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Semilinear heat equations with distributions in Morrey spaces as initial data

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Abstract. This paper is a continuous study to the paper [27]. Here we consider in Morrey spaces the Cauchy problem of the general semilinear heat equation with an external force. Both the external force and initial data belong to suitable Morrey spaces. When the norm of the external force is small, we proved the unique existence of small solution to the corresponding stationary problem. Moreover, if the initial data is close enough to the stationary solution, we verified the time-global solvability of the Cauchy problem, which leads to the stability of the small stationary solution.

Key words: semilinear heat equations, Morrey spaces, semigroup.

1. Introduction

Let us consider the Cauchy problem of the following semilinear heat equation with an external force $f(x)$ in $\mathbf{R}^n$ for $n \geq 3$:

$$\frac{\partial v}{\partial t}(t,x) = \Delta v(t,x) + v(t,x)|v(t,x)|^{\nu-1} + f(x) \quad \text{in} \quad (0,\infty)\times\mathbf{R}^n, \tag{1.1}$$

$$v(0,x) = a(x) \quad \text{on} \quad \mathbf{R}^n, \tag{1.2}$$

where $\nu > \frac{n}{n-2}$, $\nu \in \mathbf{R}$.

The corresponding stationary problem of the above equation is as follows:

$$-\Delta w(x) = w(x)|w(x)|^{\nu-1} + f(x) \quad \text{on} \quad \mathbf{R}^n. \tag{1.3}$$

There have been many researches on the Cauchy problem (1.1)–(1.2) without external forces, *i.e.* $f(x) \equiv 0$. Fujita [6] first showed that the Cauchy problem admits a time-global strong solution with $\nu > 1 + 2/n$, provided that $\|a(x)\|_{C^2(\mathbf{R}^n)}$ is sufficiently small. At the same time he also

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showed that the condition $\nu > 1 + 2/n$ is necessary for the existence of a time-global solution for nonnegative nontrivial initial data (also see Haraux and Weissler [10], Hayakawa [11], Kobayashi, Sirao and Tanaka [14]). Furthermore, Weissler [25] proved the global existence of the solution when $\|a(x)\|_{L^p(\mathbf{R}^n)}$ with $p = n(\nu - 1)/2 > 1$ is sufficiently small.

On the other hand, many authors have also studied on the Cauchy problem with measures as initial data. Brezis and Friedman [5] proved that a time-local solution exists with initial data $\delta(x)$ if and only if $\nu < 1 + 2/n$. Baras and Pierre [2] studied various capacities of the Radon measures with which as the initial data the Cauchy problem is solvable. Niwa [19] obtained a sufficient condition for the local well-posedness and the global well-posedness of the Cauchy problem with initial data in the measure spaces of the Morrey type. Kozono and Yamazaki [16] obtained time-local and time-global solutions as initial data in the Besov-type Morrey spaces.

In this paper, we are interested in studying the Cauchy problem (1.1)–(1.2) in the Morrey spaces. Zhou [27] has obtained the stability of small stationary solutions in Morrey spaces when $\nu \geq 3$ is an integer. This paper will generalize these results to the general case $\nu > \frac{n}{n-2}$, $\nu \in \mathbf{R}$ with more stationary solutions.

First, in the same way as in Zhou [27], we obtain the unique existence of small stationary solutions for the stationary problem (1.3) by using the Banach inverse mapping theorem.

Theorem 1.1 *Suppose that $n \geq 3$, $\nu > \frac{n}{n-2}$, $\nu \in \mathbf{R}$ and $\nu < q_0 \leq p_0 = \frac{n(\nu-1)}{2}$. Then we can find a positive number δ_0 and a continuous, strictly monotone-increasing function $\omega(\delta)$ on $[0, \delta_0]$ with $\omega(0) = 0$ such that:*

(1) *For every $f(x) \in \mathcal{D}'$, there exists at most one solution $w(x)$ of (1.3) in $\{w(x) \in \mathcal{M}_{p_0,q_0} \mid \|w(x)\|_{\mathcal{M}_{p_0,q_0}} < \omega(\delta_0)\}$.*

(2) *For every $f(x) \in \mathcal{M}^{-2}_{p_0,q_0}$ with $\|f(x)\|_{\mathcal{M}^{-2}_{p_0,q_0}} = \delta < \delta_0$, there exists a solution $w(x) \in \mathcal{M}_{p_0,q_0}$ of (1.3) with $\|w(x)\|_{\mathcal{M}_{p_0,q_0}} \leq \omega(\delta)$.*

Example 1.1 For $f(x) = A|x|^{-\alpha-2}$, we see easily that $w(x) = c|x|^{-\alpha}$ solves (1.3) if and only if $\alpha = 2/(\nu - 1)$ and

$$A = -c\alpha(n - 2 - \alpha) + |c|^{\nu-1}c. \tag{1.4}$$

If $\nu > n/(n - 2)$, we see that $f(x) \in L^{n/(\alpha+2),\infty}$. It follows that $f(x) \in \mathcal{M}_{n/(\alpha+2),q_1}$ for every $q_1 < n/(\alpha + 2)$. This implies $f(x) \in \mathcal{M}^{-2}_{p_0,q_0}$ for every

$q_0 < p_0$. On the other hand, $w(x) \in \mathcal{M}_{p_0,q_0}$ holds for every $q_0 < p_0$.

If A is sufficiently small, the equation (1.4) has three solutions. It follows that the condition $\|w(x)\|_{\mathcal{M}_{p_0,q_0}} < \omega(\delta_0)$ is necessary for the uniqueness of the solution in Theorem 1.1.

Next, we will consider the Cauchy problem (1.1)–(1.2) in the general case $\nu \in \mathbf{R}$, and divide it into two cases as follows.

Case 1: $\frac{n}{n-2} < \nu \le 2$, $\nu \in \mathbf{R}$.

Case 2: $\nu > 2$, $\nu \in \mathbf{R}$.

For the first case, we have the following main theorems.

Theorem 1.2 *Suppose that $n \ge 3$, $\frac{n}{n-2} < \nu \le 2$, $\nu \in \mathbf{R}$ and p_0, q_0 are the same as in Theorem 1.1. Let p, q and σ_0 be real numbers such that $p_0 < p < \frac{n-2}{n}\nu p_0$, $q_0 < q \le \frac{pq_0}{p_0}$ and $\frac{2}{p} < \sigma_0 < 2 - \frac{(n-2)\nu - n}{p}$. Then we can find positive numbers δ_1 $(\le \delta_0)$, ε_0 and M_0 satisfying the following:*

For every $f(x) \in \mathcal{M}^{-2}_{p_0,q_0}$ with $\|f(x)\|_{\mathcal{M}^{-2}_{p_0,q_0}} < \delta_1$, take the solution $w(x)$ of (1.3) in Theorem 1.1, and take $a(x) \in \mathcal{M}^{\frac{n}{p}-\frac{n}{p_0}}_{p,q}$ with $\|a(x) - w(x)\|_{\mathcal{M}^{\frac{n}{p}-\frac{n}{p_0}}_{p,q}} = \varepsilon < \varepsilon_0$, there exists a time-global solution $v(t,x)$ of (1.1)–(1.2) such that:

$$\sup_{0<t\le T'} t^{\frac{n}{2p_0}-\frac{n-2}{2p}} \|v(t,\cdot) - w\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}} < \infty, \quad \textit{for every } 0 < T' \le \infty, \tag{1.5}$$

$$\limsup_{t\to 0+} t^{\frac{n}{2p_0}-\frac{n-2}{2p}} \|v(t,\cdot) - w\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}} < M_0. \tag{1.6}$$

Moreover, the initial condition (1.2) holds in the following sense: For every s satisfying $\frac{n}{p} - \frac{(n-2)\nu}{p} \le s \le \frac{n}{p} - \frac{n}{p_0}$ and every $T' > 0$, we have

$$\sup_{0<t\le T'} t^{\frac{s}{2}+\frac{n}{2p_0}-\frac{n}{2p}} \|v(t,\cdot) - a\|_{\mathcal{M}^{s}_{p,q}} < \infty. \tag{1.7}$$

Remark 1.1 From Lemma 2.1 in the next section, the stationary solution $w(x) \in \mathcal{M}_{p_0,q_0} \subset \mathcal{M}^{\frac{n}{p}-\frac{n}{p_0}}_{p,q}$.

Theorem 1.3 *Under the same assumptions and notations as in Theorem 1.2, for every $0 < T \le \infty$, any solution of (1.1)–(1.2) on $(0,T) \times \mathbf{R}^n$ satisfying (1.5) for every $T' \in (0,T)$, (1.6) and $v(t,\cdot) - a \longrightarrow 0$ in*

$\mathcal{M}_{p,q}^{\frac{n}{p}-\frac{(n-2)\nu}{p}}$ *coincides with the restriction on* $(0,T)\times\mathbf{R}^n$ *of the time-global solution in Theorem* 1.2.

Theorem 1.4 *Under the same assumptions and notations as in Theorem* 1.2, *for every* σ *satisfying* $\frac{n}{p}-\frac{n}{p_0}\le\sigma\le\sigma_0$, *there exists a continuous, strictly monotone-increasing function* $\psi_\sigma(\varepsilon)$ *on* $[0,\varepsilon_0]$ *with* $\psi_\sigma(0)=0$ *such that*:

$$\sup_{t>0} t^{\frac{\sigma}{2}+\frac{n}{2p_0}-\frac{n}{2p}}\|v(t,\cdot)-w\|_{\mathcal{M}_{p,q}^{\sigma}}\le\psi_\sigma(\varepsilon),\quad \textit{for every}\ \ \varepsilon<\varepsilon_0. \tag{1.8}$$

Remark 1.2 The estimate (1.8) with $\sigma=\frac{n}{p}-\frac{n}{p_0}$ in Theorem 1.4 together with the fact that $\lim_{\varepsilon\to0+}\psi_\sigma(\varepsilon)=0$ asserts the Lyapunov stability of the stationary solution in the topology of $\mathcal{M}_{p,q}^{\frac{n}{p}-\frac{n}{p_0}}$. Other estimates in (1.8) give the asymptotic stability in different topologies of $\mathcal{M}_{p,q}^{\sigma}$.

Example 1.2 Let $w(x)$ be a stationary solution as shown in Theorem 1.1 (2) where $f(x)=c_0|x|^{-\frac{2\nu}{\nu-1}}$ with a small constant c_0. Suppose that p_0, q_0, p, q ($q<p$) satisfy the assumptions in Theorem 1.2. Then we can take $a(x)=w(x)+c_1(-\Delta_x)^{\frac{n}{2p_0}-\frac{n}{2p}}(|x|^{-\frac{n}{p}})$ in Theorem 1.2 provided that the coefficient c_1 is sufficiently small.

For the second case, we have the following main theorems.

Theorem 1.5 *Suppose that* $n\ge3$, $\nu>2$, $\nu\in\mathbf{R}$, $\nu<q_1\le q_0\le p_0=\frac{n(\nu-1)}{2}$. *Let* p, σ_0 *be real numbers such that* $p_0<p<2p_0$, $\frac{n}{p_0}-\frac{n}{p}<\sigma_0<\frac{2n}{p_0}-\frac{2n}{p}$. *Then we can find positive numbers* $\delta_1\,(\le\delta_0)$, ε_0 *and* M_0 *satisfying the following*:

For every $f(x)\in\mathcal{M}_{p_0,q_0}^{-2}$ *with* $\|f(x)\|_{\mathcal{M}_{p_0,q_0}^{-2}}<\delta_1$, *take the solution* $w(x)$ *of* (1.3) *in Theorem* 1.1, *and take* $a(x)\in\mathcal{M}_{p_0,q_1}$ *with* $\|a(x)-w(x)\|_{\mathcal{M}_{p_0,q_1}}=\varepsilon<\varepsilon_0$, *there exists a time-global solution* $v(t,x)$ *of* (1.1)–(1.2) *such that*:

$$\max\left\{\sup_{0<t\le T'}\|v(t,\cdot)-w\|_{\mathcal{M}_{p_0,q_1}},\ \sup_{0<t\le T'} t^{\frac{n}{2p_0}-\frac{n}{2p}}\|v(t,\cdot)-w\|_{\mathcal{M}_{p_0,q_1}^{\frac{n}{p_0}-\frac{n}{p}}}\right\}$$
$$<\infty, \tag{1.9}$$

for every $0<T'\le\infty$,

$$\max\left\{\limsup_{t\to0+}\|v(t,\cdot)-w\|_{\mathcal{M}_{p_0,q_1}},\ \limsup_{t\to0+} t^{\frac{n}{2p_0}-\frac{n}{2p}}\|v(t,\cdot)-w\|_{\mathcal{M}_{p_0,q_1}^{\frac{n}{p_0}-\frac{n}{p}}}\right\}$$
$$<M_0. \tag{1.10}$$

Moreover, the initial condition (1.2) *holds in the following sense: For every* s *satisfying* $\frac{2n}{p_0} - \frac{2n}{p} - 2 \leq s \leq 0$ *and every* $T' > 0$, *we have*

$$\sup_{0<t\leq T'} t^{\frac{s}{2}} \|v(t,\cdot) - a\|_{\mathcal{M}^s_{p_0,q_1}} < \infty. \tag{1.11}$$

Remark 1.3 As shown in Zhou [27], any stationary solution $w(x) \in \mathcal{M}_{p_0,q_0} \subset \mathcal{M}_{p_0,q_1}$.

Theorem 1.6 *Under the same assumptions and notations as in Theorem* 1.5, *for every* $0 < T \leq \infty$, *any solution of* (1.1)–(1.2) *on* $(0,T) \times \mathbf{R}^n$ *satisfying* (1.9) *for every* $T' \in (0,T)$, (1.10) *and* $v(t,\cdot) - a \longrightarrow 0$ *in* $\mathcal{M}^{\frac{2n}{p_0}-\frac{2n}{p}-2}_{p_0,q_1}$ *coincides with the restriction on* $(0,T) \times \mathbf{R}^n$ *of the time-global solution in Theorem* 1.5.

