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On purifiable torsion-free rank-one subgroups

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Abstract. First, we give a necessary and sufficient condition for a torsion-free rank-one subgroup of an arbitrary abelian group to be purifiable in a given group and show that all pure hulls of a purifiable torsion-free rank-one subgroup are isomorphic. Next, we show that if a $T(G)$-high subgroup A of an abelian group G is purifiable in G, then there exists a subgroup T' of $T(G)$ such that $G=H\oplus T'$ for every pure hull H of A in G. An abelian group G is said to be a strongly ADE decomposable group if there exists a purifiable $T(G)$-high subgroup of G. We present an example G such that not all $T(G)$-high subgroups of a strongly ADE decomposable group G are purifiable in G. Moreover, we characterize the strongly ADE decomposable groups of torsion-free rank 1. Finally, we use previous results to give a necessary and sufficient condition for an abelian group of torsion-free rank 1 to be splitting.

Key words: purifiable subgroup, strongly ADE decomposable group, height-matrix, pure hull, splitting mixed group.

A subgroup A of an arbitrary abelian group G is said to be *purifiable* in G if there exists a pure subgroup H of G containing A which is minimal among the pure subgroups of G that contain A. Such a subgroup H is said to be a *pure hull* of A in G.

Hill and Megibben [7] established properties of pure hulls of p-groups and characterized the p-groups for which all subgroups are purifiable.

Next, Benabdallah and Irwin [2] introduced the concept of almost-dense subgroups of p-groups and used it to give the structure of pure hulls of purifiable subgroups of p-groups.

Furthermore, Benabdallah and Okuyama [3] introduced a new invariant, the so-called *n-th overhangs* of a subgroup of a p-group, which are related to the n-th relative Ulm-Kaplansky invariant. Using it, they obtained a necessary condition for subgroups to be purifiable.

Benabdallah, Charles, and Mader [1] introduced the concept of maximal vertical subgroups supported by a given subsocle of a p-group and characterized the p-groups for which the necessary condition given in [3] is

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also sufficient.

As for isomorphism of pure hulls, we obtained several results in [10] and [11]. Other results about purifiable subgroups of p-groups are contained in [4], [5], [8], [9], [10], and [11].

Recently, in [13], we extended the concept of almost-dense subgroups from p-groups to arbitrary abelian groups and began to study purifiable subgroups of arbitrary abelian groups. Though we characterized the groups for which all subgroups are purifiable, we have not yet given a necessary and sufficient condition to be purifiable even for torsion-free subgroups of arbitrary abelian groups.

In this note, in Section 2, we determine the structure of pure hulls of purifiable torsion-free rank-one subgroups A of arbitrary abelian groups G. Such pure subgroups H are ADE groups. We also began to study ADE groups in [12].

Let G be an arbitrary abelian group, $g \in G$, and p_n $(n \geqq 1)$ a listing of all primes in increasing order. Then we associate the *height-matrix* $\mathbb{H}(g)$, an infinite matrix with ordinal numbers for entries, as follows;

$$\mathbb{H}(g) = \begin{pmatrix} h^*_{p_1}(g) & h^*_{p_1}(p_1 g) & \dots & h^*_{p_1}(p_1^k g) & \dots \\ \dots & & & & \\ h^*_{p_n}(g) & h^*_{p_n}(p_n g) & \dots & h^*_{p_n}(p_n^k g) & \dots \\ \dots & & & & \end{pmatrix}.$$

The element in the (n,k)-position of $\mathbb{H}(g)$ is the generalized p_n-height of $p_n^k g$, for all $n \geqq 1$ and $k \geqq 0$. The element in the (n,k)-position of $\mathbb{H}(g)$ is denoted by $\mathbb{H}_{n,k}(g)$. The nth row of $\mathbb{H}(g)$ is called the p_n-indicator of g. $\mathbb{H}_{n,k}(g) = \infty$ means that $p_n^k g$ is an element of the maximal p-divisible subgroup of G.

In Section 3, we give a necessary and sufficient condition for a torsion-free rank-one subgroup A of an arbitrary abelian group G to be purifiable in G. In fact, A is purifiable in G if and only if, for every $a \in A$ and all $n \geqq 1$, the p_n-indicator of a is one of the following two types:

(1) there exists a nonnegative integer r_n such that $\mathbb{H}_{n,r_n}(a) < \omega$ and $\mathbb{H}_{n,r_n+i}(a) = \mathbb{H}_{n,r_n}(a) + i$ for all $i \geqq 0$;
(2) there exists a nonnegative integer r_n such that $\mathbb{H}_{n,r_n}(a) = \infty$ and if $r_n > 0$, then $\mathbb{H}_{n,r_n-1}(a) < \omega$.

Moreover, we show that all pure hulls of A are isomorphic.

In Section 4, we study purifiable $T(G)$-high subgroups of arbitrary abelian groups G. We prove that if a $T(G)$-high subgroup N of G is purifiable in G, then there exists a subgroup T' of $T(G)$ such that $G = H \oplus T'$ for every pure hull H of N in G and all pure hulls of N in G are isomorphic.

An arbitrary abelian group G is said to be a *strongly ADE decomposable group* if there exists a $T(G)$-high subgroup of G to be purifiable in G. We know an ADE group that is not splitting. (See [6, Vol. 2, Example 2, p.186]). Though splitting groups are strongly ADE decomposable groups, the converse is not true.

In Section 5, we present a strongly ADE decomposable group G of torsion-free rank 1 for which not all $T(G)$-high subgroups are purifiable in G. We also characterize the groups G of torsion-free rank 1 for which all T-high subgroups of G are purifiable in G. Moreover, we give a characterization of strongly ADE decomposable groups of torsion-free rank 1. In fact, an arbitrary abelian group G of torsion-free rank 1 is ADE decomposable if and only if there exists an element $a \in G \setminus T(G)$ such that, for all $n \geqq 1$, the p_n-indicator of a is one of the following two types:

(1) there exists a nonnegative integer r_n such that $\mathbb{H}_{n,r_n}(a) < \omega$ and $\mathbb{H}_{n,r_n+i}(a) = \mathbb{H}_{n,r_n}(a) + i$ for all $i \geqq 0$;
(2) there exists a nonnegative integer r_n such that $\mathbb{H}_{n,r_n}(a) = \infty$ and if $r_n > 0$, then $\mathbb{H}_{n,r_n-1}(a) < \omega$.

In [14], Stratton established a necessary and sufficient condition for arbitrary abelian groups of torsion-free rank 1 to be splitting. We use the previous results to obtain the same result. In fact, an arbitrary abelian group G of torsion-free rank 1 is splitting if and only if there exists an element $a \in G \setminus T(G)$ such that, for all $n \geqq 1$, the p_n-indicator of a is one of the following two types:

(1) $\mathbb{H}_{n,0}(a) < \omega$ and $\mathbb{H}_{n,k}(a) = \mathbb{H}_{n,0}(a) + k$ for all $k \geqq 0$;
(2) $\mathbb{H}_{n,0}(a) = \infty$.

From the previous two characterizations, we can see that ADE decompositions are weaker than splitting.

All groups considered are arbitrary abelian groups. The terminologies and notations not expressly introduced follow the usage of [6]. Throughout this note, let p be a prime. The p-part and the torsion part $T(G)$ of any arbitrary abelian group G is denoted by G_p and T, respectively. The p-height of an element g of G means the generalized p-height, denoted by $h_p(g)$ instead of $h_p^*(g)$.

1. Notation and Basics

We recall definitions and properties mentioned in [13]. We frequently use them in this note. Throughout this section, let G be an arbitrary abelian group and A a subgroup of G.

Definition 1.1 A is said to be *p-almost-dense* in G if, for every p-pure subgroup K of G containing A, the torsion part of G/K is p-divisible. Moreover, A is said to be *almost-dense* in G if A is p-almost-dense in G for every prime p.

Proposition 1.2 [13, Proposition 1.3] *A is p-almost-dense in G if and only if, for all integers $n \geqq 0$, $A + p^{n+1}G \supseteq p^nG[p]$.*

Proposition 1.3 [13, Proposition 1.4] *The following properties are equivalent:*

(1) *A is almost-dense in G;*

(2) *for all integers $n \geqq 0$ and all primes p, $A + p^{n+1}G \supseteq p^nG[p]$;*

Definition 1.4 A is said to be *p-purifiable*[*purifiable*] in G if, among the p-pure[pure] subgroups of G containing A, there exists a minimal one. Such a minimal p-pure[pure] subgroup is called a *p-pure*[*pure*] *hull* of A.

Proposition 1.5 [13, Theorem 1.8] *There exists no proper p-pure subgroup of G containing A if and only if the following conditions hold:*

(1) *A is p-almost-dense in G;*

(2) *G/A is a p-group;*

(3) *there exists a nonnegative integer m such that $p^mG[p] \subseteq A$.*

Proposition 1.6 [13, Theorem 1.11] *There exists no proper pure subgroup of G containing A if and only if the following three conditions hold:*

(1) *A is almost-dense in G;*

(2) *G/A is torsion;*

(3) *for every prime p, there exists a nonnegative integer m_p such that*

$$p^{m_p}G[p] \subseteq A.$$

Proposition 1.7 [13, Theorem 1.12] *A is purifiable in G if and only if, for every prime p, A is p-purifiable in G.*

Definition 1.8 For every nonnegative integer n, we define the *n-th p-overhang* of A in G to be the verctor space

$$V_{p,n}(G,A) = \frac{(A + p^{n+1}G) \cap p^n G[p]}{(A \cap p^n G)[p] + p^{n+1}G[p]}.$$

Moreover, A is said to be *p-vertical* in G if $V_{p,n}(G,A) = 0$ for all $n \geqq 0$.

It is convenient to use the following notations for the numerator and the denominator of $V_{p,n}(G,A)$:

$$A_G^n(p) = (A + p^{n+1}G) \cap p^n G[p] = ((A \cap p^n G) + p^{n+1}G)[p]$$

and

$$A_n^G(p) = (A \cap p^n G)[p] + p^{n+1}G[p].$$

If A is torsion-free, then $A_n^G(p) = p^{n+1}G[p]$. Moreover, if A is torsion-free and p-almost-dense in G, then

$$V_{p,n}(G,A) = \frac{p^n G[p]}{p^{n+1}G[p]}.$$

Hence $V_{p,n}(G,A)$ is nth Ulm-Kaplansky invariant of G_p.

