 0 *each other. Then we can express the bi-germ for some coordinates as follows*:

$$g_1(x_1,y_1)=(x_1,y_1^2), \quad g_2(x_2,y_2)=(x_2^2,y_2).$$

**Lemma 3.3** *Let* $(\mathbb{R}^2,0) \xrightarrow{g_1} (\mathbb{R}^2,0) \xleftarrow{g_2} (\mathbb{R}^2,0)$ *be a bi-germ such that* $g_1(x_1,y_1)=(x_1,y_1^2)$, $g_2(x_2,y_2)=(x_2^2,y_2)$. *Let* $H_1,H_2,K:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ *be diffeomorphism germs defined by*

$$\begin{aligned} H_1(x_1,y_1)&=(x_1\alpha^2\circ g_1, y_1\beta\circ g_1),\\ H_2(x_2,y_2)&=(x_2\alpha\circ g_2, y_2\beta^2\circ g_2),\\ K(u,v)&=(u\alpha^2, v\beta^2), \end{aligned}$$

*where* $\alpha,\beta\in\mathcal{E}_{u,v}$ *with* $\alpha(0)\neq 0$, $\beta(0)\neq 0$. *Then* $g_1\circ H_1=K\circ g_1$ *and* $g_2\circ H_2=K\circ g_2$ *hold.*

**Remark** The converse of Lemma 3.3 is true if we replace $\alpha$, $\beta$ in $H_1$, $H_2$, $K$ with $\alpha_1$, $\beta_1$ in $H_1$, $\alpha_2$, $\beta_2$ in $H_2$, $\widetilde{\alpha}$, $\widetilde{\beta}$ in $K$, where $\alpha_1=\alpha_2$ (or resp. $-\alpha_2$) on $\{u\geq 0\}\cap\{v\geq 0\}$, $\widetilde{\alpha}=\pm\alpha_1$ (resp. $\pm\alpha_1$) on $\{v\geq 0\}$, $\widetilde{\alpha}=\pm\alpha_2$ (resp. $\mp\alpha_2$) on $\{u\geq 0\}$.

We now detect a normal form of type (III, III). Let $(\mu_1,g_1;\mu_2,g_2)$ be of type (III, III). By Lemma 3.2, we can suppose that $g_1(x_1,y_1)=(x_1,y_1^2)$, $g_2(x_2,y_2)=(x_2^2,y_2)$. Applying the Malgrange preparation theorem, $\mu_1$, $\mu_2$ have the following forms:

$$\mu_1=A_1\circ g_1+y_1B_1\circ g_1, \quad \mu_2=A_2\circ g_2+x_2B_2\circ g_2,$$

where $A_k,B_k\in\mathcal{E}_{u,v}$ $(k=1,2)$. Note that $B_k(0)\neq 0$ $(k=1,2)$ because the map germ $(\mu_k,g_k)$ is non-singular. Thus we can define coordinate transformations $H_1$, $H_2$, $K$ by

$$H_1(x_1,y_1)=(x_1B_2^2\circ g_1, y_1B_1\circ g_1),$$

$$H_2(x_2,y_2) = (x_2 B_2 \circ g_2, y_2 B_1^2 \circ g_2),$$
$$K(u,v) = (uB_2^2, vB_1^2).$$

Taking new coordinates by $H_1$, $H_2$, $K$, from Lemma 3.3 $(\mu_1, g_1; \mu_2, g_2)$ is equivalent to $(\mu_1{}', g_1; \mu_2{}', g_2)$ which is given by

$$\begin{aligned} \mu_1{}'(x_1{}', y_1{}') &= \alpha_1 \circ g_1(x_1{}', y_1{}') + y_1{}', \\ \mu_2{}'(x_2{}', y_2{}') &= \alpha_2 \circ g_2(x_2{}', y_2{}') + x_2{}', \end{aligned} \tag{$**$}$$

where $\alpha_k = A_k \circ K^{-1} \in \mathcal{M}_{u,v}$ $(k = 1, 2)$.