Theorem 1.7 *Under the same assumptions and notations as in Theorem* 1.5, *for every* σ *satisfying* $0 \leq \sigma \leq \sigma_0$, *there exists a continuous, strictly monotone-increasing function* $\psi_\sigma(\varepsilon)$ *on* $[0, \varepsilon_0]$ *with* $\psi_\sigma(0) = 0$ *such that*:

$$\sup_{t>0} t^{\frac{\sigma}{2}} \|v(t,\cdot) - w\|_{\mathcal{M}^\sigma_{p_0,q_1}} \leq \psi_\sigma(\varepsilon), \quad \textit{for every } \varepsilon < \varepsilon_0. \tag{1.12}$$

Remark 1.4 The estimate (1.12) with $\sigma = 0$ in Theorem 1.7 together with the fact that $\lim_{\varepsilon\to 0+}\psi_\sigma(\varepsilon) = 0$ asserts the Lyapunov stability of the stationary solution in the topology of $\mathcal{M}_{p_0,q_1}$. Other estimates in (1.12) give the asymptotic stability in different topologies of $\mathcal{M}^\sigma_{p_0,q_1}$.

Example 1.3 Let $w(x)$ be a stationary solution as shown in Example 1.1 where $f(x) = c_0|x|^{-\frac{2\nu}{\nu-1}}$ with a small constant c_0. Suppose that $n > 3$, $\nu > 2$, $\nu \in \mathbf{R}$, $\nu < q_1 \leq q_0 \leq (n-1)p_0/n = (n-1)(\nu-1)/2$. Put $x' = (x_1, \ldots, x_{n-1})$ for $x = (x_1, \ldots, x_n)$. Then we can take the function $a(x) = w(x) + C_2(-\Delta_x)^{\frac{1}{\nu-1}-\frac{n-1}{2}}\delta(x')$ in Theorem 1.5 provided that the coefficient c_2 is sufficiently small.

Remark 1.5 The critical exponent for (1.3) is $\nu = n/(n-2)$, whereas the critical exponent for (1.1) with $f(x) \equiv 0$ is $\nu = (n+2)/n$. The difference corresponds to the following fact: Brézis and Véron [1] showed that the elliptic equation

$$-\Delta u + |u|^{\nu-1}u = \delta(x)$$

on a bounded domain with smooth boundary containing the origin admits a solution if and only if $\nu < n/(n-2)$, whereas Brézis and Friedman [5] showed that

$$\frac{\partial v}{\partial t} - \Delta v + |v|^{\nu-1}v = 0$$

with the initial condition $v(0) = \delta(x)$ admits a solution if and only if $\nu < (n+2)/n$.

Remark 1.6 For ν such that $1 + 2/n < \nu < (n+2)/(n-2)$, Haraux and Weissler [10] constructed a nontrivial solution $v(t,x)$ of (1.1)–(1.2) with $f(x) \equiv a(x) \equiv 0$ of the form $v(t,x) = t^{-1/(\nu-1)}\varphi(x/\sqrt{t})$. Then we have $\|v(t,\cdot)\|_{\mathcal{M}^s_{p,q}} = Ct^{-1/(\nu-1)+n/2p-s/2}$ for every p, q and s. It follows that, even in the case $a(x) \equiv f(x) \equiv 0$, the condition (1.6) is necessary for the uniqueness in Theorem 1.3, and the condition (1.10) is necessary for the uniqueness in Theorem 1.6.

Semigroup theory will be used to prove our results in the same way as in Zhou [27]. In order to prove our main theorems in different cases, we need different estimates and propositions which can be obtained in the similar way. Therefore we obmit some unnecessary proofs. For details, see Zhou [27].

The plan of this paper is the following. In Section 2, we recall the definitions and some known results of the Sobolev-type Morrey spaces. Then we consider the stationary problem (1.3) and prove Theorem 1.1. Section 3 deals with the perturbation of the heat operater. Then we will discuss the the semigroup in Section 4. In Section 5, the equivalence between the original differential equations and the associated integral equations will be given. In Section 6, we will construct time-global solutions to the integral equation by the method of succesive approximation. Then we complete the proofs of Theorem 1.2, Theorem 1.4, Theorem 1.5 and Theorem 1.7. The uniqueness properties in Theorem 1.3 and Theorem 1.6 will be proved in the last part of this paper.

2. Morrey spaces and the stationary problem

First of all we recall the definitions of the Morrey spaces.

Definition 2.1 Let $1 \le q \le p < \infty$, $s \in \mathbf{R}$. The Morrey space $\mathcal{M}_{p,q}$ on $\mathbf{R}^n$ is defined to be the set of functions $u(x) \in L^q_{loc}(\mathbf{R}^n)$ such that

$$\|u\|_{\mathcal{M}_{p,q}} = \sup_{x_0 \in \mathbf{R}^n} \sup_{R>0} R^{n/p-n/q} \left(\int_{|x-x_0|<R} |u(x)|^q dx \right)^{1/q} < \infty.$$

Furthermore, the Sobolev-type Morrey space $\mathcal{M}^s_{p,q}$ is defined by

$$\mathcal{M}^s_{p,q} = \left\{ u(x) \in \Phi'/\mathcal{P} \mid \|u(x)\|_{\mathcal{M}^s_{p,q}} = \|(-\Delta)^{\frac{s}{2}} u\|_{\mathcal{M}_{p,q}} < \infty \right\},$$

where Φ' and $\mathcal{P}$ denote the set of tempered distributions on $\mathbf{R}^n$ and the set of polynomials with n variables, respectively.

In this paper, we only consider the spaces $\mathcal{M}^s_{p,q}$ with $s < \frac{n}{p}$. As Bourdaud [4], Kozono and Yamazaki [17] showed, they can be regarded as a subspace of Φ'.

Many properties of Morrey spaces have been shown in Zhou [27], we will not describe them here again. For more detailed properties, see Peetre [20], Taylor [22, 23], Kozono and Yamazaki [16, 17]. It is worthy to point out the following important lemma.

Lemma 2.1 *Let $1 < q \le p < \infty$, $s \in \mathbf{R}$ and $0 < \theta < 1$, then $\mathcal{M}^s_{p,q} \subset \mathcal{M}^{s-(1-\theta)n/p}_{p/\theta,q/\theta}$.*

Now we prove the Theorem 1.1 by using the Banach inverse mapping theorem. For every $w(x) \in \mathcal{M}_{p_0,q_0}$, define the mapping:

$$F(w) = -\Delta w(x) - w(x)|w(x)|^{\nu-1}.$$

Then we see that $F : \mathcal{M}_{p_0,q_0} \longrightarrow \mathcal{M}^{-2}_{p_0,q_0}$ is continuous in the same way as in [25]. Moreover, it is Fréchet differentiable at any $w_0 \in \mathcal{M}_{p_0,q_0}$ with the Fréchet derivative $-\Delta - \nu|w_0(x)|^{\nu-1}$.

The Fréchet derivative at 0 coincides with $(-\Delta)$ which is an isomorphism from $\mathcal{M}_{p_0,q_0}$ to $\mathcal{M}^{-2}_{p_0,q_0}$. By the Banach inverse mapping theorem, there exist sufficiently small positive constants γ_0 and δ_0 such that the mapping F is injective on the set

$$U = \{w(x) \in \mathcal{M}_{p_0,q_0} \mid \|w(x)\|_{\mathcal{M}_{p_0,q_0}} < \gamma_0\},$$

and the image $F(U)$ contains the set

$$V = \{f(x) \in \mathcal{M}^{-2}_{p_0,q_0} \mid \|f(x)\|_{\mathcal{M}^{-2}_{p_0,q_0}} < \delta_0\}.$$

Moreover, the inverse mapping $F^{-1} : V \longrightarrow U$ is continuous. This completes the proof of Theorem 1.1.

3. Estimates of the perturbed heat operator

In the following sections, we will consider the Cauchy problem (1.1)–(1.2). Hereafter we always assume that $n \geq 3$, $\nu > \frac{n}{n-2}$, $\nu \in \mathbf{R}$ and $\nu < q_0 \leq p_0 = \frac{n(\nu-1)}{2}$. Let $w(x) \in \mathcal{M}_{p_0,q_0}$ be the small stationary solution as in Theorem 1.1 (2) such that $\|w(x)\|_{\mathcal{M}_{p_0,q_0}} \leq \omega(\delta) < 1$.

Put $u(t,x) = v(t,x) - w(x)$, $b(x) = a(x) - w(x)$, then the system (1.1)–(1.2) is transformed into the following system:

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + (w(x) + u(t,x))|w(x) + u(t,x)|^{\nu-1} \\ - w(x)|w(x)|^{\nu-1} \quad \text{in} \quad (0,\infty) \times \mathbf{R}^n,\end{aligned} \tag{3.1}$$

$$u(0,x) = b(x) \quad \text{on} \quad \mathbf{R}^n. \tag{3.2}$$

Denote

$$\begin{aligned}&\mathcal{A}[f](x) = -\Delta_x f(x) + \mathcal{B}[f](x),\\ &\mathcal{B}[f](x) = -\nu|w(x)|^{\nu-1} f(x),\\ &G(w,u) = -(w+u)|w+u|^{\nu-1} + w|w|^{\nu-1} + \nu|w|^{\nu-1}u,\end{aligned}$$

then (3.1) becomes

$$\frac{\partial u}{\partial t}(t,x) + \mathcal{A}[u(t,\cdot)](x) + G(w(x), u(t,x)) = 0. \tag{3.3}$$

Further, for the case $\frac{n}{n-2} < \nu \leq 2$, $\nu \in \mathbf{R}$, the conditions (1.5)–(1.8) can be rewritten as

$$\sup_{0<t\leq T'} t^{\frac{n}{2p_0} - \frac{n-2}{2p}} \|u(t,\cdot)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}} < \infty, \quad \text{for every} \quad 0 < T' \leq \infty, \tag{3.4}$$

$$\limsup_{t\to 0+} t^{\frac{n}{2p_0} - \frac{n-2}{2p}} \|u(t,\cdot)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}} < M_0, \tag{3.5}$$

$$\sup_{0<t\leq T'} t^{\frac{s}{2} + \frac{n}{2p_0} - \frac{n}{2p}} \|u(t,\cdot) - b\|_{\mathcal{M}^s_{p,q}} < \infty. \tag{3.6}$$

$$\sup_{t>0} t^{\frac{\sigma}{2} + \frac{n}{2p_0} - \frac{n}{2p}} \|u(t,\cdot)\|_{\mathcal{M}^\sigma_{p,q}} \leq \psi_\sigma(\varepsilon) \tag{3.7}$$

respectively. In the meantime, for the second case $\nu > 2$, $\nu \in \mathbf{R}$, the conditions (1.9)–(1.12) can be rewritten as

$$\max\left\{\sup_{0<t\le T'}\|u(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}},\ \sup_{0<t\le T'} t^{\frac{n}{2p_0}-\frac{n}{2p}}\|u(t,\cdot)\|_{\mathcal{M}^{\frac{n}{p_0}-\frac{n}{p}}_{p_0,q_1}}\right\}<\infty, \tag{3.8}$$

for every $0<T'\le\infty$,

$$\max\left\{\limsup_{t\to 0+}\|u(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}},\ \limsup_{t\to 0+} t^{\frac{n}{2p_0}-\frac{n}{2p}}\|u(t,\cdot)\|_{\mathcal{M}^{\frac{n}{p_0}-\frac{n}{p}}_{p_0,q_1}}\right\}<M_0, \tag{3.9}$$

$$\sup_{0<t\le T'} t^{\frac{s}{2}}\|u(t,\cdot)-b\|_{\mathcal{M}^s_{p_0,q_1}}<\infty, \tag{3.10}$$

$$\sup_{t>0} t^{\frac{\sigma}{2}}\|u(t,\cdot)\|_{\mathcal{M}^\sigma_{p_0,q_1}}\le\psi_\sigma(\varepsilon),\quad \text{for every } \varepsilon<\varepsilon_0 \tag{3.11}$$

respectively. Therefore, in order to prove our main results, it is enough to find a solution of (3.2)–(3.3) satisfying the conditions (3.4)–(3.7) for every $0<T'\le\infty$ and to show its uniqueness. For the second case, it is also enough to find a solution of (3.2)–(3.3) satisfying the conditions (3.8)–(3.11) for every $0<T'\le\infty$ and to show its uniqueness. We will use different inequalities on the term $G(w(x),u(t,x))$ for different case, as the following lemma.

Lemma 3.1 *Let $G(w(x),u(t,x))$ be defined as above, we have*

(1) *If $\frac{n}{n-2}<\nu\le 2$, $\nu\in\mathbf{R}$, then $|G(w,u)|\le|u|^\nu$.*

(2) *If $\nu>2$, $\nu\in\mathbf{R}$, then $|G(w,u)|\le\frac{\nu(\nu-1)}{2}(|w|+|u|)^{\nu-2}u^2$.*

Proof. In fact, denote the function $F(\lambda)=\lambda|\lambda|^{\nu-1}$, then

$$F'(\lambda)=\nu|\lambda|^{\nu-1},\quad |F''(\lambda)|=\nu(\nu-1)|\lambda|^{\nu-2}.$$

The first inequality is deduced from

$$F(w+u)-F(w)=\int_0^1 F'(w+\theta u)u\,d\theta$$

and

$$|G(w,u)|\le\int_0^1|F'(w+\theta u)-F'(w)||u|\,d\theta$$

$$\leq \nu \int_0^1 ||w+\theta u|^{\nu-1} - |w|^{\nu-1}||u|d\theta$$
$$\leq \nu \int_0^1 |\theta u|^{\nu-1}|u|d\theta = |u|^\nu,$$

where we used the trivial inequality $(a+b)^\alpha \leq a^\alpha + b^\alpha$, for every $0<\alpha<1$ and $0 \leq a \leq b$.