Proposition 1.9 [13, Lemma 4.2 (1)] *$V_{p,m+n}(G,A) = V_{p,n}(p^m G, A \cap p^m G)$ for all $n, m \geqq 0$.*

Proposition 1.10 [13, Proposition 2.2] *For every p-pure subgroup K of G containing A,*

$$V_{p,n}(G,A) \cong V_{p,n}(K,A)$$

for all $n \geqq 0$.

Proposition 1.10 leads to the following intrinsic necessary condition for p-purifiability of subgroups.

Proposition 1.11 [13, Theorem 2.3] *If A is p-purifiable in G, then there exists a nonnegative integer m such that $V_{p,n}(G,A) = 0$ for all $n \geqq m$.*

For convenience, we call a subgroup A an *eventually p-vertical* subgroup if there exists a nonnegative integer m such that $V_{p,n}(G,A) = 0$ for all $n \geqq m$, and A is said to be *p-neat* in G if $A \cap pG = pA$.

Proposition 1.12 [13, Proposition 2.6] *Let A be p-neat in G. Then A is p-pure in G if and only if A is p-vertical in G.*

Proposition 1.13 [13, Theorem 2.8] *If $Ap = 0$, then following properties are equivalent*:
(1) *A is p-vertical in G*;
(2) $(A + p^n G)[p] = A[p] + p^n G[p]$ *for all* $n \geqq 1$;
(3) *if $a \in A$ such that $h_p(a) < \omega$, then $h_p(pa) = h_p(a) + 1$.*

Proposition 1.14 [13, Theorem 4.1 (2)] *If $A \cap p^m G$ is p-purifiable in $p^m G$ for some $m \geqq 0$, then A is p-purifiable in G.*

2. The Structure of Pure Hulls

In this section, we consider the structure of pure hulls of purifiable torsion-free rank-one subgroups of arbitrary abelian groups. Before doing it, we give a useful lemma.

Lemma 2.1 *Let G be an abelian group and A a torsion-free rank-one subgroup of G. Then we have*:
(1) *for all* $n \geqq 1$, $\dim(V_{p,n}(G, A)) \leqq 1$;
(2) $p^\omega(G/A)[p] \cap \frac{G[p]+A}{A} \subseteq \frac{p^\omega G+A}{A}[p]$;
(3) *if A is p-neat in G, then* $p^\omega(G/A)[p] = \frac{p^\omega G+A}{A}[p]$ *and*

$$\dim\left(\frac{p^\omega(G/A)[p]}{(p^\omega G[p] + A)/A}\right) \leqq 1.$$

Proof. (1) Suppose that $\dim(V_{p,n}(G, A)) > 1$. There exist $x_i \in A_G^n(p) \setminus A_n^G(p)$ for $i = 1, 2$ and a subgroup S of $V_{p,n}(G, A)$ such that

$$V_{p,n}(G, A) = \langle x_1 + A_n^G(p)\rangle \oplus \langle x_2 + A_n^G(p)\rangle \oplus S.$$

For $i = 1, 2$, there exist $a_i \in A$ and $g_i \in G$ such that $x_i = a_i + p^{n+1} g_i$. Since $r(A) = 1$, there exist integers α_i for $i = 1, 2$ such that $(\alpha_1, \alpha_2) = 1$ and $\alpha_1 a_1 + \alpha_2 a_2 = 0$. Then $\alpha_1 x_1 + \alpha_2 x_2 = p^{n+1}(\alpha_1 g_1 + \alpha_2 g_2) \in p^{n+1} G[p] \subseteq A_n^G(p)$. This is a contradiction. Hence $\dim(V_{p,n}(G, A)) \leqq 1$ for all $n \geqq 1$.

(2) Let $x + A \in p^\omega(G/A)$ with $x \in G[p]$. Without loss of generality, we may assume that $h_p(x) < \omega$. Let $r = h_p(x)$. For all $n \geqq 0$, there exist $b_n \in A$ and $h_n \in G$ such that $x = b_n + p^{r+n+1} h_n$. Since $r(A) = 1$, there exist integers β_n and γ_n such that $(\beta_n, \gamma_n) = 1$ and $\beta_n b_0 + \gamma_n b_n = 0$. Then

$$(\beta_n + \gamma_n)x = p^{r+1}(\beta_n h_0 + \gamma_n p^n h_n).$$

By a similar argument, $(\beta_n, p) = (\gamma_n, p) = 1$ and p divides $\beta_n + \gamma_n$. Hence $p^{r+1}(\beta_n h_0 + \gamma_n p^n h_n) = 0$ and $p^{r+1} h_0 \in p^\omega G$.

(3) If A is p-neat in G, then $(G/A)[p] = \frac{G[p]+A}{A}$. By (2), it is immediate that $p^\omega(G/A)[p] = \frac{p^\omega G + A}{A}[p]$. Suppose that $\dim(\frac{p^\omega(G/A)[p]}{(p^\omega G[p]+A)/A}) > 1$. We can write

$$p^\omega(G/A)[p] = \langle y_1 + A\rangle \oplus \langle y_2 + A\rangle \oplus S' \oplus \frac{p^\omega G[p] + A}{A}$$

for some $y_i \in G[p]$ for $i = 1, 2$ and some subsocle S' of G/A. Then $h_p(y_i) < \omega$ for $i = 1, 2$. Let $r_i = h_p(y_i)$ for $i = 1, 2$. For $i = 1, 2$, there exist $c_i \in A$ and $k_i \in p^\omega G$ such that

$$y_i = c_i + k_i.$$

Since $r(A) = 1$, there exist integers δ_i for $i = 1, 2$ such that $(\delta_1, \delta_2) = 1$ and $\delta_1 c_1 + \delta_2 c_2 = 0$. Then

$$\delta_1 y_1 + \delta_2 y_2 \in p^\omega G[p].$$

This is a contradiction. □

By Proposition 1.6 and [12, Proposition 2.2], we have:

Proposition 2.2 *Let G be an abelian group and A a subgroup of G. If A is purifiable in G and H is a pure hull of A in G, then we have:*
(1) *A is almost-dense in H;*
(2) *H/A is torsion;*
(3) *for every prime p, there exists a nonnegative integer t_p such that*

$$p^{t_p} H[p] \subset A;$$

(4) *if p is a prime such that $pA = A$, then $H_p = 0$.*

Standing Assumption 2.3 Let G be an abelian group and A a torsion-free rank-one subgroup of G. Suppose that A is purifiable in G. Let H be a pure hull of A in G and N a $T(H)$-high subgroup of H containing A.

We recall the definition of an ADE group.

Definition 2.4 Let A be a torsion-free group. An abelian group G is said to be an *almost dense extension group* (*ADE group*) of A if A is almost-dense and T-high in G. Such a subgroup A is called a *moho* subgroup of G.

It is immediate that H as in Proposition 2.2 is an ADE group with A as a moho subgroup. By Lemma 2.1 and Proposition 2.2, we have:

Lemma 2.5 *Assume* 2.3. *For every prime p such that $H_p \neq 0$, there exist a positive integer n_p and $y'_{pi} \in H_p$ for $1 \leqq i \leqq n_p$ such that*

$$H_p = \bigoplus_{i=1}^{n_p} \langle y'_{pi} \rangle.$$

Setting $p^{t_{pi}} = o(y'_{pi})$ for $1 \leqq i \leqq n_p$, $t_{pi} < t_{pi+1}$ for $1 \leqq i \leqq n_p - 1$.

Lemma 2.6 *Assume* 2.3. *For every prime p such that $H_p \neq 0$, let H_p be a p-group as in Lemma* 2.5. *Then $h_p^{H/N}(p^{t_{pi}-1}y'_{pi} + N) = h_p^{H/A}(p^{t_{pi}-1}y'_{pi} + A)$ for $1 \leqq i \leqq n_p$.*

Proof. Since A is almost-dense in H, $t_{pi} \leqq h_p^{H/A}(p^{t_{pi}-1}y'_{pi} + A) \leqq h_p^{H/N}(p^{t_{pi}-1}y'_{pi} + N)$. Let d_{pi} and e_{pi} be ordinals such that $d_{pi} \geqq t_{pi}$ and $e_{pi} \geqq t_{pi}$. Suppose that there exist $a_{pi} \in A$, $b_{pi} \in N$, $g_{pi} \in p^{d_{pi}}H$, and $h_{pi} \in p^{e_{pi}}H$ such that $p^{t_{pi}-1}y'_{pi} = a_{pi} + g_{pi} = b_{pi} + h_{pi}$. Since $r(N) = 1$, there exist integers α_{pi}, β_{pi} such that $(\alpha_{pi}, \beta_{pi}) = 1$ and $\alpha_{pi}a_{pi} + \beta_{pi}b_{pi} = 0$. Then

$$(\alpha_{pi} + \beta_{pi})p^{t_{pi}-1}y'_{pi} = \alpha_{pi}g_{pi} + \beta_{pi}h_{pi}.$$

By a similar argument of Lemma 2.1, $(\alpha_{pi}, p) = (\beta_{pi}, p) = 1$, p divides $\alpha_{pi} + \beta_{pi}$, and $\alpha_{pi}g_{pi} + \beta_{pi}h_{pi} = 0$. Hence $h_p^{H/N}(p^{t_{pi}-1}y'_{pi} + N) = h_p^{H/A}(p^{t_{pi}-1}y'_{pi} + A)$. □

Lemma 2.7 *Assume* 2.3. *For every prime p such that $H_p \neq 0$, let H_p be as in Lemma* 2.5. *If $h_p^{G/A}(p^{t_{pn_p}-1}y'_{pn_p} + A) < \omega$, then there exist integers c_{pi}, $a_{pi} \in A$, and $k'_{pi} \in H$ for $1 \leqq i \leqq n_p$ such that*

$$p^{t_{pi}-1}y'_{pi} = a_{pi} + p^{c_{pi}-1}k'_{pi} \quad \textit{and} \quad (H/N)_p = \bigoplus_{i=1}^{n_p} \langle k'_{pi} + N \rangle,$$

where $o(k'_{pi} + N) = p^{c_{pi}}$ for $1 \leqq i \leqq n_p$ and $t_{p1} < c_{p1} < t_{p2} < c_{p2} < \cdots < t_{pn_p} < c_{pn_p}$.