Since $\mu_1{}'|_{S_{g_1}}$ is non-singular, we have $\frac{\partial \alpha_1}{\partial u}(0) \neq 0$. Similarly $\frac{\partial \alpha_2}{\partial v}(0) \neq 0$. Conversely, for any $\alpha_1, \alpha_2 \in \mathcal{M}_{u,v}$ with $\frac{\partial \alpha_1}{\partial u}(0) \neq 0$, $\frac{\partial \alpha_2}{\partial v}(0) \neq 0$, a germ whose form is given by $(**)$ is of type $(\mathrm{III}, \mathrm{III})$. Therefore we obtain the normal form of type $(\mathrm{III}, \mathrm{III})$.

*Case of* $(\mathrm{I}, \mathrm{I})_0$, $(\mathrm{I}, \mathrm{I})_1$, $(\mathrm{I}, \mathrm{I})_2$, $(\mathrm{II}, \mathrm{I})$. The normal forms of these case have been detected essentially in [D1, 2, 3], however we shall give a process of detecting the normal forms which is more easy method by using the versal deformation theory (cf. [AGV]). Let $(\mu_1, g_1; \mu_2, g_2)$ be a topographic multigerm. Since $g_k$ $(k = 1, 2)$ is a diffeomorphism, we can suppose that $g_k = \mathrm{id}_{\mathbb{R}^2}$ $(k = 1, 2)$. We only consider the coordinate transformations preserving both $g_k = \mathrm{id}$. Hence it is sufficient to detect normal forms of the following divergent diagrams up to coordinate transformations:

$$(\mathbb{R}, 0) \xleftarrow{\quad F \quad} (\mathbb{R}^2, 0) \xrightarrow{\quad G \quad} (\mathbb{R}, 0).$$

For type $(\mathrm{I}, \mathrm{I})_0$ it is clear. For types $(\mathrm{I}, \mathrm{I})_1$ (resp. $(\mathrm{I}, \mathrm{I})_2$, $(\mathrm{II}, \mathrm{I})$) by using $R$-versal (resp. $R^+$-versal) deformation theory (cf. [AGV]) we shall obtain the normal forms. Here we describe only $(\mathrm{I}, \mathrm{I})_2$ using Theorem of bi-stability of cusps ([D1]).

In the case $(\mathrm{I}, \mathrm{I})_2$: We may suppose that $G(x, y) = y$. The generic condition on this type means that

$$\frac{\partial F}{\partial x}(0) = \frac{\partial^2 F}{\partial x^2}(0) = 0, \quad \frac{\partial^3 F}{\partial x^3}(0) \neq 0, \quad \frac{\partial^2 F}{\partial x \partial y}(0) \neq 0; \quad \frac{\partial F}{\partial y}(0) \neq 0.$$

Hence $F(x, 0)$ has $A_2$-singularity. So that let $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be the diffeomorphism germ such that $F(\psi(x), 0) = \pm x^3$. Then trivially the diagram $(F, y)$ is equivalent to $(F', y')$, where $F'(x', y') = \pm F(\psi(x'), y')$. Due to the condition $\frac{\partial^2 F}{\partial x \partial y}(0) \neq 0$, we can easily verify that $F'$ is an $R^+$-infinitesimally versal (hence $R^+$-versal) deformation of $x'^3$. On the other hand, $x^3+xy$ is also an $R^+$-versal deformation of $x^3$. Therefore by the uniqueness of versal deformation, the diagram $(F', y')$ is equivalent to

$$(x^3+xy+\alpha(y), y)$$

for some $\alpha \in \mathcal{M}_y$. For our case $\alpha$ must be non-singular and we may suppose that $\frac{d\alpha}{dy}(0)=1$. Here we need the following crucial result by Dufour [D1].

**Lemma 3.4** *For any $\alpha \in \mathcal{M}_y$, the diagram $(x^3+xy+y+y^2\alpha(y), y)$ is equivalent to $(x^3+xy+y, y)$.*

That is, we obtain the normal form of $(\mathrm{I}, \mathrm{I})_2$.

This completes the proof of Theorem B in the case of $r=2$. □

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