The second inequality is easy to be calculated as follows.

$$\begin{aligned}|G(w,u)| &\leq \int_0^1 |F'(w+\theta u) - F'(w)||u|d\theta \\ &\leq \int_0^1 \left(\int_0^\theta |F''(w+\tau u)||u|d\tau\right)|u|d\theta \\ &\leq \int_0^1 \nu(\nu-1)(1-\tau)(|w|+\tau|u|)^{\nu-2}u^2 d\tau \\ &= \frac{\nu(\nu-1)}{2}(|w|+|u|)^{\nu-2}u^2.\end{aligned}$$

□

Lemma 3.2 *Suppose that* $\frac{n}{n-2} < \nu \leq 2$, $\nu \in \mathbf{R}$, $p_0 < p < \infty$, $q_0 < q \leq \frac{pq_0}{p_0}$, *then for every* $s \leq 2$ *such that* $0 < s < \frac{n}{p}$, *the operators* $\mathcal{A}$ *and* $\mathcal{B}$ *are bounded from* $\mathcal{M}^s_{p,q}$ *to* $\mathcal{M}^{s-2}_{p,q}$.

Proof. In view of Propostion 2.1 (Zhou [27]), it is enough to prove the conclusion for the operator $\mathcal{B}$. Put

$$\frac{n}{p_1} = \frac{n}{p} - s, \quad q_2 = \frac{p_1 q}{p},$$
$$\frac{1}{p_2} = \frac{1}{p_1} + \frac{\nu-1}{p_0}, \quad \frac{1}{q_3} = \frac{1}{q_2} + \frac{\nu-1}{q_0}.$$

Let $f(x)$ be an element of $\mathcal{M}^s_{p,q}$. Then Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4 (Zhou [27]) imply the following inclusion relations:

$$\mathcal{B}[f](x) = -\nu|w(x)|^{\nu-1}f(x) \in \mathcal{M}_{p_2,q_3} \subset \mathcal{M}_{p_2,\frac{p_2 q}{p}} \subset \mathcal{M}^\sigma_{p,q},$$

where

$$\sigma = -\left(1-\frac{p_2}{p}\right)\frac{n}{p_2} = \frac{n}{p} - \left(\frac{n}{p_1} + \frac{n(\nu-1)}{p_0}\right) = s-2.$$

□

For the second case, the operators $\mathcal{A}$ and $\mathcal{B}$ have the following property.

Lemma 3.3 *Suppose that $\nu > 2$, $\nu \in \mathbf{R}$, $\nu < q_1 \le q_0$, $0 < s < \frac{n}{p_0}$. Then the operators $\mathcal{A}$ and $\mathcal{B}$ are bounded from $\mathcal{M}^s_{p_0,q_1}$ to $\mathcal{M}^{s-2}_{p_0,q_1}$.*

In order to discuss the semigroup generated by the perturbed heat operator $\mathcal{A}$, we need some estimates of the resolvent of $\mathcal{A}$. For the proof we make use of the following lemmas. Firstly we have the following lemma (see Zhou [27] Lemma 3.2).

Lemma 3.4 *Let p and q be real numbers such that $1 < q \le p$. Then, for every positive number ε with $\varepsilon < \pi/2$ and nonnegative numbers a and b with $a \le b$, there exists a positive constant $C = C_{n,\varepsilon,a,b}$ such that the estimate*

$$\|(-\Delta_x)^a(\lambda + \Delta_x)^{-b}\|_{\mathcal{L}(\mathcal{M}_{p,q},\mathcal{M}_{p,q})} \le C|\lambda|^{a-b} \tag{3.12}$$

holds for every $\lambda \in \mathbf{C} \setminus [0, +\infty)$ with $|\arg \lambda| \ge \varepsilon$.

Next, it is easy to get the following two lemmas from Proposition 2.1 (Zhou [27]), Lemma 3.2 and Lemma 3.3.

Lemma 3.5 *Let ν, p_0, q_0, p, q be the same as in Lemma 3.2 and suppose that $\theta \in (0,1)$ satisfies $0 < \theta < \frac{n}{2p}$. Then the operator $(-\Delta_x)^{\theta-1}\mathcal{B}(-\Delta_x)^{-\theta}$ is bounded on $\mathcal{M}_{p,q}$.*

Lemma 3.6 *Let ν, p_0, q_1 be the same as in Theorem 1.5. Suppose that $0 < \theta < \frac{n}{2p_0}$. Then the operator $(-\Delta_x)^{\theta-1}\mathcal{B}(-\Delta_x)^{-\theta}$ is bounded on $\mathcal{M}_{p_0,q_1}$.*

Now we have the main propositions on the estimates of the resolvent of $\mathcal{A}$.

Proposition 3.1 *Let ν, p_0, q_0, p, q be the same as in Lemma 3.2. Suppose that s, $\sigma < 2$ are real numbers satisfying*

$$s, \sigma \in \left(-2,\ \frac{n}{p}\right) \quad \textit{and} \quad |s - \sigma| \le 2. \tag{3.13}$$

Then, for every $\varepsilon > 0$, we can take a positive number $\gamma_1 = \gamma_1(s,\sigma)$, if $w(x) \in \mathcal{M}_{p_0,q_0}$ satisfies $\|w(x)\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}} \le \gamma_1$, then every $\lambda \in \mathbf{C} \setminus [0, +\infty)$ with $|\arg \lambda| \ge \varepsilon$ belongs to the resolvent of the operator $\mathcal{A}$ in $\mathcal{M}^s_{p,q}$. In the meantime, the following estimate

$$\begin{aligned}&\|(-\Delta_x)^{\sigma/2}(\lambda+\Delta_x)^{-1}\mathcal{B}(\lambda-\mathcal{A})^{-1}(-\Delta_x)^{-s/2}\|_{\mathcal{L}(\mathcal{M}_{p,q},\mathcal{M}_{p,q})}\\&\quad\le\frac{C|\lambda|^{(\sigma-s)/2-1}}{\gamma_1-\|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}}}\end{aligned}\tag{3.14}$$

holds with a positive constant C.

Furthermore, if $\sigma\ge s$, *the operator* $(-\Delta_x)^{\sigma/2}(\lambda-\mathcal{A})^{-1}(-\Delta_x)^{-s/2}$ *can be extended to a bounded operator on* $\mathcal{M}_{p,q}$ *and enjoys the estimate*

$$\begin{aligned}&\|(-\Delta_x)^{\sigma/2}(\lambda-\mathcal{A})^{-1}(-\Delta_x)^{-s/2}\|_{\mathcal{L}(\mathcal{M}_{p,q},\mathcal{M}_{p,q})}\\&\quad\le\frac{C|\lambda|^{(\sigma-s)/2-1}}{\gamma_1-\|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}}}\end{aligned}\tag{3.15}$$

with a positive constant C.

Proof. We prove in the similar way as in Zhou [27]. First we prove (3.14).

Since $\frac{1}{2}\max\{s,\sigma\}<\min\{1,\frac{n}{2p}\}\le 1$, $\frac{1}{2}\max\{s,\sigma\}\le 1+\frac{1}{2}\min\{s,\sigma\}$ and $0<1+\frac{1}{2}\min\{s,\sigma\}$, hence we can find a positive number $\theta\in(0,1)$ satisfying

$$\frac{1}{2}\max\{s,\sigma\}\le\theta\le 1+\frac{1}{2}\min\{s,\sigma\}\quad\text{and}\quad 0<\theta<\frac{n}{2p}.$$

Then Lemma 3.5 implies that the operator

$$\Theta=(-\Delta_x)^{\theta-1}\mathcal{B}(-\Delta_x)^{-\theta}$$

is bounded on $\mathcal{M}_{p,q}$ and the estimate $\|\Theta\|_{\mathcal{L}(\mathcal{M}_{p,q},\mathcal{M}_{p,q})}\le C_1\|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}}$ holds with some positive constant C_1.

On the other hand, the operator $(-\Delta_x)(\lambda+\Delta_x)^{-1}$ is also bounded on $\mathcal{M}_{p,q}$ for every $\lambda\in\mathbf{C}\setminus[0,+\infty)$, $|\arg\lambda|\ge\varepsilon$ and the estimate

$$\|(-\Delta_x)(\lambda+\Delta_x)^{-1}\|_{\mathcal{L}(\mathcal{M}_{p,q},\mathcal{M}_{p,q})}\le C_2$$

holds with some positive constant C_2.

In the same way, Lemma 3.4 implies that there exists a positive constant C_3, we have the following estimates:

$$\begin{aligned}\|(-\Delta_x)^{\frac{\sigma}{2}+1-\theta}(\lambda+\Delta_x)^{-1}\|_{\mathcal{L}(\mathcal{M}_{p,q},\mathcal{M}_{p,q})}&\le C_3|\lambda|^{\frac{\sigma}{2}-\theta},\\ \|(-\Delta_x)^{\theta-\frac{s}{2}}(\lambda+\Delta_x)^{-1}\|_{\mathcal{L}(\mathcal{M}_{p,q},\mathcal{M}_{p,q})}&\le C_3|\lambda|^{\theta-1-\frac{s}{2}},\end{aligned}$$

for every $\lambda \in \mathbf{C} \setminus [0, +\infty)$, $|\arg \lambda| \geq \varepsilon$.

Therefore, if $\|w\|_{\mathcal{M}_{p_0,q_0}}^{\nu-1} < \gamma_1(s,\sigma) = \frac{1}{C_1 C_2}$, the series

$$\begin{aligned}\mathbf{I} &= (-\Delta_x)^{\frac{\sigma}{2}+1-\theta}(\lambda+\Delta_x)^{-1}\Theta \\ &\quad \times \sum_{j=0}^{\infty}\{(\lambda+\Delta_x)^{-1}(-\Delta_x)\Theta\}^j(-\Delta_x)^{\theta-\frac{s}{2}}(\lambda+\Delta_x)^{-1}\end{aligned}$$

converges in $\mathcal{L}(\mathcal{M}_{p,q}, \mathcal{M}_{p,q})$ and the operator norm of the limit is dominated by

$$\begin{aligned}&C_3|\lambda|^{\frac{\sigma}{2}-\theta}C_1\|w\|_{\mathcal{M}_{p_0,q_0}}^{\nu-1}\frac{1}{1-C_1C_2\|w\|_{\mathcal{M}_{p_0,q_0}}^{\nu-1}}C_3|\lambda|^{\theta-1-\frac{s}{2}} \\ &\quad \leq \frac{C|\lambda|^{\frac{\sigma-s}{2}-1}}{\gamma_1-\|w\|_{\mathcal{M}_{p_0,q_0}}^{\nu-1}}.\end{aligned}$$

Moreover, the limit is

$$\begin{aligned}\mathbf{I} &= (-\Delta_x)^{\frac{\sigma}{2}}(\lambda+\Delta_x)^{-1}\mathcal{B}(-\Delta_x)^{-\theta} \\ &\quad \times \sum_{j=0}^{\infty}\{(\lambda+\Delta_x)^{-1}(-\Delta_x)^{\theta}\mathcal{B}(-\Delta_x)^{-\theta}\}^j(-\Delta_x)^{\theta-\frac{s}{2}}(\lambda+\Delta_x)^{-1} \\ &= (-\Delta_x)^{\frac{\sigma}{2}}(\lambda+\Delta_x)^{-1}\mathcal{B}(-\Delta_x)^{-\theta} \\ &\quad \times \{1-(\lambda+\Delta_x)^{-1}(-\Delta_x)^{\theta}\mathcal{B}(-\Delta_x)^{-\theta}\}^{-1}(-\Delta_x)^{\theta-\frac{s}{2}}(\lambda+\Delta_x)^{-1} \\ &= (-\Delta_x)^{\frac{\sigma}{2}}(\lambda+\Delta_x)^{-1}\mathcal{B}(\lambda-\mathcal{A})^{-1}(-\Delta_x)^{-\frac{s}{2}}.\end{aligned}$$

This yields (3.14). Observe the equality

$$\begin{aligned}&(-\Delta_x)^{\frac{\sigma}{2}}(\lambda-\mathcal{A})^{-1}(-\Delta_x)^{-\frac{s}{2}} \\ &= (-\Delta_x)^{\frac{\sigma}{2}}(\lambda+\Delta_x)^{-1}(-\Delta_x)^{-\frac{s}{2}} \\ &\qquad + (-\Delta_x)^{\frac{\sigma}{2}}(\lambda+\Delta_x)^{-1}\mathcal{B}(\lambda-\mathcal{A})^{-1}(-\Delta_x)^{-\frac{s}{2}} \\ &= (-\Delta_x)^{\frac{\sigma-s}{2}}(\lambda+\Delta_x)^{-1} + (-\Delta_x)^{\frac{\sigma}{2}}(\lambda+\Delta_x)^{-1}\mathcal{B}(\lambda-\mathcal{A})^{-1}(-\Delta_x)^{-\frac{s}{2}},\end{aligned}$$

the estimate (3.15) follows immediately from the above discussion and Lemma 3.4. □

Proposition 3.2 *Let ν, p_0, q_0, q_1 be the same as in Lemma 3.3. Suppose that s, σ are real numbers satisfying*

$$s, \sigma \in \left(-2, \frac{n}{p_0}\right) \quad \text{and} \quad |s-\sigma| \leq 2. \tag{3.16}$$

Then, for every $\varepsilon > 0$, we can take a positive number $\gamma_1 = \gamma_1(s,\sigma)$, if $w(x) \in \mathcal{M}_{p_0,q_0}$ satisfies $\|w(x)\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}} \le \gamma_1$, then every $\lambda \in \mathbf{C}\setminus[0,+\infty)$ with $|\arg\lambda| \ge \varepsilon$ belongs to the resolvent of the operator $\mathcal{A}$ in $\mathcal{M}^s_{p_0,q_1}$. In the meantime, the following estimate

$$\begin{aligned}&\|(-\Delta_x)^{\sigma/2}(\lambda+\Delta_x)^{-1}\mathcal{B}(\lambda-\mathcal{A})^{-1}(-\Delta_x)^{-s/2}\|_{\mathcal{L}(\mathcal{M}_{p_0,q_1},\mathcal{M}_{p_0,q_1})}\\ &\quad\le \frac{C|\lambda|^{(\sigma-s)/2-1}}{\gamma_1-\|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}}}\end{aligned} \tag{3.17}$$

holds with a positive constant C.