Proof. For $1 \leqq i \leqq n_p$, let $d_{pi} = h_p^{H/N}(p^{t_{pi}-1}y'_{pi} + N)$. By Lemma 2.6, there exist $a_{pi} \in A$ and $k''_{pi} \in p^{d'_{pi}}H$ such that

$$p^{t_{pi}-1}y'_{pi} = a_{pi} + k''_{pi}.$$

By Proposition 2.2, we have $t_{pi} < d_{pi} + 1$. Suppose that $t_{pi+1} \leqq d_{pi} + 1$. Since $r(A) = 1$, there exist integers α_{pi}, α_{pi+1} such that $(\alpha_{pi}, \alpha_{pi+1}) = 1$ and $\alpha_{pi}a_{pi} + \alpha_{pi+1}a_{pi+1} = 0$. Hence

$$\alpha_{pi}p^{t_{pi}-1}y'_{pi} + \alpha_{pi+1}p^{t_{pi+1}-1}y'_{pi+1} = \alpha_{pi}k''_{pi} + \alpha_{pi+1}k''_{pi+1}.$$

Since $(\alpha_{pi}, \alpha_{pi+1}) = 1$, p divides α_{pi} and $(\alpha_{pi+1}, p) = 1$. Then $\alpha_{pi+1}p^{t_{pi+1}-1}y'_{pi+1} \in p^{t_{pi+1}}H$. This is a contradiction. Hence $d_{pi} + 1 < t_{pi+1}$ for $1 \leqq i \leqq n_p - 1$ and $h_p^{H/N}(p^{t_{pi}-1}y'_{pi} + N) = h_p^{G/A}(p^{t_{pi}-1}y'_{pi} + A) = d_{pi}$ for $1 \leqq i \leqq n_p$. Then, for $1 \leqq i \leqq n_p$, there exists $k'_{pi} \in H$ such that $k''_{pi} = p^{d_{pi}}k'_{pi}$. Moreover, for $1 \leqq i \leqq n_p$, we have $o(k'_{pi} + N) = p^{d_{pi}+1}$. Let $c_{pi} = d_{pi} + 1$. Since N is $T(H)$-high in H, we have $(H/N)[p] = \frac{H[p]+N}{N} \cong H[p]$. Therefore $(H/N)_p = \bigoplus_{i=1}^{n_p}\langle k'_{pi} + N\rangle$. □

Lemma 2.8 *Assume* 2.3. *For every prime p such that* $H_p \neq 0$, *let* H_p *be as in Lemma* 2.5. *If* $p^{t_{pn_p}-1}y'_{pn_p} + A \in p^{\omega}(G/A)[p]$, *then there exist integers* c_{pi}, $a_{pi} \in A$, *and* $k'_{pi} \in H$ *for* $1 \leqq i \leqq n_p - 1$ *and a subgroup* $D^{(p)}$ *of* H *such that*

$$p^{t_{pi}-1}y'_{pi} = a_{pi} + p^{c_{pi}-1}k'_{pi} \quad \text{and}$$
$$(H/N)_p = \bigoplus_{i=1}^{n_p-1}\langle k_{pi} + N\rangle \oplus D^{(p)}/N,$$

where $o(k'_{pi} + N) = p^{c_{pi}}$ *for* $1 \leqq i \leqq n_p - 1$, $t_{p1} < c_{p1} < t_{p2} < c_{p2} < \cdots < t_{pn_p}$, $D^{(p)}/N \cong \mathbf{Z}[p^{\infty}]$, *and* $(D^{(p)}/N)[p] = \langle p^{t_{pn_p}-1}y'_{pn_p} + N\rangle$.

Proof. By Lemma 2.7, for $1 \leqq i \leqq n_p - 1$, $h_p^{G/A}(p^{t_{pi}-1}y'_{pi} + A) < \omega$. Let $c_{pi} - 1 = h_p(p^{t_{pi}-1}y'_{pi} + A)$. Then there exist $a_{pi} \in A$ and $k'_{pi} \in H$ such that

$$p^{t_{pi}-1}y'_{pi} = a_{pi} + p^{c_{pi}-1}k'_{pi}.$$

By a similar proof of Lemma 2.7, we have $o(k'_{pi} + N) = p^{c_{pi}}$ for $1 \leqq i \leqq n_p - 1$ and $t_{p1} < c_{p1} < t_{p2} < c_{p2} < \cdots < t_{pn_p}$. Since $(H/N)[p] = \frac{H[p]+N}{N} \cong H[p]$, there exists a subgroup $D^{(p)}$ of H such that

$$(H/N)_p = \bigoplus_{i=1}^{n_p-1}\langle k'_{pi} + N\rangle \oplus D^{(p)}/N,$$

where $D^{(p)}/N \cong \mathbf{Z}[p^{\infty}]$. By Lemma 2.1 (3), it is immediate that $(D^{(p)}/N)[p] = \langle p^{t_{pn_p}-1}y'_{pn_p} + N\rangle$. □

Let N be a neat subgroup of G, $b \in N$, and $m_p = h_p^N(b)$ for every prime p. If $m_p < \infty$, then there exists $b_p \in N$ such that $h_p^N(b_p) = 0$. Since N is neat in G, $h_p(b_p) = 0$.

Theorem 2.9 *Assume* 2.3. *For every prime p such that $H_p \neq 0$, let H_p be as in Lemma* 2.5.

(1) *If $h_p^{G/A}(p^{t_{pn_p}-1}y'_{pn_p} + A) < \omega$, then there exist $b_p \in N$, $h_{pi} \in H$ and $y_{pi} \in H_p$ for $1 \leqq i \leqq n_p$ such that*
   (1) $(H/N)_p = \bigoplus_{i=1}^{n_p} \langle h_{pi} + N \rangle$;
   (2) $H_p = \bigoplus_{i=1}^{n_p} \langle y_{pi} \rangle$, *where* $o(y_{pi}) = p^{t_{pi}}$ *for* $1 \leqq i \leqq n_p$;
   (3) *setting* $o(h_{pi} + N) = p^{c_{pi}}$ *for* $1 \leqq i \leqq n_p$, $t_{p1} < c_{p1} < t_{p2} < c_{p2} < \cdots < t_{pn_p} < c_{pn_p}$;
   (4) $y_{p1} = b_p + p^{c_{p1}-t_{p1}} h_{p1}$, $p^{t_{p1}-1} b_p \in A$, $y_{pi} = h_{pi-1} + p^{c_{pi}-t_{pi}} h_{pi}$, $p^{t_{pi}-1} h_{pi-1} \in A$ *for* $2 \leqq i \leqq n_p$;
   (5) *for* $1 \leqq i < n_p$, $h_p(p^s h_{pi}) = s$ *for* $0 \leqq s < t_{pi+1}$ *and* $h_p(p^s h_{pn_p}) = s$ *for all* $s \geqq 0$.

(2) *If $p^{t_{pn_p}-1} y'_{pn_p} + A \in p^\omega(G/A)[p]$, then there exist $b_p \in N$, $h_{pi} \in H_p$ for $i \geqq 1$, $y_{pi} \in H[p]$ for $1 \leqq i \leqq n_p$, and a subgroup $D^{(p)}$ of H such that*
   (1) $(H/N)_p = \bigoplus_{i=1}^{n_p-1} \langle h_{pi} + N \rangle \oplus D^{(p)}/N$, *where* $o(h_{pi} + N) = p^{c_{pi}}$ *for* $1 \leqq i \leqq n_p - 1$ *and* $D^{(p)}/N \cong \mathbf{Z}[p^\infty]$ *such that*

$$D^{(p)}/N = \langle h_{pi} + N \mid i \geqq n_p,\ ph_{pi+1} = h_{pi},\ p^{t_{pn_p}+1} h_{pn_p} \in A \rangle;$$

   (2) $H_p = \bigoplus_{i=1}^{n_p} \langle y_{pi} \rangle$, *where* $o(y_{pi}) = p^{t_{pi}}$ *for* $1 \leqq i \leqq n_p$;
   (3) $t_{p1} < c_{p1} < t_{p2} < c_{p2} < \cdots < t_{pn_p}$;
   (4) $y_{p1} = b_p + p^{c_{p1}-t_{p1}} h_{p1}$, $p^{t_{p1}-1} b_p \in A$, $y_{pi} = h_{pi-1} + p^{c_{pi}-t_{pi}} h_{pi}$, $p^{t_{pi-1}-1} h_{pi-1} \in A$ *for* $1 \leqq i \leqq n_p - 1$ *and* $y_{pn_p} = h_{pn_p-1} + ph_{pn_p}$;
   (5) *for* $1 \leqq i \leqq n_p - 1$, $h_p(p^s h_{pi}) = s$ *for* $0 \leqq s < t_{pi+1}$ *and* $h_p(h_{pn_p}) = \infty$.

Moreover, for every prime p such that $H_p \neq 0$ and $1 \leqq i \leqq n_p$, let

$$e_{pi} = \begin{cases} t_{p1} & \text{if } i = 1, \\ t_{p1} + \sum_{j=2}^{i} (t_{pj} - c_{pj-1}) & \text{if } i > 1. \end{cases}$$

Then

$$p^{t_{pi}-1} y_{pi} = (-1)^{i-1} p^{e_{pi}-1} b_p + p^{c_{pi}-1} h_{pi}.$$

Let $c_{pn_p} = \infty$ *if* $p^\omega(G/A)[p] \neq 0$. *Then*

$$h_p(p^i b_p) = \begin{cases} i & \text{for } 0 \leqq i < e_{p1}, \\ i + c_{pk} - e_{pk} & \text{for } e_{pk} \leqq i < e_{pk+1} \text{ and } 2 \leqq k < n_p - 1, \\ i + c_{pn_p} - e_{pn_p} & \text{for } i \geqq e_{pn_p}. \end{cases}$$

Proof. (1) Let p be a prime such that $h_p(p^{t_{pn_p}-1}y'_{pn_p} + A) < \omega$. By Proposition 2.2, $pN \neq N$. By hypothesis and Lemma 2.7, there exist integers $c_{pi}, a_{pi} \in A$, and $k'_{pi} \in H$ for $1 \leqq i \leqq n_p$ such that

$$p^{t_{pi}-1}y'_{pi} = a_{pi} + p^{c_{pi}-1}k'_{pi} \quad \text{and} \quad (H/N)_p = \bigoplus\nolimits_{i=1}^{n_p} \langle k'_{pi} + N \rangle,$$

where $o(k_{pi} + N) = p^{c_{pi}}$ for $1 \leqq i \leqq n_p$ and $t_{p1} < c_{p1} < t_{p2} < c_{p2} < \cdots < t_{pn_p} < c_{pn_p}$. For convenience, we replace k'_{pi}, y'_{pi}, c_{pi}, a_{pi} and t_{pi} with k'_i, y'_i, c_i, a_i and t_i, respectively. Let $b'_p \in N$ such that $h_p(b'_p) = 0$.

By the structure of H/N, we also write

$$y'_1 = \alpha_1 k'_1 + \sum_{i=2}^{n_p} \alpha_i k'_i + \frac{v_p}{u_p} b'_p,$$

where every α_i, u_p, and v_p are integers for $1 \leqq i \leqq n_p$ such that $(u_p, v_p) = (u_p, p) = 1$. Since

$$\begin{aligned} p^{t_1-1}u_p y'_1 &= u_p a_1 + p^{c_1-1}u_p k'_1 \\ &= p^{t_1-1}\alpha_1 u_p k'_1 + \sum_{i=2}^{n_p} p^{t_1-1}\alpha_i u_p k'_i + p^{t_1-1}v_p b'_p, \end{aligned}$$

we have $p^{t_1-1}\alpha_1 - p^{c_1-1} = p^{c_1}\beta_1$ and $p^{t_1-1}\alpha_i = p^{c_i}\beta_i$ for some integers β_i and $1 \leqq j \leqq n_p$. Then

$$\begin{aligned} p^{t_1-1}u_p y'_1 &= p^{c_1-1}(1 + \beta_1 p)u_p k'_1 + \sum_{i=2}^{n_p} p^{c_i}\beta_i u_p k'_i + p^{t_1-1}v_p b'_p \\ &= p^{t_1-1}v_p b'_p + p^{c_1-1}u_p\Big\{(1 + \beta_1 p)k'_1 + \sum_{i=2}^{n_p} p^{c_i-c_1+1}\beta_i k'_i\Big\}. \end{aligned}$$

Hence $(v_p, p) = 1$. Let $h'_1 = (1 + \beta_1 p)u_p k'_1 + \sum_{i=2}^{n_p} p^{c_i-c_1+1}\beta_i u_p k'_i$. Then we have

$$(H/N)_p = \langle h'_1 + N \rangle \oplus \Big(\bigoplus\nolimits_{i=2}^{n_p} \langle k'_i + N \rangle\Big)$$

and

$$p^{t_1-1}u_p y_1' = u_p a_1 + u_p p^{c_1-1}k_1' = p^{t_1-1}v_p b_p' + p^{c_1-1}h_1'.$$

Since $r(N) = 1$, there exist integers γ_1 and δ_1 such that $(\gamma_1, \delta_1) = 1$ and $\gamma_1 u_p a_1 = \delta_1 p^{t_1-1}v_p b_p'$. Then $(\gamma_1 - \delta_1)p^{t_1-1}u_p y_1' = p^{c_1-1}(\gamma_1 u_p k_1' - \delta_1 h_1') = 0$ and hence $(\gamma_1, p) = (\delta_1, p) = 1$.

Let $z_1 = -\delta_1 u_p y_1' + \delta_1 v_p b_p' + p^{c_1-t_1}\delta_1 h_1'$. Then $z_1 \in H[p^{t_1-1}]$. Let $y_1 = \delta_1 u_p y_1' + z_1$, $b_p = \delta_1 v_p b_p'$, and $h_1 = \delta_1 h_1'$. Then $y_1 = b_p + p^{c_1-t_1}h_1$, $H_p = \langle y_1\rangle \oplus (\bigoplus_{i=2}^{n_p}\langle y_i'\rangle)$, $p^{t_1-1}b_p \in A$, and $(H/N)_p = \langle h_1 + N\rangle \oplus (\bigoplus_{i=2}^{n_p}\langle k_i' + N\rangle)$.

It is immediate that $h_p(p^s h_1) = s$ for $0 \leqq s < t_1$. If $p^{c_1}h_1 \in p^{c_1+1}H$, then there exist $g_p \in H$ and $x_p \in H[p]$ such that $x_p = p^{c_1-1}h_1 - p^{c_1}g_p$. Since $h_p(x_p) = c_1 - 1$ and $t_1 < c_1 < t_2$, this is a contradiction. Hence $h_p(p^{c_1}h_1) = c_1$. By induction and a similar proof, we have $h_p(p^s h_1) = s$ for $0 \leqq s < t_2$.

There exist integers μ_2 and ν_2 such that $\mu_2 p^{t_1-1}b_p + \nu_2 a_2 = 0$ and $(\mu_2, \nu_2) = 1$. Since $\mu_2 p^{t_1-1}y_1 + \nu_2 p^{t_2-1}y_2' = \mu_2 p^{c_1-1}h_1 + \nu_2 p^{c_2-1}k_2'$ and $t_1 < c_1 < t_2 < c_2$, we have $\mu_2 = p\mu_2'$ for some integer μ_2'. Then $(\nu_2, p) = 1$ and $\nu_2 p^{t_2-1}y_2' = \mu_2' p^{c_1}h_1 + \nu_2 p^{c_2-1}k_2'$. Hence $h_p(\mu_2 p^{c_1}h_1) = t_2 - 1$. Since $h_p(p^s h_1) = s$ for $s < t_2$, there exists an integer μ_2'' such that $\mu_2 = p^{t_2-c_1}\mu_2''$ and $(\mu_2'', p) = 1$. Then we can write

$$\nu_2 p^{t_2-1}y_2' = \mu_2'' p^{t_2-1}h_1 + \nu_2 p^{c_2-1}k_2'.$$

Since $(\mu_2'', p) = 1$, there exist integers γ_2 and δ_2 such that $h_1 = \gamma_2\mu_2'' h_1 + \delta_2 p^{c_2-1}h_1$. Since $(\gamma_2, p) = 1$, we have

$$0 \neq \nu_2\gamma_2 p^{t_2-1}y_2' = p^{t_2-1}h_1 + p^{c_2-1}(\nu_2\gamma_2 k_2' - \delta_2 p^{t_2-1}h_1).$$

Let $h_2 = \nu_2\gamma_2 k_2' - \delta_2 p^{t_2-1}h_1$. Then we have

$$(H/N)_p = \langle h_1 + N\rangle \oplus \langle h_2 + N\rangle \oplus \left(\bigoplus_{i=3}^{n_p}\langle k_i' + N\rangle\right).$$

Let $z_2 = -\nu_2\gamma_2 y_2' + h_1 + p^{c_2-t_2}h_2$. Then $z_2 \in G[p^{t_2-1}]$. Let $y_2 = \nu_2\gamma_2 y_2' + z_2$. Then $H_p = \langle y_1\rangle \oplus \langle y_2\rangle \oplus (\bigoplus_{i=3}^{n_p}\langle y_i'\rangle)$. Hence $y_2 = h_1 + p^{c_2-t_2}h_2$. Moreover, since $p^{t_1}b_p = -p^{c_1}h_1$, we have $p^{t_2-1}h_1 \in A$. By a similar proof, $h_p(p^s h_2) = s$ for $0 \leqq s < t_3$.

Suppose by induction that there exist $y_i \in H_p$ and $k_i \in H$ for $1 \leqq i \leqq r$ and $b_p \in N$ such that $H_p = \oplus(\bigoplus_{i=1}^{r}\langle y_i\rangle) \oplus (\bigoplus_{i=r+1}^{n_p}\langle y_i'\rangle)$, $(H/N)_p = (\bigoplus_{i=1}^{r}\langle k_i + N\rangle) \oplus (\bigoplus_{i=r+1}^{n_p}\langle k_i' + N\rangle)$, $y_1 = b_p + p^{c_1-t_1}h_1$, $p^{t_1-1}b_p \in A$,

$y_i = h_{i-1} + p^{c_i - t_i} h_i$, and $p^{t_i - 1} h_{i-1} \in A$ for $2 \leqq i \leqq r$. Then, by a similar proof, there exist $y_{r+1} \in H_p$ and $h_{r+1} \in H$ such that $H_p = \left(\bigoplus_{i=1}^{r+1} \langle y_i \rangle\right) \oplus \left(\bigoplus_{i=r+2}^{n_p} \langle y_i' \rangle\right)$, $(H/N)_p = \left(\bigoplus_{i=1}^{r+1} \langle k_i + N \rangle\right) \oplus \left(\bigoplus_{i=r+2}^{n_p} \langle k_i' + N \rangle\right)$, $y_{r+1} = h_r + p^{c_{r+1} - t_{r+1}} h_{r+1}$, and $p^{t_{r+1} - 1} h_r \in A$.

It is immediate that $p^{t_r - 1} h_r \in A$. By a same argument, for $1 \leqq i \leqq n_p$, we have $h_p(p^s h_i) = s$ for $0 \leqq s < t_{i+1}$ and $h_p(p^s h_{n_p}) = s$ for all $s \geqq 0$.

(2) Let p be a prime such that $p^{t_{pn_p} - 1} y'_{pn_p} + A \in p^\omega (G/A)[p]$. By Lemma 2.8 and a similar proof, there exist $b_p \in N$, $h_i \in H$, $y_i \in H_p$ for $1 \leqq i \leqq n_p - 1$, and a subgroup $D^{(p)}$ of H such that

(1) $(H/N)_p = \bigoplus_{i=1}^{n_p - 1} \oplus \langle h_i + N \rangle \oplus D^{(p)}/N$, where $o(h_i + N) = p^{c_i}$ for $1 \leqq i \leqq n_p - 1$, $D^{(p)}/N \cong \mathbf{Z}[p^\infty]$ and $(D^{(p)}/N)[p] = \langle p^{t_{n_p} - 1} y'_{n_p} + N \rangle$;

(2) $H_p = \bigoplus_{i=1}^{n_p - 1} \langle y_i \rangle \oplus \langle y'_{n_p} \rangle$, where $o(y_i) = p^{t_i}$ for $1 \leqq i \leqq n_p - 1$;

(3) $t_1 < c_1 < t_2 < c_2 < \cdots < t_{n_p}$;

(4) $y_1 = b_p + p^{c_{pi} - t_{pi}} h_1$, $p^{t_1 - 1} b_p \in A$, $y_i = h_{i-1} + p^{c_i - t_i} h_i$ for $2 \leqq i \leqq n_p - 1$, $p^{t_i - 1} h_{i-1} \in A$ for $2 \leqq i \leqq n_p$;

(5) for $1 \leqq i \leqq n_p - 1$, $h_p(p^s h_i) = s$ for all $s < t_{i+1}$.