Furthermore, if $\sigma \ge s$, the operator $(-\Delta_x)^{\sigma/2}(\lambda-\mathcal{A})^{-1}(-\Delta_x)^{-s/2}$ can be extended to a bounded operator on $\mathcal{M}_{p_0,q_1}$ and enjoys the estimate

$$\begin{aligned}&\|(-\Delta_x)^{\sigma/2}(\lambda-\mathcal{A})^{-1}(-\Delta_x)^{-s/2}\|_{\mathcal{L}(\mathcal{M}_{p_0,q_1},\mathcal{M}_{p_0,q_1})}\\ &\quad\le \frac{C|\lambda|^{(\sigma-s)/2-1}}{\gamma_1-\|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}}}\end{aligned} \tag{3.18}$$

with a positive constant C.

Proof. Since $\frac{1}{2}\max\{s,\sigma\} < \frac{n}{2p_0} = \frac{1}{\nu-1} < 1$, $\frac{1}{2}\max\{s,\sigma\} \le 1 + \frac{1}{2}\min\{s,\sigma\}$ and $0 < 1 + \frac{1}{2}\min\{s,\sigma\}$, hence we can find a positive number $\theta \in (0,1)$ satisfying

$$\frac{1}{2}\max\{s,\sigma\} \le \theta \le 1 + \frac{1}{2}\min\{s,\sigma\} \quad \text{and} \quad 0 < \theta < \frac{n}{2p_0}.$$

Then we can finish the rest part of our proof in the same way as the proof of Proposition 3.1, by using Lemma 3.4 and Lemma 3.6. $\square$

4. The semigroup generated by the perturbed operator

In this section, we will give some properties of the semigroup $\exp(-t\mathcal{A})$ generated by the pertubed heat operator $\mathcal{A}$. It is defined by the formula

$$\exp(-t\mathcal{A}) = \int_\Gamma \exp(-t\lambda)(\lambda-\mathcal{A})^{-1}d\lambda,$$

for $t \in \mathbf{C}$ such that $\operatorname{Re} t > 0$, the contour Γ satisfies the condition

$$\begin{aligned}&\Gamma \subset \{\lambda \in \mathbf{C}\setminus\{0\} \mid |\arg\lambda| > \varepsilon\} \quad \text{connects}\\ &\qquad\qquad \exp(-i\omega)\infty \quad \text{to} \quad \exp(i\omega)\infty,\end{aligned} \tag{4.1}$$

where $0 < \varepsilon < \omega < \pi/2 - |\arg t|$. In the following sections, we will use this semigroup to construct our time-global solutions by succesive approximation.

First, we have some properties about the boundedness of the semigroup $\exp(-t\mathcal{A})$ on the Morrey spaces. For the case $\frac{n}{n-2} < \nu \le 2$, $\nu \in \mathbf{R}$, we have the following

Theorem 4.1 *Let ν, p_0, q_0, p, q be the same as in Lemma 3.2, s, $\sigma < 2$ satisfy (3.13). Suppose that $w(x) \in \mathcal{M}_{p_0,q_0}$ satisfies $\|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}} < \gamma_1(s,\sigma)$. Then there exists a positive constant C such that for every $t > 0$, we have:*

$$\|\exp(-t\mathcal{A})\|_{\mathcal{L}(\mathcal{M}^s_{p,q},\mathcal{M}^\sigma_{p,q})} \le \frac{Ct^{(s-\sigma)/2}}{\gamma_1 - \|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}}}, \quad \text{if } s \le \sigma, \tag{4.2}$$

$$\|\exp(-t\mathcal{A}) - 1\|_{\mathcal{L}(\mathcal{M}^s_{p,q},\mathcal{M}^\sigma_{p,q})} \le \frac{Ct^{(s-\sigma)/2}}{\gamma_1 - \|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}}}, \quad \text{if } \sigma \le s. \tag{4.3}$$

For the second case $\nu > 2$, $\nu \in \mathbf{R}$, the boundedness is as follows.

Theorem 4.2 *Let ν, p_0, q_0, q_1 be the same as in Lemma 3.3, s, σ satisfy the condition (3.16). Suppose that $w(x) \in \mathcal{M}_{p_0,q_0}$ satisfies $\|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}} < \gamma_1(s,\sigma)$. Then there exists a positive constant C such that for every $t > 0$, we have:*

$$\|\exp(-t\mathcal{A})\|_{\mathcal{L}(\mathcal{M}^s_{p_0,q_1},\mathcal{M}^\sigma_{p_0,q_1})} \le \frac{Ct^{\frac{s-\sigma}{2}}}{\gamma_1 - \|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}}}, \quad \text{if } s \le \sigma, \tag{4.4}$$

$$\|\exp(-t\mathcal{A}) - 1\|_{\mathcal{L}(\mathcal{M}^s_{p_0,q_1},\mathcal{M}^\sigma_{p_0,q_1})} \le \frac{Ct^{\frac{s-\sigma}{2}}}{\gamma_1 - \|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}}}, \quad \text{if } \sigma \le s. \tag{4.5}$$

The two theorems above-mentioned can be proved in the same way as in Theorem 4.1 (Zhou [27]), by using Proposition 3.1 and Proposition 3.2.

As in the usual analytical semigroup theory, $\exp(-t\mathcal{A})$ is independent of the choice of ω, and the semigroup property $\exp(-(t+s)\mathcal{A}) = \exp(-t\mathcal{A}) \cdot \exp(-s\mathcal{A})$ holds for every t and s such that $\operatorname{Re} t$, $\operatorname{Re} s > 0$.

Before we construct the time-global solutions to the Cauchy problem (3.2)–(3.3), we must prove the equivalence between differential equations and integral equations. However, we need the following propositions of the strong continuity and the strong differentiability of $\exp(-t\mathcal{A})f$. The proofs are similar to the proof of Proposition 4.1 (Zhou [27]). For the first case $\frac{n}{n-2} < \nu \leq 2$, $\nu \in \mathbf{R}$, we have

Proposition 4.1 *Let ν, p_0, q_0, p, q be the same as in Theorem 4.1, and assume that $0 < \sigma + 2 < s < \min\{2, n/p\}$. Moreover, suppose that $w(x) \in \mathcal{M}_{p_0,q_0}$ satisfies the estimate*

$$\|w\|_{\mathcal{M}_{p_0,q_0}}^{\nu-1} < \min\{\gamma_1(s, s-2), \gamma_1(s-2, \sigma+2)\}.$$

Then, for every $f(x) \in \mathcal{M}_{p,q}^{s}$, we have

$$\frac{\exp(-t\mathcal{A})f - f}{t} + \mathcal{A}f \in \mathcal{M}_{p,q}^{\sigma}, \quad \textit{for every} \quad t > 0,$$

$$\lim_{t \to 0+} \frac{\exp(-t\mathcal{A})f - f}{t} + \mathcal{A}f = 0 \quad \textit{in the topology of} \ \mathcal{M}_{p,q}^{\sigma}.$$

For the second case $\nu > 2$, $\nu \in \mathbf{R}$, this property is described as follows.

Proposition 4.2 *Let ν, p_0, q_0, q_1 be the same as in Theorem 4.2, and assume that $0 < \sigma + 2 < s < n/p_0$. Moreover, suppose that $w(x) \in \mathcal{M}_{p_0,q_0}$ satisfies the estimate*

$$\|w\|_{\mathcal{M}_{p_0,q_0}}^{\nu-1} < \min\{\gamma_1(s, s-2), \gamma_1(s-2, \sigma+2)\}.$$

Then, for every $f(x) \in \mathcal{M}_{p_0,q_1}^{s}$, we have

$$\frac{\exp(-t\mathcal{A})f - f}{t} + \mathcal{A}f \in \mathcal{M}_{p_0,q_1}^{\sigma}, \quad \textit{for every} \quad t > 0,$$

$$\lim_{t \to 0+} \frac{\exp(-t\mathcal{A})f - f}{t} + \mathcal{A}f = 0 \quad \textit{in the topology of} \ \mathcal{M}_{p_0,q_1}^{\sigma}.$$

5. Equivalence between differential equations and integral equations

In this section, by virtue of the estimates of the semigroup $\exp(-t\mathcal{A})$ established in the previous section, we will prove the equivalence of the original differential equations and the associated integral equations in the similar way as in Zhou [27]. The integral equation of (3.3) is as follows:

$$u(t, \cdot) = \exp(-t\mathcal{A})b - \int_0^t \exp(-(t-\tau)\mathcal{A})G(w(\cdot), u(\tau, \cdot))d\tau. \quad (5.1)$$

From the definition of the Morrey spaces, we have the following result on complex interpolation. See Lemma 2.5 of Kozono and Yamazaki [17]. For complex interpolation, see Bergh and Löfström [3] or Triebel [24], for example.

Lemma 5.1 *Suppose that $1 < q \le p < \infty$, $s_1, s_2 \in \mathbf{R}$ and $0 < \theta < 1$, and put $s = (1-\theta)s_1 + \theta s_2$. Then the space $\mathcal{M}^s_{p,q}$ coincides with the complex interpolation space $[\mathcal{M}^{s_1}_{p,q}, \mathcal{M}^{s_2}_{p,q}]_\theta$.*

Now we consider the Cauchy problem (3.2)–(3.3).

Case 1: $\frac{n}{n-2} < \nu \le 2$, $\nu \in \mathbf{R}$.

Firstly we determine the positive number δ_1 in Theorem 1.2. Let σ_0 be a constant in Theorem 1.2. Denote $\sigma_1 = \frac{\nu}{2p_0}$, then there exists a real number σ_2 such that

$$\sigma_1 < \sigma_2 < 2/p\ .$$

Therefore we can take a positive constant $\delta_1 \le \delta_0$ sufficiently small so that, for every $f \in \mathcal{M}^{-2}_{p_0,q_0}$ such that $\|f\|_{\mathcal{M}^{-2}_{p_0,q_0}} < \delta_1$, the stationary solution of (1.3) given in Theorem 1.1 enjoys the estimate

$$\begin{aligned}
&\|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}} \\
&< \min\Big\{\gamma_1(\sigma_0-2,\sigma_0-2),\ \gamma_1(\sigma_0,\sigma_0-2),\ \gamma_1(\sigma_0-2,\sigma_0),\ \gamma_1(\sigma_0,\sigma_0),\\
&\qquad \gamma_1(\sigma_1-2,\sigma_1-2),\ \gamma_1(\sigma_1,\sigma_1-2),\ \gamma_1(\sigma_1-2,\sigma_1),\ \gamma_1(\sigma_1,\sigma_1),\\
&\qquad \gamma_1\Big(\frac{2}{p},\frac{2}{p}-2\Big),\ \gamma_1\Big(\frac{2}{p}-2,\sigma_2\Big)\Big\} \le 1.
\end{aligned}$$

In the sequel we assume that $f(x)$ and $w(x)$ are as above. Then Theorem 4.1 and Lemma 5.1 imply the following proposition.

Proposition 5.1 *The estimate*

$$\|\exp(-t\mathcal{A})\|_{\mathcal{L}(\mathcal{M}^s_{p,q},\mathcal{M}^\sigma_{p,q})} \le Ct^{(s-\sigma)/2}$$

is valid for every s and σ satisfying $\sigma_1 - 2 \le s \le \sigma \le \sigma_0$ and $\sigma \le s+2$. Moreover, the estimate

$$\|\exp(-t\mathcal{A}) - 1\|_{\mathcal{L}(\mathcal{M}^s_{p,q},\mathcal{M}^\sigma_{p,q})} \le Ct^{(s-\sigma)/2}$$

is valid for every s and σ satisfying $\sigma_1 - 2 \le \sigma \le s \le \sigma_0$ and $s \le \sigma + 2$.

The following theorem shows the equivalence between differential equations and integral equations.

Theorem 5.1 *Let $0 < T \le \infty$. Suppose that $b(x) \in \mathcal{M}_{p,q}^{\frac{n}{p}-\frac{n}{p_0}}$ and that $u(t,x)$ is a function on $(0,T)\times \mathbf{R}^n$ satisfying* (3.4) *for every $T' \in (0,T)$. Then the following three cone ditions on the function $u(t,x)$ are equivalent*:

(1) *$u(t,x)$ satisfies the differential equation* (3.3) *on $(0,T)$, and the condition* (3.6) *for every $s \in \left[\frac{n}{p} - \frac{(n-2)\nu}{p},\ \frac{n}{p} - \frac{n}{p_0}\right]$ and every $T' \in (0,T)$.*

(2) *$u(t,x)$ satisfies the differential equation* (3.3) *on $(0,T)$, and $u(t,\cdot) - b \longrightarrow 0$ in the topology of $\mathcal{M}_{p,q}^{\frac{n}{p}-\frac{(n-2)\nu}{p}}$ as $t \to 0+$.*

(3) *$u(t,x)$ satisfies the integral equation* (5.1) *on $(0,T)$.*

In order to prove the theorem, we need the following lemma.