By Lemma 2.1 (2), Lemma 2.6 and Lemma 2.8, there exist $a_{n_p} \in A$ and $d_p \in p^\omega H$ such that $p^{t_{n_p} - 1} y'_{n_p} = a_{n_p} + d_p$. Since $r(A) = 1$ and $p^{t_{n_p - 1} - 1} h_{n_p - 2} \in A$, there exist integers μ_{n_p} and ν_{n_p} such that $(\mu_{n_p}, \nu_{n_p}) = 1$ and $\mu_{n_p} p^{t_{n_p - 1} - 1} h_{n_p - 2} + \nu_{n_p} a_{n_p} = 0$. Then

$$\mu_{n_p} p^{t_{n_p - 1} - 1} y_{n_p - 1} + \nu_{n_p} p^{t_{n_p} - 1} y'_{n_p} = \mu_{n_p} p^{c_{n_p - 1} - 1} h_{n_p - 1} + \nu_{n_p} d_p.$$

Since $t_{n_p - 1} < c_{n_p - 1} < t_{n_p}$, we have $\mu_{n_p} = p \mu'_{n_p}$ for some integer μ'_{n_p}. Then $(\nu_{n_p}, p) = 1$ and $\nu_{n_p} p^{t_{n_p} - 1} y'_{n_p} = \mu'_{n_p} p^{c_{n_p - 1}} h_{n_p - 1} + \nu_{n_p} d_p$ and hence $h_p(\mu'_{n_p} p^{c_{n_p - 1}} h_{n_p - 1}) = t_{n_p} - 1$. Since $h_p(p^s h_{n_p - 1}) = s$ for $0 \leqq s < t_{n_p}$, there exists an integer μ''_{n_p} such that $\mu_{n_p} = p^{t_{n_p} - c_{n_p - 1}} \mu''_{n_p}$ and $(\mu''_{n_p}, p) = 1$. Hence we can write

$$\nu_{n_p} p^{t_{n_p} - 1} y'_{n_p} = \mu''_{n_p} p^{t_{n_p} - 1} h_{n_p - 1} + \nu_{n_p} d_p.$$

Since $d_p \in p^\omega H$, there exists $d'_p \in H$ such that $d_p = p^{2t_{n_p}} d'_p$. Since $(\mu''_{n_p}, p) = 1$, there exist integers γ_{n_p} and δ_{n_p} such that $h_{n_p - 1} = \gamma_{n_p} \mu''_{n_p} h_{n_p - 1} + \delta_{n_p} p^{2t_{n_p}} h_{n_p - 1}$. Note that $p^{t_{n_p} - 1} h_{n_p - 1} \in A$. Since $(\gamma_{n_p}, p) = 1$, we have

$$0 \neq \nu_{n_p} \gamma_{n_p} p^{t_{n_p} - 1} y'_{n_p} = p^{t_{n_p} - 1} h_{n_p - 1} + p^{2t_{n_p}} (\nu_{n_p} \gamma_{n_p} d'_p - \delta_{n_p} p^{t_{n_p} - 1} h_{n_p - 1}).$$

Let $h'_{n_p} = \nu_{n_p}\gamma_{n_p}d'_p - \delta_{n_p}p^{t_{n_p}-1}h_{n_p-1}$. By the proof of Lemma 2.1 (2), $h_p(p^{2t_{n_p}}h'_{n_p}) \geqq \omega$. For every $i \geqq 0$, there exists $h'_{n_p+i} \in H$ such that $p^{2t_{n_p}}h'_{n_p} = p^{t_{n_p}}(p^{t_{n_p}}h'_{n_p}) = p^{t_{n_p}}(p^{t_{n_p}+i}h'_{n_p+i})$. Let $h_{n_p+i} = p^{t_{n_p}}h'_{n_p+i}$ for every $i \geqq 0$. Then $p^{2t_{n_p}}h'_{n_p} = p^{t_{n_p}}h_{n_p} = p^{t_{n_p}+i}h_{n_p+i}$ for all $i \geqq 1$. Since $ph_{n_p+i+1} - h_{n_p+i} \in p^{t_{n_p}}H_p = 0$, $ph_{n_p+i+1} = h_{n_p+i}$ for $i \geqq 0$. Note that $\nu_{n_p}\gamma_{n_p}p^{t_{n_p}-1}y'_{n_p} = p^{t_{n_p}-1}h_{n_p-1} + p^{t_{n_p}}h_{n_p}$.

Let $z_{n_p} = -\nu_{n_p}\gamma_{n_p}y'_{n_p} + h_{n_p-1} + ph_{t_{n_p}}$. Then $z_{n_p} \in G[p^{t_{n_p}-1}]$. Let $y_{n_p} = \nu_{n_p}\gamma_{n_p}y'_{n_p} + z_{n_p}$. Then $y_{n_p} = h_{n_p-1} + ph_{n_p}$, $H_p = \bigoplus_{i=1}^{n_p}\langle y_i\rangle$, and $p^{t_{n_p}-1}h_{n_p-1} \in A$. Let $H^{(p)} = \langle h_{n_p+i} \mid i \geqq 0\rangle$. Then it is easy to see that $H^{(p)}$ is p-divisible. Then $h_p(h_{n_p}) = \infty$. Let $D^{(p)}/N = \langle h_{n_p+i} + N \mid i \geqq 0\rangle$. Hence the assertion holds.

For $1 \leqq i \leqq n_p$, let

$$e_{pi} = \begin{cases} t_{p1} & \text{if } i = 1, \\ t_{p1} + \sum_{j=2}^{i}(t_{pj} - c_{pj-1}) & \text{if } i > 1. \end{cases}$$

By an easy induction, we have

$$p^{t_{pi}-1}y_{pi} = (-1)^{i-1}p^{e_{pi}-1}b_p + p^{c_{pi}-1}h_{pi}$$

and

$$h_p(p^ib_p) = \begin{cases} i & \text{for } 0 \leqq i < e_{p1}, \\ i + c_{pk} - e_{pk} & \text{for } e_{pk} \leqq i < e_{pk+1} \text{ and } 2 \leqq k < n_p - 1, \\ i + c_{pn_p} - e_{pn_p} & \text{for } i \geqq e_{pn_p}. \end{cases}$$

□

Proposition 2.10 *Let G be an abelian group and A a torsion-free rank-one subgroup of G. Suppose that A is purifiable in G. Let H and K be pure hulls of A in G. Then, for every prime p, $H_p \neq 0$ if and only if $K_p \neq 0$. For every prime p such that $H_p \neq 0$, let $H_p = \bigoplus_{i=1}^{n_p}\langle x_{pi}\rangle$, where $x_{pi} \in H_p$ and $o(x_{pi}) = p^{t_{pi}}$, $K_p = \bigoplus_{i=1}^{n'_p}\langle y_{pi}\rangle$, where $y_{pi} \in K_p$ and $o(y_{pi}) = p^{t'_{pi}}$, $c_{pi} = h_p^{G/A}(p^{t_{pi}-1}x_{pi} + A) + 1$ for $1 \leqq i \leqq n_p$, and $c'_{pi} = h_p^{G/A}(p^{t'_{pi}-1}y_{pi} + A) + 1$ for $1 \leqq i \leqq n'_p$. Then $n_p = n'_p$, $t_{pi} = t'_{pi}$, and $c_{pi} = c'_{pi}$ for $1 \leqq i \leqq n_p$.*

Proof. By Proposition 1.10, for all $n \geqq 0$,

$$V_{p,n}(H, A) \cong V_{p,n}(G, A) \cong V_{p,n}(K, A).$$

Hence, by Lemma 2.1 (1), $n_p = n'_p$ and $t_{pi} = t'_{pi}$ for $1 \leqq i \leqq n_p$. By Theorem 2.9, for $1 \leqq i \leqq n_p$, if $c_{pi} < \infty$ and $c'_{pi} < \infty$, then there exist $a'_{pi}, a''_{pi} \in A$, $h_{pi} \in H$, and $k_{pi} \in K$ such that $p^{t_{pi}-1}x_{pi} = a'_{pi} + p^{c_{pi}-1}h_{pi}$, $p^{t_{pi}-1}y_{pi} = a''_{pi} + p^{c'_{pi}-1}k_{pi}$, $h_p(pa'_{pi}) = c_{pi}$, and $h_p(pa''_{pi}) = c'_{pi}$. Since $r(A) = 1$, there exist integers α_{pi} and β_{pi} such that $(\alpha_{pi}, \beta_{pi}) = 1$ and $\alpha_{pi}a'_{pi} = \beta_{pi}a''_{pi}$. Since $h_p(a'_{pi}) = h_p(a''_{pi})$, we have $(\alpha_{pi}, p) = (\beta_{pi}, p) = 1$. Since $\alpha_{pi}pa'_p = \beta_{pi}pa''_{pi}$, we have $c_{pi} = c'_{pi}$. If $c_{pn_p} < \infty$ and $c'_{pn_p} = \infty$, then, by Lemma 2.1 (2), there exist $b_p, b'_p \in A, h_{pn_p} \in H$, and $k_{pn_p} \in p^{\omega}K$ such that $p^{t_{pn_p}-1}x_{pn_p} = b_p + p^{c_{pn_p}-1}h_{pn_p}$, $p^{t_{pn_p}-1}y_{pn_p} = b'_p + k_{pn_p}$, $h_p(pb_p) = c_{pn_p}$, and $h_p(pb'_p) = \infty$. By a similar proof, this is a contradiction. Hence $c_{pi} = c'_{pi}$ for $1 \leqq i \leqq n_p$. □

Assume 2.3. By Proposition 2.10, we define the p-coordinate of the *torsion system* of A as the following sequence:

$$\mathbf{T}_p^A(G) = \begin{cases} (t_{p1}, t_{p2}, \ldots, t_{pn_p}) & \text{if } A \text{ is not } p\text{-vertical in } G, \\ (0) & \text{if } A \text{ is } p\text{-vertical in } G \end{cases}$$

and the p-coordinate of the *quotient system* of A as the following sequence:

$$\mathbf{Q}_p^A(G) = \begin{cases} (c_{p1}, c_{p2}, \ldots, c_{pn_p}) & \text{if } A \text{ is not } p\text{-vertical in } G, \\ (0) & \text{if } A \text{ is } p\text{-vertical in } G, \end{cases}$$

where $c_{pn_p} = h_p^{G/A}(p^{t_{pn_p}-1}y_{pn_p} + A)$. By the structure of H, $c_{pn_p} < \omega$ or $c_{pn_p} = \infty$.