Lemma 5.2 *Let $u(t,x)$ be a function on $(0,T)\times\mathbf{R}^n$ satisfying* (3.4) *for every $T' \in (0,T)$ and* (5.1) *on $(0,T)$, and suppose that $b(x) \in \mathcal{M}_{p,q}^{\frac{n}{p}-\frac{n}{p_0}}$. Then, for every $s \in \left[\frac{n}{p} - \frac{(n-2)\nu}{p},\ \sigma_0\right]$, there exists a constant C_s such that*

$$\|u(t,\cdot) - \exp(-t\mathcal{A})b\|_{\mathcal{M}_{p,q}^s} \le C_s M^\nu t^{\frac{n}{2p}-\frac{n}{2p_0}-\frac{s}{2}}, \tag{5.2}$$

$$\|u(t,\cdot) - b\|_{\mathcal{M}_{p,q}^s} \le C_s\big(M^\nu + \|b\|_{\mathcal{M}_{p,q}^{\frac{n}{p}-\frac{n}{p_0}}}\big)t^{\frac{n}{2p}-\frac{n}{2p_0}-\frac{s}{2}} \quad \textit{if} \ \ s \le \frac{n}{p} - \frac{n}{p_0}, \tag{5.3}$$

$$\|u(t,\cdot)\|_{\mathcal{M}_{p,q}^s} \le C_s\big(M^\nu + \|b\|_{\mathcal{M}_{p,q}^{\frac{n}{p}-\frac{n}{p_0}}}\big)t^{\frac{n}{2p}-\frac{n}{2p_0}-\frac{s}{2}} \quad \textit{if} \ \ s \ge \frac{n}{p} - \frac{n}{p_0} \tag{5.4}$$

hold for every $t \in (0,T')$, where

$$M = \sup_{0<t\le T'} t^{\frac{n}{2p_0}-\frac{n-2}{2p}} \|u(t,\cdot)\|_{\mathcal{M}_{p,q}^{\frac{2}{p}}}.$$

Proof. We first prove (5.2). Denote $a = \frac{n}{n-2}$. If $u(t,\cdot) \in \mathcal{M}_{p,q}^{\frac{2}{p}}$, then Lemma 2.1 implies that $u(t,\cdot) \in \mathcal{M}_{ap,aq}$. It follows from Lemma 3.1 (1) that $G(w(\cdot),u(t,\cdot)) \in \mathcal{M}_{ap/\nu,aq/\nu} \subset \mathcal{M}_{p,q}^{\frac{n}{p}-\frac{(n-2)\nu}{p}}$.

Since $\sigma_1 - 2 < \frac{n}{p} - \frac{(n-2)\nu}{p} \le s \le \sigma_0$ and $s - \left(\frac{n}{p} - \frac{(n-2)\nu}{p}\right) < \sigma_0 - \big(\frac{n}{p} -$

$\frac{(n-2)\nu}{p}) < 2$, we have the following estimate in view of Proposition 5.1.

$$\begin{aligned}
&\|u(t,\cdot) - \exp(-t\mathcal{A})b\|_{\mathcal{M}^s_{p,q}} \\
&\leq \int_0^t \|\exp(-(t-\tau)\mathcal{A})G(w(\cdot),u(\tau,\cdot))\|_{\mathcal{M}^s_{p,q}} d\tau \\
&\leq C\int_0^t (t-\tau)^{\frac{1}{2}(\frac{n}{p}-\frac{(n-2)\nu}{p})-\frac{s}{2}} \|G(w(\cdot),u(\tau,\cdot))\|_{\mathcal{M}^{\frac{n}{p}-\frac{(n-2)\nu}{p}}_{p,q}} d\tau \\
&\leq C\int_0^t (t-\tau)^{\frac{1}{2}(\frac{n}{p}-\frac{(n-2)\nu}{p})-\frac{s}{2}} (\|u(\tau,\cdot)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}})^{\nu} d\tau \\
&\leq CM^{\nu}\int_0^t (t-\tau)^{\frac{1}{2}(\frac{n}{p}-\frac{(n-2)\nu}{p})-\frac{s}{2}} \tau^{\nu(\frac{n-2}{2p}-\frac{n}{2p_0})} d\tau \leq C_s M^{\nu} t^{\frac{n}{2p}-\frac{n}{2p_0}-\frac{s}{2}},
\end{aligned}$$

which yields the estimate (5.2).

If $s \leq \frac{n}{p} - \frac{n}{p_0}$, we have $\sigma_1 - 2 < \frac{n}{p} - \frac{(n-2)\nu}{p} \leq s \leq \frac{n}{p} - \frac{n}{p_0} < \sigma_0$ and $\frac{n}{p} - \frac{n}{p_0} - s < 2$, hence the following inequality holds from Proposition 5.1:

$$\|(\exp(-t\mathcal{A}) - 1)b\|_{\mathcal{M}^s_{p,q}} \leq Ct^{\frac{n}{2p}-\frac{n}{2p_0}-\frac{s}{2}} \|b\|_{\mathcal{M}^{\frac{n}{p}-\frac{n}{p_0}}_{p,q}},$$

which together with (5.2) implies (5.3).

Finally, if $s \geq \frac{n}{p} - \frac{n}{p_0}$, we have $\sigma_1 - 2 < \frac{n}{p} - \frac{n}{p_0} \leq s \leq \sigma_0$ and $s - (\frac{n}{p} - \frac{n}{p_0}) < 2$. Therefore Proposition 5.1 follows that

$$\|\exp(-t\mathcal{A})b\|_{\mathcal{M}^s_{p,q}} \leq Ct^{\frac{n}{2p}-\frac{n}{p_0}-\frac{s}{2}} \|b\|_{\mathcal{M}^{\frac{n}{p}-\frac{n}{p_0}}_{p,q}},$$

which together with (5.2) implies (5.4). □

Proof of Theorem 5.1. (1) $\Longrightarrow$ (2) is trivial.

Next we show the implication (3) $\Longrightarrow$ (1). Suppose that $u(t,x)$ is a solution of (5.1) on $(0,T)$. Then the estimate (3.6) for every $T' \in (0,T)$ follows from (5.3) for every $s \in \left[\frac{n}{p} - \frac{(n-2)\nu}{p}, \frac{n}{p} - \frac{n}{p_0}\right]$.

Step 1. We verify that the function $u(t,\cdot)$ is Hölder continuous from $[\varepsilon, T']$ to $\mathcal{M}^{\sigma_2}_{p,q}$ for every ε and T' such that $0 < \varepsilon < T' < T$.

Let $\varepsilon \leq \tau \leq T'$, then we have

$$\begin{aligned}
u(t,\cdot) - u(\tau,\cdot) = {} & (\exp(-(t-\tau)\mathcal{A}) - 1)u(\tau,\cdot) \\
& - \int_\tau^t \exp(-(t-s)\mathcal{A})G(w(\cdot),u(s,\cdot))ds.
\end{aligned}$$

From Proposition 5.1, we get the following estimate.

$$
\begin{aligned}
&\|u(t,\cdot)-u(\tau,\cdot)\|_{\mathcal{M}^{\sigma_2}_{p,q}} \\
&\quad \le \|\exp(-(t-\tau)\mathcal{A})-1\|_{\mathcal{L}\left(\mathcal{M}^{\frac{2}{p}}_{p,q},\mathcal{M}^{\sigma_2}_{p,q}\right)}\|u(\tau,\cdot)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}} \\
&\quad\quad + \int_\tau^t \|\exp(-(t-s)\mathcal{A})G(w(\cdot),u(s,\cdot))\|_{\mathcal{M}^{\sigma_2}_{p,q}}ds \\
&\quad \le CM(t-\tau)^{\frac{1}{p}-\frac{\sigma_2}{2}}\tau^{\frac{n-2}{2p}-\frac{n}{2p_0}} \\
&\quad\quad + C\int_\tau^t (t-s)^{\frac{1}{2}\left(\frac{n}{p}-\frac{(n-2)\nu}{p}\right)-\frac{\sigma_2}{2}}\|G(w(\cdot),u(s,\cdot))\|_{\mathcal{M}^{\frac{n}{p}-\frac{(n-2)\nu}{p}}_{p,q}}ds \\
&\quad \le CM(t-\tau)^{\frac{1}{p}-\frac{\sigma_2}{2}}\tau^{\frac{n-2}{2p}-\frac{n}{2p_0}} \\
&\quad\quad + CM^\nu(t-\tau)^{1-\frac{(n-2)\nu-n}{2p}-\frac{\sigma_2}{2}}\tau^{\nu\left(\frac{n-2}{2p}-\frac{n}{2p_0}\right)},
\end{aligned}
$$

which implies that the function $u(t,\cdot)$ is Hölder continuous from $[\varepsilon,T']$ to $\mathcal{M}^{\sigma_2}_{p,q}$.

Step 2. We verify that the function $G(w(\cdot),u(t,\cdot))$ is Hölder continuous from $[\varepsilon,T']$ to $\mathcal{M}^{\sigma_2-\frac{(n-2)(\nu-1)}{p}}_{p,q}$ for every ε and T' such that $0<\varepsilon<T'<T$. Let $\varepsilon\le\tau\le T'$, then we have the following estimate from the definition of $G(w(\cdot),u(t,\cdot))$.

$$
\begin{aligned}
&|G(w(\cdot),u(t,\cdot))-G(w(\cdot),u(\tau,\cdot))| \\
&\quad\quad \le \nu(|u(t,\cdot)|^{\nu-1}+|u(\tau,\cdot)|^{\nu-1})|u(t,\cdot)-u(\tau,\cdot)|.
\end{aligned}
$$

Then the Hölder continuity of $G(w(\cdot),u(t,\cdot))$ in $\mathcal{M}^{\sigma_2-\frac{(n-2)(\nu-1)}{p}}_{p,q}$ can be easily deduced from that of $u(t,\cdot)$ in the same way as in Zhou [27].

Step 3. We prove that $u(t,x)$ satisfies the differential equation (3.3). Let t_0 be an arbitrary point of (ε,T'), and let t_1, t_2 be points of (ε,T') such that $t_1<t_2$. Then we have

$$
\begin{aligned}
&\frac{u(t_2)-u(t_1)}{t_2-t_1} \\
&= \frac{\exp(-(t_2-t_1)\mathcal{A})-1}{t_2-t_1}u(t_0)+\frac{\exp(-(t_2-t_1)\mathcal{A})-1}{t_2-t_1}(u(t_1)-u(t_0)) \\
&\quad - \frac{1}{t_2-t_1}\int_{t_1}^{t_2}\exp(-(t_2-\tau)\mathcal{A})\big(G(w(\cdot),u(\tau,\cdot))-G(w(\cdot),u(t_0,\cdot))\big)d\tau
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{t_2-t_1}\int_{t_1}^{t_2}(\exp(-(t_2-\tau)\mathcal{A})-1)G(w(\cdot),u(t_0,\cdot))d\tau \\
&-G(w(\cdot),u(t_0,\cdot)) \\
&=\mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_3+\mathbf{I}_4-G(w(\cdot),u(t_0,\cdot)).
\end{aligned}
$$

Now let t_1, $t_2 \to t_0$. Since

$$\|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}} < \min\left\{\gamma_1\left(\frac{2}{p},\frac{2}{p}-2\right),\ \gamma_1\left(\frac{2}{p}-2,\sigma_2\right)\right\},$$

Proposition 4.1 implies that $\mathbf{I}_1$ tends to $-\mathcal{A}u(t_0)$ in $\mathcal{M}^{\sigma_2-2}_{p,q}$. Next, Proposition 5.1 implies that the operator

$$\frac{\exp(-t\mathcal{A})-1}{t}$$

is uniformly bounded in $\mathcal{L}(\mathcal{M}^{\sigma_2}_{p,q},\mathcal{M}^{\sigma_2-2}_{p,q})$. Observing the Hölder continuity of $u(t,x)$ in $\mathcal{M}^{\sigma_2}_{p,q}$, we know that $\mathbf{I}_2$ tends to 0 in $\mathcal{M}^{\sigma_2-2}_{p,q}$.

Further, we get the following estimate in the same way as in (5.2).

$$\|\mathbf{I}_4\|_{\mathcal{M}^{\sigma_2-2}_{p,q}} \le C(t_2-t_1)^{\frac{1}{2}\left(\frac{n}{p}-\frac{(n-2)\nu}{p}\right)-\frac{\sigma_2}{2}+1}\left(\|u(t_0)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}}\right)^{\nu},$$

hence $\mathbf{I}_4$ tends to 0 in $\mathcal{M}^{\sigma_2-2}_{p,q}$ as t_1, $t_2 \to t_0$.

Next, we discuss the term $\mathbf{I}_3$. Proposition 5.1 implies that the operator $\exp(-t\mathcal{A})$ is uniformly bounded in $\mathcal{L}\big(\mathcal{M}^{\sigma_2-\frac{(n-2)(\nu-1)}{p}}_{p,q},\mathcal{M}^{\sigma_2-\frac{(n-2)(\nu-1)}{p}}_{p,q}\big)$. It follows from this fact and the Hölder continuity of $G(w(\cdot),u(t,\cdot))$ in $\mathcal{M}^{\sigma_2-\frac{(n-2)(\nu-1)}{p}}_{p,q}$ that $\mathbf{I}_3$ tends to 0 in $\mathcal{M}^{\sigma_2-\frac{(n-2)(\nu-1)}{p}}_{p,q}$.

From the discussion above we have proved the equality

$$\lim_{t_1,t_2\to t_0}\frac{u(t_2)-u(t_1)}{t_2-t_1} = -\mathcal{A}u(t_0)-G(w(\cdot),u(t_0,\cdot))$$

holds in Φ'. That means $u(t,x)$ satisfies (3.3) on (ε,T'). Since ε and T' are arbitrary, $u(t,x)$ satisfies (3.3) on $(0,T)$. This shows that (3) $\Longrightarrow$ (1).

Finally, we will prove the implication (2) $\Longrightarrow$ (3). Suppose that $u(t,x)$ satisfies (3.4) for every $T' \in (0,T)$, (3.3) and $u(t,x)-b \longrightarrow 0$ in the topology of $\mathcal{M}^{\frac{n}{p}-\frac{(n-2)\nu}{p}}_{p,q}$ as $t \to 0+$.

Put

$$v(t,\cdot) = -\int_0^t \exp(-(t-\tau)\mathcal{A})G(w(\cdot),u(\tau,\cdot))d\tau.$$

Then we can prove that $u(t,x) = \exp(-t\mathcal{A})b + v(t,x)$ satisfies (5.1) in the same way as in Zhou [27]. That means (2) $\Longrightarrow$ (3).

This completes the proof of Theorem 5.1. $\square$

Next we consider the second case.

Case 2: $\nu > 2$, $\nu \in \mathbf{R}$.

Let ν, p_0, q_0, q_1 be the same as in Theorem 1.5. We will use the same method for the first case to prove our results. Firstly we determine the positive number δ_1 in Theorem 1.5. Let σ_0 be a constant in Theorem 1.5. Denote $\sigma_1 = \frac{n}{2p_0} - \frac{n}{2p}$, then there exist a real number σ_2 such that

$$\frac{n}{2p_0} - \frac{n}{2p} < \sigma_2 < \frac{n}{p_0} - \frac{n}{p}.$$

Therefore we can take a positive constant $\delta_1 \leq \delta_0$ sufficiently small so that, for every $f \in \mathcal{M}^{-2}_{p_0,q_0}$ such that $\|f\|_{\mathcal{M}^{-2}_{p_0,q_0}} < \delta_1$, the stationary solution of (1.3) given in Theorem 1.1 enjoys the estimate

$$\begin{aligned}
&2\|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}} \\
&< \min\Big\{\gamma_1(\sigma_0-2,\sigma_0-2),\ \gamma_1(\sigma_0,\sigma_0-2),\ \gamma_1(\sigma_0-2,\sigma_0),\ \gamma_1(\sigma_0,\sigma_0), \\
&\qquad \gamma_1(\sigma_1-2,\sigma_1-2),\ \gamma_1(\sigma_1,\sigma_1-2),\ \gamma_1(\sigma_1-2,\sigma_1),\ \gamma_1(\sigma_1,\sigma_1), \\
&\qquad \gamma_1\Big(\frac{n}{p_0}-\frac{n}{p},\frac{n}{p_0}-\frac{n}{p}-2\Big),\ \gamma_1\Big(\frac{n}{p_0}-\frac{n}{p}-2,\sigma_2\Big)\Big\} \leq 1.
\end{aligned}$$

Hereafter we assume that $f(x)$ and $w(x)$ are as above for the second case. Then we have the following proposition by using Theorem 4.2 and Lemma 5.1.

Proposition 5.2 *The estimate*

$$\|\exp(-t\mathcal{A})\|_{\mathcal{L}(\mathcal{M}^s_{p_0,q_1},\mathcal{M}^\sigma_{p_0,q_1})} \leq Ct^{(s-\sigma)/2}$$

is valid for every s and σ satisfying $\sigma_1 - 2 \leq s \leq \sigma \leq \sigma_0$ and $\sigma \leq s+2$. Moreover, the estimate

$$\|\exp(-t\mathcal{A}) - 1\|_{\mathcal{L}(\mathcal{M}^s_{p_0,q_1},\mathcal{M}^\sigma_{p_0,q_1})} \leq Ct^{(s-\sigma)/2}$$

is valid for every s and σ satisfying $\sigma_1 - 2 \leq \sigma \leq s \leq \sigma_0$ and $s \leq \sigma + 2$.

Then we also have the following theorem about the equivalence between differential equations and integral equations when $\nu > 2$, $\nu \in \mathbf{R}$.

Theorem 5.2 *Let $0 < T \le \infty$. Suppose that $b(x) \in \mathcal{M}_{p_0,q_1}$ and that $u(t,x)$ is a function on $(0,T) \times \mathbf{R}^n$ satisfying* (3.8) *for every $T' \in (0,T)$. Then the following three conditions on the function $u(t,x)$ are equivalent*:

(1) *$u(t,x)$ satisfies the differential equation* (3.3) *on $(0,T)$, and the condition* (3.10) *for every $s \in \left[\frac{2n}{p_0} - \frac{2n}{p} - 2, 0\right]$ and every $T' \in (0,T)$.*

(2) *$u(t,x)$ satisfies the differential equation* (3.3) *on $(0,T)$, and $u(t,\cdot) - b \longrightarrow 0$ in the topology of $\mathcal{M}_{p_0,q_1}^{\frac{2n}{p_0}-\frac{2n}{p}-2}$ as $t \to 0+$.*

(3) *$u(t,x)$ satisfies the integral equation* (5.1) *on $(0,T)$.*

As before we need the following lemma to prove Theorem 5.2.

Lemma 5.3 *Let $u(t,x)$ be a function on $(0,T) \times \mathbf{R}^n$ satisfying* (3.8) *for every $T' \in (0,T)$ and* (5.1) *on $(0,T)$, and suppose that $b(x) \in \mathcal{M}_{p_0,q_1}$. Then, for every $s \in \left[\frac{2n}{p_0} - \frac{2n}{p} - 2,\ \sigma_0\right]$, there exists a constant C_s such that*

$$\|u(t,\cdot) - \exp(-t\mathcal{A})b\|_{\mathcal{M}_{p_0,q_1}^s} \le C_s(M^\nu + \|w\|_{\mathcal{M}_{p_0,q_0}}^{\nu-2} M^2) t^{-\frac{s}{2}}, \quad (5.5)$$

$$\|u(t,\cdot) - b\|_{\mathcal{M}_{p_0,q_1}^s} \le C_s\big(M^\nu + \|w\|_{\mathcal{M}_{p_0,q_0}}^{\nu-2} M^2 + \|b\|_{\mathcal{M}_{p_0,q_1}}\big) t^{-\frac{s}{2}} \quad \text{if } s \le 0, \quad (5.6)$$

$$\|u(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}^s} \le C_s\big(M^\nu + \|w\|_{\mathcal{M}_{p_0,q_0}}^{\nu-2} M^2 + \|b\|_{\mathcal{M}_{p_0,q_1}}\big) t^{-\frac{s}{2}} \quad \text{if } s \ge 0 \quad (5.7)$$

hold for every $t \in (0,T')$, where

$$M = \max\left\{ \sup_{0<t\le T'} \|u(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}},\ \sup_{0<t\le T'} t^{\frac{n}{2p_0}-\frac{n}{2p}} \|u(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}^{\frac{n}{p_0}-\frac{n}{p}}} \right\}.$$

Proof. If $u(t,\cdot) \in \mathcal{M}_{p_0,q_1}^{\frac{n}{p_0}-\frac{n}{p}}$, Lemma 2.1 implies that $u(t,\cdot) \in \mathcal{M}_{p,\frac{pq_1}{p_0}}$. It follows from Lemma 3.1 (2) that $G(w(\cdot),u(t,\cdot)) \in \mathcal{M}_{p_0,q_1}^{\frac{2n}{p_0}-\frac{2n}{p}-2}$.

Since $\sigma_1 - 2 < \frac{2n}{p_0} - \frac{2n}{p} - 2 \le s \le \sigma_0$ and $s - \left(\frac{2n}{p_0} - \frac{2n}{p} - 2\right) \le \sigma_0 - \left(\frac{2n}{p_0} - \frac{2n}{p} - 2\right) < 2$, Proposition 5.2 implies that

$$\begin{aligned}
&\|u(t,\cdot) - b\|_{\mathcal{M}_{p_0,q_1}^s} \\
&\le \int_0^t \|\exp(-(t-\tau)\mathcal{A}) G(w(\cdot),u(\tau,\cdot))\|_{\mathcal{M}_{p_0,q_1}^s} d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_0^t (t-\tau)^{\frac{1}{2}(\frac{2n}{p_0}-\frac{2n}{p}-2)-\frac{s}{2}}\|G(w(\cdot),u(\tau,\cdot))\|_{\mathcal{M}_{p_0,q_1}^{\frac{2n}{p_0}-\frac{2n}{p}-2}}d\tau\\
&\leq C\int_0^t (t-\tau)^{\frac{n}{p_0}-\frac{n}{p}-\frac{s}{2}-1}(\|w(\cdot)\|_{\mathcal{M}_{p_0,q_0}}+\|u(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}})^{\nu-2}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\left(\|u(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}^{\frac{n}{p_0}-\frac{n}{p}}}\right)^2 d\tau\\
&\leq C\int_0^t (t-\tau)^{\frac{n}{p_0}-\frac{n}{p}-\frac{s}{2}-1}\tau^{\frac{n}{p}-\frac{n}{p_0}}(\|w(\cdot)\|_{\mathcal{M}_{p_0,q_0}}+M)^{\nu-2}M^2 d\tau\\
&\leq C_s(M^\nu+\|w\|_{\mathcal{M}_{p_0,q_0}}^{\nu-2}M^2)t^{-\frac{s}{2}}.
\end{aligned}
$$

holds for every $t\in(0,T')$, which yields the estimate (5.5).

If $s\leq 0$, then $\sigma_1-2<\frac{2n}{p_0}-\frac{2n}{p}-2\leq s\leq 0<\sigma_0$ and $-s<2-\sigma_1<2$, hence we have the following estimate in view of Proposition 5.2.

$$
\|(\exp(-t\mathcal{A})-1)b\|_{\mathcal{M}_{p_0,q_1}^s}\leq C_s t^{-\frac{s}{2}}\|b\|_{\mathcal{M}_{p_0,q_1}},
$$

which together with (5.5) implies (5.6).

Finally, if $s\geq 0$, we have $\sigma_1-2<\frac{2n}{p_0}-\frac{2n}{p}-2\leq 0\leq s\leq\sigma_0<2$, then it follows from Proposition 5.2 that

$$
\|\exp(-t\mathcal{A})b\|_{\mathcal{M}_{p_0,q_1}^s}\leq C_s t^{-\frac{s}{2}}\|b\|_{\mathcal{M}_{p_0,q_1}},
$$

which together with (5.5) implies (5.7). □

Now we prove Theorem 5.2 in the same way as in that of Theorem 5.1.

Proof of Theorem 5.2. It is enough to prove the implication (3) $\Longrightarrow$ (1). Suppose that $u(t,x)$ is a solution of (5.1) on $(0,T)$. Then the estimate (3.10) for every $T'\in(0,T)$ follows from (5.6) for every $s\in\left[\frac{2n}{p_0}-\frac{2n}{p}-2,0\right]$.

Suppose that σ_1, σ_2 and σ_0 satisfy the assumptions above-mentioned. Then we can also prove that the function $u(t,x)$ is Hölder continuous from $[\varepsilon,T']$ to $\mathcal{M}_{p_0,q_1}^{\sigma_2}$ for every ε and T' such that $0<\varepsilon<T'<T$.

Next, we have the following inequality by calculating directly.

$$
\begin{aligned}
&|G(w(\cdot),u(t,\cdot))-G(w(\cdot),u(\tau,\cdot))|\\
&\leq C(|w(\cdot)|+|u(t,\cdot)|+|u(\tau,\cdot)|^{\nu-2})(|u(t,\cdot)|+|u(\tau,\cdot)|)|u(t,\cdot)-u(\tau,\cdot)|.
\end{aligned}
$$

From this fact and the Hölder continuity of $u(t,x)$ in $\mathcal{M}_{p_0,q_1}^{\sigma_2}$, it is also easy to prove that $G(w(\cdot),u(t,\cdot))$ is Hölder continuous from $[\varepsilon,T']$ to $\mathcal{M}_{p_0,q_1}^{\sigma_2-2}$ for every ε and T' such that $0<\varepsilon<T'<T$.

Now we prove that $u(t,x)$ satisfies the differential equation (3.3). Let t_0 be an arbitrary point of (ε, T'), and let t_1, t_2 be points of (ε, T') such that $t_1 < t_2$. Then we have

$$\begin{aligned}
&\frac{u(t_2)-u(t_1)}{t_2-t_1} \\
&= \frac{\exp(-(t_2-t_1)\mathcal{A})-1}{t_2-t_1}u(t_0) + \frac{\exp(-(t_2-t_1)\mathcal{A})-1}{t_2-t_1}(u(t_1)-u(t_0)) \\
&\quad - \frac{1}{t_2-t_1}\int_{t_1}^{t_2} \exp(-(t_2-\tau)\mathcal{A})\big(G(w(\cdot),u(\tau,\cdot)) - G(w(\cdot),u(t_0,\cdot))\big)d\tau \\
&\quad - \frac{1}{t_2-t_1}\int_{t_1}^{t_2} (\exp(-(t_2-\tau)\mathcal{A})-1)G(w(\cdot),u(t_0,\cdot))d\tau \\
&\quad - G(w(\cdot),u(t_0,\cdot)) \\
&= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4 - G(w(\cdot),u(t_0,\cdot)).
\end{aligned}$$

Let t_1, $t_2 \to t_0$. Since

$$\|w\|^{\nu-1}_{\mathcal{M}_{p_0,q_0}} < \min\left\{\gamma_1\Big(\frac{n}{p_0}-\frac{n}{p}, \frac{n}{p_0}-\frac{n}{p}-2\Big), \gamma_1\Big(\frac{n}{p_0}-\frac{n}{p}-2, \sigma_2\Big)\right\},$$

Proposition 4.2 implies that $\mathbf{I}_1$ tends to $-\mathcal{A}u(t_0)$ in $\mathcal{M}^{\sigma_2-2}_{p,q}$. Next, Proposition 5.2 implies that the operator

$$\frac{\exp(-t\mathcal{A})-1}{t}$$

is uniformly bounded in $\mathcal{L}(\mathcal{M}^{\sigma_2}_{p_0,q_1}, \mathcal{M}^{\sigma_2-2}_{p_0,q_1})$. Observing the Hölder continuity of $u(t,x)$ in $\mathcal{M}^{\sigma_2}_{p_0,q_1}$, we know that $\mathbf{I}_2$ tends to 0 in $\mathcal{M}^{\sigma_2-2}_{p_0,q_1}$.

Further, we get the following estimate in the same way as in (5.5).

$$\begin{aligned}
\|\mathbf{I}_4\|_{\mathcal{M}^{\sigma_2-2}_{p_0,q_1}} &\le C(\|w\|_{\mathcal{M}_{p_0,q_0}} + \|u(t_0)\|_{\mathcal{M}_{p_0,q_1}})^{\nu-2} \\
&\quad \times \Big(\|u(t_0)\|_{\mathcal{M}^{\frac{n}{p_0}-\frac{n}{p}}_{p_0,q_1}}\Big)^2 (t_2-t_1)^{\frac{n}{p_0}-\frac{n}{p}-\frac{\sigma_2}{2}},
\end{aligned}$$

hence $\mathbf{I}_4$ tends to 0 in $\mathcal{M}^{\sigma_2-2}_{p_0,q_1}$ as t_1, $t_2 \to t_0$.