Since we can make the matrix from the quotient system and the torsion system, we define the p-coordinate of the *QT-matrices* of A, denoted by $\mathbf{QT}_p^A(G)$, as follows: if A is not p-vertical in G, then let

$$\mathbf{QT}_p^A(G) = \begin{pmatrix} c_{p1}, c_{p2}, \ldots, c_{pn_p} \\ t_{p1}, t_{p2}, \ldots, t_{pn_p} \end{pmatrix}$$

and if A is p-vertical in G, then let

$$\mathbf{QT}_p^A(G) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In view of Theorem 2.9, to have the property that $p^{t_{p1}-1}b_p \in A$ and $p^{t_{pi}-1}h_{pi-1} \in A$ for $2 \leqq i \leqq n_p$, we need to choose a special element $b_p \in N$. To prove that all pure hulls of A in G are isomorphic, we do not need to do it. Hence, by similar calculations, we establish the following corollary:

Corollary 2.11 *Assume* 2.3. *For every prime p such that $H_p \neq 0$, let H_p be as in Lemma* 2.5, $b \in N$, *and for every prime p, let $m_p = h_p^N(b)$. If p is a prime such that $H_p \neq 0$, then there exists $b'_p \in N$ such that $b = p^{m_p} b'_p$ and $H_p(b'_p) = 0$.*

(1) *If $h_p(p^{t_{pn_p}-1} y'_{pn_p} + N) < \omega$, then there exist $g_{pi} \in H$ and $x_{pi} \in H_p$ for $1 \leqq i \leqq n_p$ such that*

(1) $(H/N)_p = \bigoplus_{i=1}^{n_p} \langle g_{pi} + N \rangle$;

(2) $H_p = \bigoplus_{i=1}^{n_p} \langle x_{pi} \rangle$, *where* $o(x_{pi}) = p^{t_{pi}}$ *for* $1 \leqq i \leqq n_p$;

(3) *setting* $o(h_{pi} + N) = p^{c_{pi}}$ *for* $1 \leqq i \leqq n_p$, $t_{p1} < c_{p1} < t_{p2} < c_{p2} < \cdots < t_{pn_p} < c_{pn_p}$;

(4) $x_{p1} = b'_p + p^{c_{p1}-t_{p1}} g_{p1}$ *and* $x_{pi} = h_{pi-1} + p^{c_{pi}-t_{pi}} g_{pi}$ *for* $2 \leqq i \leqq n_p$;

(5) *for* $1 \leqq i \leqq n_p - 1$, $h_p(p^s g_{pi}) = s$ *for* $0 \leqq s < t_{pi+1}$ *and* $h_p(p^s g_{pn_p}) = s$ *for all* $s \geqq 0$.

(2) *If $h_p(p^{t_{pn_p}-1} y'_{pn_p} + N) \in p^\omega(G/N)[p]$, then there exist $g_{pi} \in H$ for $1 \leqq i \leqq n_p - 1$, $x_{pi} \in H[p]$ for $1 \leqq i \leqq n_p$, and a subgroup $D^{(p)}$ of H such that*

(1) $(H/N)_p = \bigoplus_{i=1}^{n_p-1} \langle g_{pi} + N \rangle \oplus D^{(p)}/N$, *where* $o(g_{pi} + N) = p^{c_{pi}}$ *for* $1 \leqq i \leqq n_p - 1$ *and* $D^{(p)}/N \cong \mathbf{Z}[p^\infty]$ *such that*

$$D^{(p)}/N = \langle g_{pi} + N \mid i \geqq n_p,\ pg_{pi+1} = h_{pi},\ p^{t_{pn_p}+1} g_{pn_p} \in N \rangle;$$

(2) $H_p = \bigoplus_{i=1}^{n_p} \langle x_{pi} \rangle$, *where* $o(x_{pi}) = p^{t_{pi}}$ *for* $1 \leqq i \leqq n_p$;

(3) $t_{p1} < c_{p1} < t_{p2} < c_{p2} < \cdots < t_{pn_p}$;

(4) $x_{p1} = b'_p + p^{c_{p1}-t_{p1}} g_{p1}$ *and* $x_{pi} = g_{pi-1} + p^{c_{pi}-t_{pi}} g_{pi}$ *for* $1 \leqq i \leqq n_p - 1$ *and* $x_{pn_p} = g_{pn_p-1} + p g_{pn_p}$;

(5) *for* $1 \leqq i \leqq n_p - 1$, $h_p(p^s g_{pi}) = s$ *for* $0 \leqq s < t_{pi+1}$ *and* $h_p(g_{pn_p}) = \infty$.

Moreover, for every prime p such that $H_p \neq 0$ and $1 \leqq i \leqq n_p$, let

$$e_{pi} = \begin{cases} t_{p1} & \text{if } i = 1, \\ t_{p1} + \sum_{j=2}^{i} (t_{pj} - c_{pj-1}) & \text{if } i > 1. \end{cases}$$

Then

$$p^{t_{pi}-1} y_{pi} = (-1)^{i-1} p^{e_{pi}-1} b'_p + p^{c_{pi}-1} h_{pi}. \tag{2.11.1.}$$

Let $c_{pn_p} = \infty$ if $p^\omega(G/A)[p] \neq 0$. Then

$$h_p(p^i b'_p) = \begin{cases} i & \text{for } 0 \leqq i < e_{p1}, \\ i + c_{pk} - e_{pk} & \text{for } e_{pk} \leqq i < e_{pk+1} \text{ and } 2 \leqq k < n_p - 1, \\ i + c_{pn_p} - e_{pn_p} & \text{for } i \geqq e_{pn_p}. \end{cases}$$

3. Torsion-Free Rank-one Subgroups

Our major goals of this section are to give a necessary and sufficient condition for torsion-free rank-one subgroups of arbitrary abelian groups to be purifiable in a given group and to show that if A is a purifiable torsion-free rank-one subgroup of an arbitrary abelian group, then all pure hulls of A are isomorphic.

Lemma 3.1 *Let G be an arbitrary abelian group and A a torsion-free rank-one subgroup of G. Let p be a prime such that $A \neq pA$ and $a_p \in A$ such that $h_p^A(a_p) = 0$. Then the following hold.*

(1) *If there exists an integer r such that $h_p(p^r a_p) = d < \omega$, then $A \cap p^d G$ is p-vertical in $p^d G$ if and only if $h_p(p^{r+n} a_p) = d + n$ for $n \geqq 0$.*

(2) *If there exists a nonnegative integer r such that $h_p(p^r a_p) = d < \omega$ and $h_p(p^{r+n} a_p) = h_p(p^r a_p) + n$ for $n \geqq 0$, then A is p-purifiable in G.*

(3) *If there exists a nonnegative integer r such that $h_p(p^r a_p) = d < \omega$ and $h_p(p^{r+1} a_p) = \infty$, then $A \cap p^{d+1} G$ is p-vertical in $p^{d+1} G$.*

Proof. (1) ($\Rightarrow$) Suppose by induction that $h_p(p^{r+i} a_p) = d + i$ for all $0 \leqq i \leqq k$. If $h_p(p^{r+k+1} a_p) \geqq d + k + 2$, then there exists $g \in G$ such that $p^{r+k+1} a_p = p^{d+k+2} g$. Since $A \cap p^d G$ is p-vertical in $p^d G$, by Proposition 1.13 (2), we have $p^{r+k} a_p - p^{d+k+1} g \in ((A \cap p^d G) + p^{d+k+1} G)[p] = p^{d+k+1} G[p]$. This is a contradiction. Hence $h_p(p^{r+i} a_p) = d + i$ for all $i \geqq 0$.

($\Leftarrow$) Note that if $b_p \in A$ such that $h_p^A(b_p) = 0$, then $h_p(p^{r+n} b_p) = d+n$ for $n \geqq 0$. It suffices to prove that $((A \cap p^d G) + p^{d+n} G)[p] = p^{d+n} G[p]$ for all $n \geqq 1$. Let $a + p^{d+n} g \in ((A \cap p^d G) + p^{d+n} G)[p]$ such that $a \in A \cap p^d G$ and $g \in G$. If $h_p(a) < d + n$, then there exists $a'_p \in A$ such that $h_p^A(a'_p) = 0$ and $a = p^t a'_p$ for some integer t. Then $t \geqq r$ and hence $h_p(pa) < d + n + 1$. But $pa = -p^{d+n+1} g$. This is a contradiction. Hence $h_p(a) \geqq d + n$ and $A \cap p^d G$ is p-vertical in $p^d G$.

(2) We prove that $A \cap p^d G$ is p-neat in $p^d G$. Let $px \in A \cap p^d G$ with $x \in p^d G$. Since $r(A) = 1$, there exist integers α and β such that $(\alpha, \beta) = 1$ and $\alpha a_p = \beta px$. Then $(\beta, p) = 1$. Let $\alpha = p^s \alpha'$ for some integer α' such that $(\alpha', p) = 1$. Note that $h_p(p^s a_p) = h_p(px) \geqq d + 1$. If $s < r$, then

$d+1 \leqq h_p(p^s a_p) < h_p(p^r a_p) = d$. This is a contradiction. Hence $r \leqq s$. Since $h_p(p^{s-r}p^r a_p) = d+s-r \geqq d+1$, we have $s \geqq r+1$. Hence $p^s a_p = p^{s-r-1}pp^r a_p \in p(A \cap p^d G)$. Since $\alpha' p^s a_p = \beta px$ and $(\alpha', p) = (\beta, p) = 1$, $px \in p(A \cap p^d G)$. Hence $A \cap p^d G$ is p-neat in $p^d G$. By Proposition 1.12, $A \cap p^d G$ is p-pure in $p^d G$. By Proposition 1.14, A is p-purifiable in G.

(3) We show that $((A \cap p^{d+1}G) + p^{d+n+1}G)[p] = p^{d+n+1}G[p]$ for all $n \geqq 1$. Let $a + p^{d+n+1}g \in ((A \cap p^{d+1}G) + p^{d+n+1}G)[p]$ such that $a \in A \cap p^{d+1}G$ and $g \in G$. Since $r(A) = 1$, there exist integers γ and δ such that $(\gamma, \delta) = 1$ and $\gamma a_p = \delta a$. Then $(\delta, p) = 1$. Let $\gamma = p^{s'}\gamma'$ for some integer γ' such that $(\gamma', p) = 1$. By a similar argument, we have $s' \geqq r+1$. Hence $h_p(a) = \infty$ and $((A \cap p^{d+1}G) + p^{d+n+1}G)[p] = p^{d+n+1}G[p]$ for all $n \geqq 1$. □

Now we give a necessary and sufficient condition for a torsion-free rank-one subgroup of an arbitrary abelian group to be purifiable in a given group.

Theorem 3.2 *Let G be an abelian group and A a torsion-free rank-one subgroup of G. Then the following properties are equivalent:*

(1) *A is purifiable in G;*

(2) *for every prime p such that $A \neq pA$, there exists $a_p \in A$ such that $h_p^A(a_p) = 0$ and one of the following two conditions holds:*
  - (i) *there exists a nonnegative integer r_p such that $h_p(p^{r_p}a_p) < \omega$ and $h_p(p^{r_p+n}a_p) = h_p(p^{r_p}a_p) + n$ for $n \geqq 0$;*
  - (ii) *there exists a nonnegative integer r_p such that $h_p(p^{r_p}a_p) = \infty$ and if $r_p > 0$, then $h_p(p^{r_p-1}a_p) < \omega$.*

(3) *for every prime p and every $a \in A$, one of the following two conditions holds:*
  - (i) *there exists a nonnegative integer $k_p(a)$ such that $h_p(p^{k_p(a)}a) < \omega$ and $h_p(p^{k_p(a)+n}a) = h_p(p^{k_p(a)}a) + n$ for $n \geqq 0$;*
  - (ii) *there exists a nonnegative integer $k_p(a)$ such that $h_p(p^{k_p(a)}a) = \infty$ and if $k_p(a) > 0$, then $h_p(p^{k_p(a)-1}a) < \omega$.*

(4) *for every prime p,*
  - (i) *A is eventually p-vertical in G and*
  - (ii) *for every $a \in A$, if $h_p(a) \geqq \omega$, then $h_p(a) = \infty$.*

Proof. (1) ($\Rightarrow$) (2) Let H be a pure hull of A in G and N a $T(H)$-high subgroup of H containing A. Suppose that $A \neq pA$. Then there exists $a_p \in A$ such that $h_p^A(a_p) = 0$.

Suppose that $H_p = 0$. By Proposition 1.10, both of A and N are p-vertical in G. By Proposition 1.12, N is p-pure in G. Since N is torsion-free, the assertion holds.

Suppose that $H_p \neq 0$. Let

$$\mathbf{QT}_p^A(G) = \begin{pmatrix} c_{p1}, c_{p2}, \ldots, c_{pn_p} \\ t_{p1}, t_{p2}, \ldots, t_{pn_p} \end{pmatrix}.$$

For convenience, let

$$e_p = \begin{cases} t_{p1} & \text{if } n_p = 1, \\ t_{p1} + \sum_{j=2}^{n_p}(t_{pj} - c_{pj-1}) & \text{if } n_p > 1. \end{cases}$$

If $c_{pn_p} < \omega$, then, by Theorem 2.9, there exist $b_p \in N$ and $h_{pn_p} \in H$ such that $(-1)^{n_p} p^{e_p} b_p = p^{c_{pn_p}} h_{pn_p}$ and $h_p(p^{c_{pn_p}+n} h_{pn_p}) = c_{pn_p} + n$ for all $n \geqq 0$. Since $(-1)^{n_p} p^{e_p - 1} b_p \in A$ and $A \neq pA$, there exist $a'_p \in A$ and an integer r_p such that $h_p^A(a'_p) = 0$ and $(-1)^{n_p} p^{e_p-1} b_p = p^{r_p} a'_p$. Since $r(A) = 1$, there exist integers α_p and β_p such that $(\alpha_p, \beta_p) = (\alpha_p, p) = (\beta_p, p) = 1$ and $\alpha_p a'_p = \beta_p a_p$. Then $h_p(p^{r_p+1} a_p) = c_{pn_p}$ and $h_p(p^{r_p+n+1} a_p) = c_{pn} + n$ for all $n \geqq 0$.

If $c_{pn_p} = \infty$, then, by Theorem 2.9, there exist $b_p \in N$ and $h_{pn_p} \in H$ such that $h_p(p^{e_p-1} b_p) = t_{pn_p} - 1$, $(-1)^{n_p} p^{e_p} b_p = p^{t_{pn_p}+1} h_{pn_p}$, and $h_p(h_{pn_p}) = \infty$. By a same argument, the assertion holds.

(2) (⇒) (3) If $A = pA$, then, for every $a \in A$, $h_p^A(a) = \infty$. Without loss of generality, we may assume that $A \neq pA$. By hypothesis (2), there exists $a_p \in A$ such that $h_p^A(a_p) = 0$ and one of (i) and (ii) holds. Let $a \in A$. Since $r(A) = 1$, there exist integers γ_p and δ_p such that $(\gamma_p, \delta_p) = (\gamma_p, p) = 1$ and $\gamma_p a = \delta_p a_p$. Hence the assertion holds.

(3) (⇒) (2) Trivial.

(2) (⇒) (4) It is sufficient to show that, for every prime p, A is eventually p-vertical in G. If $A = pA$, then A is p-vertical in G. Without loss of generality, we may assume that $A \neq pA$. By hypothesis (2), there exists $a_p \in A$ such that $h_p^A(a_p) = 0$ and one of (i) and (ii) holds. If the condition (i) holds, then, by Lemma 3.1 (1), A is eventually p-vertical in G. Suppose that the condition (ii) holds. If $r_p = 0$, then, for all $a \in A$, $h_p(a) = \infty$. Hence A is p-vertical in G. If $r_p > 0$, then, by Lemma 3.1 (3), A is eventually p-vertical in G.

(4) (⇒) (2) For every prime p such that $A \neq pA$, let $a_p \in A$ such that $h_p^A(a_p) = 0$. By (ii), without loss of generality, we may assume that

$h_p(p^n a_p) < \omega$ for all $n \geqq 0$. Since A is eventually p-vertical in G, then, by Proposition 1.9, there exists integers r_p and d_p such that $h_p(p^{r_p} a_p) = d_p < \omega$ and $A \cap p^{d_p} G$ is p-vertical in $p^{d_p} G$. By Lemma 3.1 (1), the assertion holds.

(2) ($\Rightarrow$) (1) If $A = pA$, then A is p-vertical and p-neat in G. By Proposition 1.12, A is p-pure in G. Hence, without loss of generality, we may assume that $A \neq pA$. By hypothesis (2), there exists $a_p \in A$ such that $h_p^A(a_p) = 0$ and one of (i) and (ii) holds.

Suppose that (i) is satisfied. Let $d_p = h_p(p^{r_p} a_p)$. By Lemma 3.1 (2), $A \cap p^{d_p} G$ is p-purifiable in $p^{d_p} G$. Hence, by Proposition 1.14, A is p-purifiable in G.

Suppose that (ii) is satisfied. Let

$$d_p = \begin{cases} 0 & \text{if } r_p = 0, \\ h_p(p^{r_p - 1} a_p) + 1 & \text{if } r_p > 0. \end{cases}$$

Note that $h_p(g) = \infty$ means that g is an element of the maximal p-divisible subgroup of G. Since $h_p(p^{r_p} a_p) = \infty$, there exist an element $g_{pi} \in G$ for $i \geqq 1$ such that $p^{r_p} a_p = p^i g_{pi}$ and $p g_{pi+1} = g_{pi}$ for all $i \geqq 1$. Let

$$L = \langle g_{pi}, A \cap p^{d_p} G \mid i \geqq 1 \rangle.$$

We prove that L is p-pure in $p^{d_p} G$. Let $p^n g \in L$ such that $g \in p^{d_p} G$ and n is an integer. Then we can write

$$p^n g = \lambda_p g_{pm} + a'$$

for some integers m, λ_p and $a' \in A \cap p^{d_p} G$. Since $r(A) = 1$, there exist integers γ_p'' and δ_p'' such that $(\gamma_p'', \delta_p'') = 1$ and $\gamma_p'' a' = \delta_p'' a_p$. Then $(\gamma_p'', p) = 1$. Let $\delta_p'' = p^{u_p} \tau_p$ for some integer τ_p such that $(\tau_p, p) = 1$. Then $u_p \geqq r_p$ and

$$\begin{aligned} \gamma_p'' p^n g = \gamma_p'' \lambda_p g_{pm} + \gamma_p'' a' &= \gamma_p'' \lambda_p p^n g_{pm+n} + \tau_p p^{u_p - r_p} p^{r_p} a_p \\ &= \gamma_p'' \lambda_p p^n g_{pm+n} + \tau_p p^{u_p - r_p} p^n g_{pn} \in p^n L. \end{aligned}$$

Hence L is a p-pure subgroup of $p^{d_p} G$ containing $A \cap p^{d_p} G$. Since $A \cap p^{d_p} G$ is p-vertical in $p^{d_p} G$ by Lemma 3.1 (3) and $\frac{L}{A \cap p^{d_p} G}$ is a divisible p-group, we have $L[p] = 0$. By Proposition 1.5, L is a p-pure hull of $A \cap p^{d_p} G$ in $p^{d_p} G$. Hence A is p-purifiable in G. Since A is p-purifiable in G for every prime p, by Proposition 1.7, A is purifiable in G. $\square$

We recall the height-matrix introduced in [6, Vol. 2, p.198]. Let G be an arbitray abelian group, $p_n (n \geqq 1)$ a listing of all primes in increasing order, and $g \in G$. Then we associate with g the *height-matrix* $\mathbb{H}(g)$, an infinite matrix with ordinal numbers for entries, as follows;

$$\mathbb{H}(g) = \begin{pmatrix} h_{p_1}(g) & h_{p_1}(p_1 g) & \ldots & h_{p_1}(p_1^k g) & \ldots \\ \ldots & & & & \\ h_{p_n}(g) & h_{p_n}(p_n g) & \ldots & h_{p_n}(p_n^k g) & \ldots \\ \ldots & & & & \end{pmatrix}.$$

The element in the (n, k)-position of $\mathbb{H}(g)$ is the generalized p_n-height of $p_n^k g$, for all $n \geqq 1$ and $k \geqq 0$. The element in the (n, k)-position of $\mathbb{H}(g)$ is denoted by $\mathbb{H}_{n,k}(g)$. The nth row of $\mathbb{H}(g)$ is called the *p_n-indicator* of a. $\mathbb{H}_{n,k}(g) = \infty$ means that $p^k g$ is an element of the maximal p-divisible subgroup of G.