Finally, Proposition 5.2 implies that the operator $\exp(-t\mathcal{A})$ is uniformly bounded in $\mathcal{L}(\mathcal{M}^{\sigma_2-2}_{p_0,q_1}, \mathcal{M}^{\sigma_2-2}_{p_0,q_1})$. Moreover, $G(w(\cdot),u(t,\cdot))$ is Hölder continuous in $\mathcal{M}^{\sigma_2-2}_{p_0,q_1}$, it follows that $\mathbf{I}_3$ tends to 0 in $\mathcal{M}^{\sigma_2-2}_{p_0,q_1}$.

From the discussion above we have proved the equality

$$\lim_{t_1,t_2\to t_0} \frac{u(t_2)-u(t_1)}{t_2-t_1} = -\mathcal{A}u(t_0) - G(w(\cdot), u(t_0,\cdot))$$

holds in $\mathcal{M}^{\sigma_2-2}_{p_0,q_1} \subset \Phi'$. That means $u(t,x)$ satisfies (3.3) on (ε, T'). Since ε and T' are arbitrary, $u(t,x)$ satisfies (3.3) on $(0,T)$. This shows that $(3) \Longrightarrow (1)$.

The rest part of our proof can be finished in the same way as the proof of Theorem 5.1. □

6. Stability of the stationary solution

In this section, we will construct the solution of the integral equation (5.1) by succesive approximation. Then we show this solution satisfies all of the properties required in our theorems.

Firstly we consider the case $\nu \leq 2$.

Case 1: $\frac{n}{n-2} < \nu \leq 2$, $\nu \in \mathbf{R}$.

Define the sequence of functions $\{u_j(t,x)\}_{j=0}^{\infty}$ inductively by

$$\begin{aligned} u_0(t,\cdot) &= \exp(-t\mathcal{A})b, \\ u_{j+1}(t,\cdot) &= u_0(t,\cdot) - \int_0^t \exp(-(t-\tau)\mathcal{A})G(w(\cdot), u_j(\tau,\cdot))d\tau. \end{aligned}$$

Put $A_j = \sup_{t>0} t^{\frac{n}{2p_0}-\frac{n-2}{2p}} \|u_j(t,\cdot)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}}$,
$B_j = \sup_{t>0} t^{\frac{n}{2p_0}-\frac{n-2}{2p}} \|u_{j+1}(t,\cdot) - u_j(t,\cdot)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}}$. Then we have the following estimates.

Lemma 6.1 *There exist positive constants C_1, C_2 which only depend on ν, p_0, q_0, p, q such that*

(1) $A_0 \leq C_1\varepsilon$.

(2) $A_{j+1} \leq C_1\varepsilon + C_2 A_j{}^{\nu}$, *for every* $j = 0, 1, 2, \dots$.

Proof. Since $\sigma_1 - 2 < \frac{n}{p} - \frac{n}{p_0} < \frac{2}{p} < \sigma_0$ and $\frac{2}{p} - \left(\frac{n}{p} - \frac{n}{p_0}\right) < 2$, Propositions 5.1 implies that

$$\|u_0(t,\cdot)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}} \leq C_1 t^{\frac{1}{2}\left(\frac{n}{p}-\frac{n}{p_0}-\frac{2}{p}\right)} \|b\|_{\mathcal{M}^{\frac{n}{p}-\frac{n}{p_0}}_{p,q}} = C_1\varepsilon t^{-\left(\frac{n}{2p_0}-\frac{n-2}{2p}\right)},$$

which yields (1).

Next, we consider A_j. We get the following estimate in the same way as in (5.2).

$$\begin{aligned}
&\|u_{j+1}(t,\cdot)-u_0(t,\cdot)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}} \\
&\le \int_0^t \|\exp(-(t-\tau)\mathcal{A})G(w(\cdot),u_j(\tau,\cdot))\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}}\,d\tau \\
&\le C\int_0^t (t-\tau)^{\frac{1}{2}(\frac{n}{p}-\frac{(n-2)\nu}{p}-\frac{2}{p})}\|G(w(\cdot),u_j(\tau,\cdot))\|_{\mathcal{M}^{\frac{n}{p}-\frac{(n-2)\nu}{p}}_{p,q}}\,d\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{(n-2)(\nu-1)}{2p}}\left(\|u_j(\tau,\cdot)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}}\right)^{\nu}d\tau \\
&\le CA_j{}^{\nu}\int_0^t (t-\tau)^{-\frac{(n-2)(\nu-1)}{2p}}\tau^{-\nu(\frac{n}{2p_0}-\frac{n-2}{2p})}d\tau \\
&= CA_j{}^{\nu}t^{1-\frac{(n-2)(\nu-1)}{2p}-\nu(\frac{n}{2p_0}-\frac{n-2}{2p})}\int_0^1 (1-\tau)^{-\frac{(n-2)(\nu-1)}{2p}}\tau^{-\nu(\frac{n}{2p_0}-\frac{n-2}{2p})}d\tau \\
&= C_2A_j{}^{\nu}t^{-(\frac{n}{2p_0}-\frac{n-2}{2p})},
\end{aligned}$$

for every $j=0,1,2,\dots$. This implies $A_{j+1}\le C_1\varepsilon+C_2A_j{}^{\nu}$. □

Now we decide the positive numbers ε_0 and M_0 required in Theorem 1.2. Put $\varepsilon_0<\frac{\nu-1}{C_1\nu}(C_2\nu)^{-\frac{1}{\nu-1}}$ be a positive constant. Define the functions $\psi_\varepsilon(x)=C_1\varepsilon+C_2x^{\nu}$ and $M(\varepsilon)=\frac{\nu}{\nu-1}C_1\varepsilon$ for $\varepsilon\in[0,\varepsilon_0]$. Here C_1, C_2 are the constants in Lemma 6.1. Since $\psi_\varepsilon(0)=C_1\varepsilon>0$ and the function $\psi_\varepsilon(x)-x$ takes its minimum $C_1\varepsilon-\frac{\nu-1}{\nu}(C_2\nu)^{-\frac{1}{\nu-1}}\le 0$ at $x=x_0=(C_2\nu)^{-\frac{1}{\nu-1}}$, there exists a number $x_1\in(0,x_0]$ such that $\psi_\varepsilon(x_1)=x_1$. On the other hand, since the function $\psi_\varepsilon(x)$ is convex, we have the following estimate.

$$x_1=\psi_\varepsilon(x_1)\le\left(1-\frac{x_1}{x_0}\right)\psi_\varepsilon(0)+\frac{x_1}{x_0}\psi_\varepsilon(x_0)=C_1\varepsilon+\frac{1}{\nu}x_1.$$

Hence $x_1\le M(\varepsilon)$. Since $A_0=C_1\varepsilon<\psi_\varepsilon(x_1)=x_1$, the monotonicity of $\psi_\varepsilon(x)$ implies that $A_j<x_1$ for every $j\in\mathbf{N}$. It follows that $\sup_{j\in\mathbf{N}}A_j\le x_1\le M(\varepsilon)\le M(\varepsilon_0)=M_0$.

In order to prove the convergence of the sequence $\{u_j(t,x)\}_{j=0}^{\infty}$, we make use of the following lemma.

Lemma 6.2 *There exists a positive constant C_3 which only depends on ν, p_0, q_0, p, q such that $B_{j+1} \leq C_3 M(\varepsilon)^{\nu-1} B_j$, for every $j \in \mathbf{N}$.*

Proof. We obtain exactly in the same way as in the proof of Theorem 5.1 and Lemma 6.1 that

$$\begin{aligned}
&\|u_{j+2}(t,\cdot) - u_{j+1}(t,\cdot)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}} \\
&\leq \int_0^t \|\exp(-(t-\tau)\mathcal{A})\big(G(w(\cdot),u_{j+1}(t,\cdot)) - G(w(\cdot),u_j(t,\cdot))\big)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}} d\tau \\
&\leq C\int_0^t (t-\tau)^{\frac{1}{2}(\frac{n}{p}-\frac{(n-2)\nu}{p}-\frac{2}{p})} \|G(w(\cdot),u_{j+1}(t,\cdot)) \\
&\qquad\qquad - G(w(\cdot),u_j(t,\cdot))\|_{\mathcal{M}^{\frac{n}{p}-\frac{(n-2)\nu}{p}}_{p,q}} d\tau \\
&\leq C\int_0^t (t-\tau)^{-\frac{(n-2)(\nu-1)}{2p}} \big(\|u_{j+1}(t,\cdot)\|^{\nu-1}_{\mathcal{M}^{\frac{2}{p}}_{p,q}} + \|u_j(t,\cdot)\|^{\nu-1}_{\mathcal{M}^{\frac{2}{p}}_{p,q}}\big) \\
&\qquad\qquad \|u_{j+1}(t,\cdot) - u_j(t,\cdot)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}} d\tau \\
&\leq C\int_0^t (t-\tau)^{-\frac{(n-2)(\nu-1)}{2p}} \tau^{-\nu(\frac{n}{2p_0}-\frac{n-2}{2p})} (A^{\nu-1}_{j+1} + A^{\nu-1}_j) B_j d\tau \\
&\leq C_3 M(\varepsilon)^{\nu-1} B_j t^{-(\frac{n}{2p_0}-\frac{n-2}{2p})},
\end{aligned}$$

which implies the conclusion. □

Let $\varepsilon_0 < \frac{\nu-1}{C_1\nu}(2C_3)^{-\frac{1}{\nu-1}}$ if necessary. Then we have $B_{j+1} \leq \frac{1}{2}B_j$, for every $j \in \mathbf{N}$. From this fact, we conclude that $\sum_{j=0}^{\infty} B_j < \infty$, which implies that the sequence $u_j(t,x)$ converges to a function $u(t,x)$ as $j \to \infty$ such that

$$\sup_{t>0} t^{\frac{n}{2p_0}-\frac{n-2}{2p}} \|u(t,\cdot)\|_{\mathcal{M}^{\frac{2}{p}}_{p,q}} \leq M(\varepsilon) < M_0.$$

It follows that $u(t,x)$ satisfies (3.4) for every $T' \in (0,\infty)$ and (3.5). It is also easy to see that $u(t,x)$ enjoys the integral equation (5.1) on $(0,\infty)$. Hence Theorem 5.1 implies that $u(t,x)$ satisfies (3.3).

Moreover, the estimate (5.3) in Lemma 5.2 implies that $u(t,x)$ satisfies the estimate (3.6) for every $T' \in (0,\infty)$, and the estimate (5.4) implies that $u(t,x)$ satisfies the estimate (3.7) with $\psi_\sigma(\varepsilon) = C_\sigma(\varepsilon + M(\varepsilon)^\nu)$.

From the discussion above, we have proved the existence of time-global solution of the Cauchy problem (3.2)–(3.3) with all the properties required in Theorem 1.2 and Theorem 1.4. Next we show the uniqueness of such solution $u(t,x)$ in the similar way as in Zhou [27].

Proof of Theorem 1.3. Let $\tilde{u}(t,x)$ be another solution of (3.3) on $(0,T) \times \mathbf{R}^n$ satisfying (3.4) for every $T' \in (0,T)$ and (3.5) such that $\tilde{u}(t,\cdot) - b \longrightarrow 0$ in the topology of $\mathcal{M}_{p,q}^{\frac{n}{p}-\frac{(n-2)\nu}{p}}$ as $t \to 0+$. Then Theorem 5.1 implies that $\tilde{u}(t,x)$ also solves the integral equation (5.1) on $(0,T)$.

Let T_0 be a positive number less than T such that

$$M = \sup_{0<t\le T_0} t^{\frac{n}{2p_0}-\frac{n-2}{2p}} \|\tilde{u}(t,\cdot)\|_{\mathcal{M}_{p,q}^{\frac{2}{p}}} < M_0.$$

Putting $\bar{u}(t,x) = u(t,x) - \tilde{u}(t,x)$ and $A = \sup_{0<t\le T_0} t^{\frac{n}{2p_0}-\frac{n-2}{2p}} \|\bar{u}(t,\cdot)\|_{\mathcal{M}_{p,q}^{\frac{2}{p}}}$, we obtain $A < \frac{1}{2}A$ in the same way as in Lemma 6.2. Hence we have $\tilde{u}(t,x) \equiv u(t,x)$ on $(0,T_0] \times \mathbf{R}^n$.

Next, take $T' \in (0,T)$ arbitrarily, $s_0 \equiv \frac{n}{2p_0} - \frac{n-2}{2p}$, let

$$M' = \sup_{0<t\le T'} t^{s_0} \|\tilde{u}(t,\cdot)\|_{\mathcal{M}_{p,q}^{\frac{2}{p}}}$$

and

$$A(\tau) = \sup_{0<t\le \tau} t^{s_0} \|\bar{u}(t,\cdot)\|_{\mathcal{M}_{p,q}^{\frac{2}{p}}}$$

for every $\tau \in (0,T']$. Then we have

$$t^{-s_0}A(t) \le C_3{}' \int_0^t (t-\tau)^{(\nu-1)s_0-1}\tau^{-\nu s_0}(M_0^{\nu-1} + M'^{\nu-1})A(\tau)d\tau$$

in the same way as in Lemma 6.2. Now suppose that $A(T_1) = 0$ with some T_1 such that $T_0 \le T_1 < T'$, then we have

$$A(t) \le A(t)C_4 t^{s_0} \int_{T_1}^t (t-\tau)^{(\nu-1)s_0-1}\tau^{-\nu s_0}d\tau.$$

Choose a positive number δ such that

$$0 < \delta < \frac{T_0}{2}\min\left\{1, \left(\frac{(\nu-1)s_0}{C_4}\right)^{\frac{1}{(\nu-1)s_0}}\right\}.$$

Then $t = \min\{T_1 + \delta,\ T'\}$ satisfies $t \le 2T_1$. It follows that

$$A(t) \le A(t)\frac{C_4}{(\nu-1)s_0}\left(\frac{2\delta}{T_1}\right)^{(\nu-1)s_0} < A(t).$$

Therefore we obtain that $A(T_1) = 0$ implies $A(t) = 0$. Starting at $T_1 = T_0$ and repeating this process, we arrive at $t = T'$ after finite steps, which implies that $u(t,x) \equiv \tilde{u}(t,x)$ on $(0,T'] \times \mathbf{R}^n$. Since $T' \in (0,T)$ is arbitrary, it follows that $u(t,x) \equiv \tilde{u}(t,x)$ on $(0,T) \times \mathbf{R}^n$.

This completes the proof of Theorem 1.3. □

In the following part, we will prove the main results for the second case.

Case 2: $\nu > 2$, $\nu \in \mathbf{R}$.

As well as the fisrt case, we construct the time-global solution of the integral equation (5.1) by the same method of succesive approximation.

Define the sequence of functions $\{u_j(t,x)\}_{j=0}^{\infty}$ inductively by

$$\begin{aligned} u_0(t,\cdot) &= \exp(-t\mathcal{A})b, \\ u_{j+1}(t,\cdot) &= u_0(t,\cdot) - \int_0^t \exp(-(t-\tau)\mathcal{A})G(w(\cdot),u_j(\tau,\cdot))d\tau. \end{aligned}$$

Put

$$A_j = \max\left\{\sup_{t>0}\|u_j(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}},\ \sup_{t>0} t^{\frac{n}{2p_0}-\frac{n}{2p}}\|u_j(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}^{\frac{n}{p_0}-\frac{n}{p}}}\right\},$$

$$\begin{aligned} B_j = \max\Bigg\{&\sup_{t>0}\|u_{j+1}(t,\cdot)-u_j(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}}, \\ &\sup_{t>0} t^{\frac{n}{2p_0}-\frac{n}{2p}}\|u_{j+1}(t,\cdot)-u_j(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}^{\frac{n}{p_0}-\frac{n}{p}}}\Bigg\}. \end{aligned}$$

Suppose that ν, p_0, q_0, p, q_1 and σ_0 satisfy the assumptions of Theorem 1.5. Let $f(x)$ and $w(x)$ be the same as in the last section. For any given function $b(x) \in \mathcal{M}_{p_0,q_1}$ such that $\|b(x)\|_{\mathcal{M}_{p_0,q_1}} = \varepsilon < \varepsilon_0$, we have the following estimates.

Lemma 6.3 *There exist positive constant C_1, C_2 which only depend on ν, p_0, q_0, p, q_1 such that*

(1) $A_0 \le C_1\varepsilon$.

(2) $A_{j+1} \le C_1\varepsilon + C_2(\|w\|_{\mathcal{M}_{p_0,q_0}} + A_j)^{\nu-2}A_j^2$, *for every* $j = 0,1,2,\ldots$.

Proof. (1) It follows from Proposition 5.2 that $\|u_0(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}} \le C_1\varepsilon$ and

$$\|u_0(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}^{\frac{n}{p_0}-\frac{n}{p}}} \le C_1\varepsilon t^{-(\frac{n}{2p_0}-\frac{n}{2p})}.$$

This implies the conclusion.

(2) We get the following estimates in the same way as in (5.2).

$$\begin{aligned}
&\|u_{j+1}(t,\cdot)-u_0(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}}\\
&\le \int_0^t \|\exp(-(t-\tau)\mathcal{A})G(w(\cdot),u_j(\tau,\cdot))\|_{\mathcal{M}_{p_0,q_1}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{\frac{n}{p_0}-\frac{n}{p}-1}\|G(w(\cdot),u_j(\tau,\cdot))\|_{\mathcal{M}_{p_0,q_1}^{\frac{2n}{p_0}-\frac{2n}{p}-2}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{\frac{n}{p_0}-\frac{n}{p}-1}(\|w\|_{\mathcal{M}_{p_0,q_0}}+\|u_j\|_{\mathcal{M}_{p_0,q_1}})^{\nu-2}\Big(\|u_j\|_{\mathcal{M}_{p_0,q_1}^{\frac{n}{p_0}-\frac{n}{p}}}\Big)^2 d\tau\\
&\le C(\|w\|_{\mathcal{M}_{p_0,q_0}}+A_j)^{\nu-2}A_j^2\int_0^t (t-\tau)^{\frac{n}{p_0}-\frac{n}{p}-1}\tau^{\frac{n}{p}-\frac{n}{p_0}}\,d\tau\\
&\le C_2(\|w\|_{\mathcal{M}_{p_0,q_0}}+A_j)^{\nu-2}A_j^2
\end{aligned}$$

and

$$\begin{aligned}
&\|u_{j+1}(t,\cdot)-u_0(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}^{\frac{n}{p_0}-\frac{n}{p}}}\\
&\le \int_0^t \|\exp(-(t-\tau)\mathcal{A})G(w(\cdot),u_j(\tau,\cdot))\|_{\mathcal{M}_{p_0,q_1}^{\frac{n}{p_0}-\frac{n}{p}}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{\frac{n}{2p_0}-\frac{n}{2p}-1}\|G(w(\cdot),u_j(\tau,\cdot))\|_{\mathcal{M}_{p_0,q_1}^{\frac{2n}{p_0}-\frac{2n}{p}-2}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{\frac{n}{2p_0}-\frac{n}{2p}-1}(\|w\|_{\mathcal{M}_{p_0,q_0}}+\|u_j\|_{\mathcal{M}_{p_0,q_1}})^{\nu-2}\Big(\|u_j\|_{\mathcal{M}_{p_0,q_1}^{\frac{n}{p_0}-\frac{n}{p}}}\Big)^2 d\tau\\
&\le C(\|w\|_{\mathcal{M}_{p_0,q_0}}+A_j)^{\nu-2}A_j^2\int_0^t (t-\tau)^{\frac{n}{2p_0}-\frac{n}{2p}-1}\tau^{\frac{n}{p}-\frac{n}{p_0}}\,d\tau\\
&\le C_2(\|w\|_{\mathcal{M}_{p_0,q_0}}+A_j)^{\nu-2}A_j^2 t^{-(\frac{n}{2p_0}-\frac{n}{2p})},
\end{aligned}$$

for every $j=0,1,2,\ldots$. This gives the conclusion (2). □

For the convergence of the sequence $\{u_j(t,x)\}_{j=0}^{\infty}$, we need the following lemma.

Lemma 6.4 *There exists a positive constant C_3 which only depends on ν, p_0, q_0, p, q_1 such that*:

$$B_{j+1} \le C_3(\|w\|_{\mathcal{M}_{p_0,q_0}} + A_j + A_{j+1})^{\nu-1} B_j,$$

for every $j \in \mathbf{N}$.

Proof. Fisrtly we have the folloing inequality

$$\begin{aligned}
&|G(w(\cdot), u_{j+1}(t,\cdot)) - G(w(\cdot), u_j(t,\cdot))| \\
&\quad \le C(|w(\cdot)| + |u_{j+1}(t,\cdot)| + |u_j(t,\cdot)|)^{\nu-2} \\
&\qquad\qquad |u_{j+1}(t,\cdot) + u_j(t,\cdot)||u_{j+1}(t,\cdot) - u_j(t,\cdot)|.
\end{aligned}$$

Then we obtain exactly in the same way as in the proof of Theorem 5.2 and Lemma 6.3 that

$$\begin{aligned}
&\|u_{j+2}(t,\cdot) - u_{j+1}(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}} \\
&\le \int_0^t \|\exp(-(t-\tau)\mathcal{A})\big(G(w(\cdot), u_{j+1}(t,\cdot)) - G(w(\cdot), u_j(t,\cdot))\big)\|_{\mathcal{M}_{p_0,q_1}} d\tau \\
&\le C\int_0^t (t-\tau)^{\frac{n}{p_0}-\frac{n}{p}-1} \|\big(G(w(\cdot), u_{j+1}(t,\cdot)) - G(w(\cdot), u_j(t,\cdot))\big)\|_{\mathcal{M}^{\frac{2n}{p_0}-\frac{2n}{p}-2}_{p_0,q_1}} d\tau \\
&\le C\int_0^t (t-\tau)^{\frac{n}{p_0}-\frac{n}{p}-1} (\|w\|_{\mathcal{M}_{p_0,q_0}} + \|u_{j+1}\|_{\mathcal{M}_{p_0,q_1}} + \|u_j\|_{\mathcal{M}_{p_0,q_1}})^{\nu-2} \\
&\qquad \times \big(\|u_{j+1}\|_{\mathcal{M}^{\frac{n}{p_0}-\frac{n}{p}}_{p_0,q_1}} + \|u_j\|_{\mathcal{M}^{\frac{n}{p_0}-\frac{n}{p}}_{p_0,q_1}}\big)\|u_{j+1} - u_j\|_{\mathcal{M}^{\frac{n}{p_0}-\frac{n}{p}}_{p_0,q_1}} d\tau \\
&\le C(\|w\|_{\mathcal{M}_{p_0,q_0}} + A_{j+1} + A_j)^{\nu-2}(A_{j+1} + A_j)B_j \int_0^t (t-\tau)^{\frac{n}{p_0}-\frac{n}{p}-1}\tau^{\frac{n}{p}-\frac{n}{p_0}} d\tau \\
&\le C_3(\|w\|_{\mathcal{M}_{p_0,q_0}} + A_j + A_{j+1})^{\nu-1} B_j,
\end{aligned}$$

and

$$\begin{aligned}
&\|u_{j+2}(t,\cdot) - u_{j+1}(t,\cdot)\|_{\mathcal{M}^{\frac{n}{p_0}-\frac{n}{p}}_{p_0,q_1}} \\
&\le \int_0^t \|\exp(-(t-\tau)\mathcal{A})\big(G(w(\cdot), u_{j+1}(t,\cdot)) - G(w(\cdot), u_j(t,\cdot))\big)\|_{\mathcal{M}^{\frac{n}{p_0}-\frac{n}{p}}_{p_0,q_1}} d\tau \\
&\le C\int_0^t (t-\tau)^{\frac{n}{2p_0}-\frac{n}{2p}-1} \|\big(G(w(\cdot), u_{j+1}(t,\cdot)) - G(w(\cdot), u_j(t,\cdot))\big)\|_{\mathcal{M}^{\frac{2n}{p_0}-\frac{2n}{p}-2}_{p_0,q_1}} d\tau \\
&\le C(\|w\|_{\mathcal{M}_{p_0,q_0}} + A_{j+1} + A_j)^{\nu-2}(A_{j+1} + A_j)B_j \int_0^t (t-\tau)^{\frac{n}{2p_0}-\frac{n}{2p}-1}\tau^{\frac{n}{p}-\frac{n}{p_0}} d\tau
\end{aligned}$$

$$\leq C_3(\|w\|_{\mathcal{M}_{p_0,q_0}} + A_j + A_{j+1})^{\nu-1} B_j t^{-(\frac{n}{2p_0} - \frac{n}{2p})},$$

for every $j = 0, 1, 2, \dots$, which implies the conclusion. □

Now we decide the positive numbers ε_0 and M_0 required in Theorem 1.5. Observing that the constants C_2 and C_3 are independent of the norm $\|w\|_{\mathcal{M}_{p_0,q_0}} (\leq 1/2)$, there exists a contant $C_4 \geq 2$ such that:

$$\begin{aligned} A_{j+1} &\leq C_1\varepsilon + 2C_4 A_j^2, \\ B_{j+1} &\leq C_4(\|w\|_{\mathcal{M}_{p_0,q_0}} + A_j + A_{j+1})^{\nu-1} B_j, \end{aligned}$$

for every $j = 0, 1, 2, \dots$. Here $A_j < 1/2$ will be determined as follows. Let the constant δ_1 smaller if neccessary such that $\|w\|_{\mathcal{M}_{p_0,q_0}} \leq 1/2C_4$, if $\varepsilon < \varepsilon_0 = 1/8C_1C_4$, then it follows from the above inequality that

$$A_j \leq \psi(\varepsilon) = \frac{1 - \sqrt{1 - 8C_1C_4\varepsilon}}{4C_4} < M_0 = \psi(\varepsilon_0) = \frac{1}{4C_4} < \frac{1}{2},$$

for every j by induction.

Moreover, we get the estimate of B_j:

$$B_{j+1} \leq C_4 \left(\frac{1}{2C_4} + \frac{1}{2C_4} \right)^{\nu-1} B_j \leq \left(\frac{1}{C_4} \right)^{\nu-2} B_j,$$

for every j, where $(\frac{1}{C_4})^{\nu-2} < 1$.

From this fact, we conclude that $\sum_{j=0}^{\infty} B_j < \infty$, which implies that the sequence $u_j(t,x)$ converges to a function $u(t,x)$ in $\mathcal{M}_{p_0,q_1} \cap \mathcal{M}_{p_0,q_1}^{\frac{n}{p_0} - \frac{n}{p}}$ as $j \to \infty$ such that

$$\max\left\{ \sup_{t>0} \|u(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}}, \sup_{t>0} t^{\frac{n}{2p_0} - \frac{n}{2p}} \|u(t,\cdot)\|_{\mathcal{M}_{p_0,q_1}^{\frac{n}{p_0} - \frac{n}{p}}} \right\} \leq \psi(\varepsilon_0) < M_0.$$

It follows that $u(t,x)$ satisfies (3.8) for every $T' \in (0,\infty)$ and (3.9). It is also easy to see that $u(t,x)$ enjoys the integral equation (5.1) on $(0,\infty)$. Hence Theorem 5.1 implies that $u(t,x)$ satisfies the differential equation (3.3).

Moreover, the estimate (5.6) in Lemma 5.3 implies that $u(t,x)$ satisfies the estimate (3.10) for every $T' \in (0,\infty)$, and the estimate (5.7) implies that $u(t,x)$ satisfies the estimate (3.11) with $\psi_\sigma(\varepsilon) = C_\sigma(\psi(\varepsilon)^\nu + \omega(\delta)^{\nu-2}\psi(\varepsilon)^2 + \varepsilon)$.

Therefore we have completed the proofs of Theorem 1.5 and Theorem 1.7. The uniqueness in Theorem 1.6 is easy to be obtained in the same way as in Theorem 1.3.

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