We can rephrase Theorem 3.2 in terms of height matrices as follows. A is purifiable in G if and only if, for every $a \in A$ and all $n \geqq 1$, the p_n-indicator of a in the height matrix $\mathbb{H}(a)$ is one of the following two types:

(1) there exists a nonnegative integer r_n such that $\mathbb{H}_{n,r_n+i}(a) < \omega$ and $\mathbb{H}_{n,r_n+i}(a) = \mathbb{H}_{n,r_n}(a) + i$ for all $i \geqq 0$;

(2) there exists a nonnegative integer r_n such that $\mathbb{H}_{n,r_n}(a) = \infty$ and if $r_n > 0$, then $\mathbb{H}_{n,r_n-1}(a) < \omega$.

In the latter half of this section, we show that all pure hulls of a torsion-free rank-one subgroup of an arbitrary abelian group are isomorphic. To do this, we need the following lemma.

Lemma 3.3 *Let G be an abelian group and A a torsion-free rank-one subgroup of G. Suppose that A is purifiable in G. Let H and K be pure hulls of A in G, M a $T(H)$-high subgroup of H containing A, and N a $T(K)$-high subgroup of K containing A. Then $M \cong N$.*

Proof. Let $u \in M$ and $v \in N$. For every prime p, let $m_p = h_p^M(u)$ and $n_p = h_p^N(v)$. By hypothesis, there exist integers r, s such that $ru = sv \in A$.

Suppose that $m_p = \infty$ and $n_p < \infty$. Then, by Proposition 2.2 (4) and Proposition 2.10, $H_p = K_p = 0$. Hence A is p-vertical in K and there exists $v'_p \in N$ such that $h_p(v'_p) = 0$ and $v = p^{n_p} v'_p$. Then $h_p(ru) = \infty$. On the other hand, by Proposition 1.13 (3), $h_p(sp^{n_p} v'_p) < \infty$. This is a contradiction. Hence $m_p = \infty$ if and only if $n_p = \infty$.

Without loss of generality, we may assume that $m_p < \infty$ and $n_p < \infty$. There exists $u_p \in M$ and $v_p \in N$ such that $h_p(u_p) = h_p(v_p) = 0$, $u = p^{m_p}u_p$, and $v = p^{n_p}v_p$. Note that $ru = rp^{m_p}u_p = sp^{n_p}v_p = sv \in A$ and $h_p(ru) = h_p(sv)$.

If p is a prime such that $H_p = 0$ and $(r,p) = (s,p) = 1$, then $K_p = 0$ and hence $m_p = n_p$.

Let p be a prime such that $H_p \neq 0$. By Proposition 2.10, $K_p \neq 0$. For every prime p such that $H_p \neq 0$, let

$$\mathbf{QT}_p^A(G) = \begin{pmatrix} c_{p1}, c_{p2}, \ldots, c_{pn_p} \\ t_{p1}, t_{p2}, \ldots, t_{pn_p} \end{pmatrix}$$

and

$$e_p = \begin{cases} t_{p1} & \text{if } n_p = 1, \\ t_{p1} + \sum_{j=2}^{n_p}(t_{pj} - c_{pj-1}) & \text{if } n_p > 1. \end{cases}$$

By Theorem 2.9, there exist $b_p \in M$ and $b'_p \in N$ such that $h_p(b_p) = h_p(b'_p) = 0$, $h_p(p^i b_p) = h_p(p^i b'_p) < \infty$ for $0 \leqq i < e_p$ and $p^{e_p-1}b_p, p^{e_p-1}b'_p \in A$. Since $r(A) = 1$, there exist integers α_p and β_p such that $(\alpha_p, \beta_p) = 1$ and $\alpha_p p^{e_p-1}b_p = \beta_p p^{e_p-1}b'_p$. Then $(\alpha_p, p) = (\beta_p, p) = 1$. Without loss of generality, we may assume that $b_p = u_p$, $b'_p = v_p$, and $\alpha_p p^{e_p-1}u_p = \beta_p p^{e_p-1}v_p \in A$.

Suppose that p is a prime such that $(r,p) = (s.p) = 1$ and $m_p < e_p$. Since $h_p(rp^{m_p}u_p) = h_p(sp^{n_p}v_p)$ and $h_p(p^i u_p) = h_p(p^i v_p) < \infty$ for $0 \leqq i < e_p$, we have $m_p = n_p$. Suppose that p is a prime such that $(r,p) = (s,p) = 1$ and $m_p \geqq e_p$. Then $n_p \geqq e_p$ and $\beta_p rp^{m_p - e_p}\alpha_p p^{e_p}u_p = \alpha_p sp^{n_p - e_p}\beta_p p^{e_p}v_p$. Since $\alpha_p p^{e_p}u_p = \beta_p p^{e_p}v_p \in A$, we have $m_p = n_p$. Hence $M \cong N$. □

Theorem 3.4 *Let G be an abelian group and A a torsion-free rank-one subgroup of G. If A is purifiable in G, then all pure hulls of A are isomorphic.*

Proof. Let H and K be pure hulls of A in G, M a $T(H)$-high subgroup of H containing A, and N a $T(K)$-high subgroup of K containing A. By Lemma 3.3, $M \cong N$. We have an isomorphism $\phi : M \to N$, choose $u \in M$, and let $v = \phi(u)$. Then, for every prime p, $h_p^M(u) = h_p^N(v) = m_p$. By Proposition 2.2 (4), if p is a prime such that $H_p \neq 0$, then $m_p < \infty$. Therefore there exists $u_p \in M$ and $v_p \in N$ such that $h_p(u_p) = h_p(v_p) = 0$, $u = p^{m_p}u_p$, $v = p^{m_p}v_p$, and $\phi(u_p) = v_p$.

For a prime p such that $H_p \neq 0$, let

$$\mathbf{QT}_p^A(G) = \begin{pmatrix} c_{p1}, c_{p2}, \ldots, c_{pn_p} \\ t_{p1}, t_{p2}, \ldots, t_{pn_p} \end{pmatrix}.$$

By Corollary 2.11, we have the following.

(1) If $c_{pn_p} < \infty$, then, for $1 \leqq i \leqq n_p$, there exist $x_{pi} \in H_p$, $x'_{pi} \in K_p$, $g_{pi} \in H$, and $g'_{pi} \in K$ such that
  (1) $(H/M)_p = \bigoplus_{i=1}^{n_p} \langle g_{pi} + M \rangle$, where $o(g_{pi} + M) = p^{c_{pi}}$ for $1 \leqq i \leqq n_p$;
  (2) $H_p = \bigoplus_{i=1}^{n_p} \langle x_{pi} \rangle$, where $o(x_{pi}) = p^{t_{pi}}$ for $1 \leqq i \leqq n_p$;
  (3) $x_{p1} = u_p + p^{c_{p1}-t_{p1}} g_{p1}$, $x_{pi} = g_{pi-1} + p^{c_{pi}-t_{pi}} g_{pi}$ for $2 \leqq i \leqq n_p$,

and
  (1) $(K/N)_p = \bigoplus_{i=1}^{n_p} \langle g'_{pi} + N \rangle$, where $o(g'_{pi} + N) = p^{c_{pi}}$ for $1 \leqq i \leqq n_p$;
  (2) $K_p = \bigoplus_{i=1}^{n_p} \langle x'_{pi} \rangle$, where $o(x'_{pi}) = p^{t_{pi}}$ for $1 \leqq i \leqq n_p$;
  (3) $x'_{p1} = v_p + p^{c_{p1}-t_{p1}} g'_{p1}$, $x'_{pi} = g'_{pi-1} + p^{c_{pi}-t_{pi}} g'_{pi}$ for $2 \leqq i \leqq n_p$.

(2) If $c_{pn_p} = \infty$, then there exist $x_{pi} \in H_p$ and $x'_{pi} \in K_p$ for $1 \leqq i \leqq n_p$, and $g_{pi} \in H$ and $g'_{pi} \in K$ for $i \geqq 1$ such that
  (1) $(H/M)_p = \bigoplus_{i=1}^{n_p-1} \langle g_{pi} + M \rangle \oplus D^{(p)}/N$, where $o(g_{pi} + M) = p^{c_{pi}}$ for $1 \leqq i \leqq n_p - 1$ and $D^{(p)}/M \cong \mathbf{Z}[p^\infty]$ such that

$$D^{(p)}/M = \langle g_{pi} + M \mid i \geqq n_p,\ pg_{pi+1} = g_{pi},\ p^{t_{pn_p}+1} g_{pn_p} \in M \rangle;$$

  (2) $H_p = \bigoplus_{i=1}^{n_p} \langle x_{pi} \rangle$, where $o(x_{pi}) = p^{t_{pi}}$ for $1 \leqq i \leqq n_p$;
  (3) $x_{p1} = u_p + p^{c_{p1}-t_{p1}} g_{p1}$, $x_{pi} = g_{pi-1} + p^{c_{pi}-t_{pi}} g_{pi}$ for $1 \leqq i \leqq n_p - 1$ and $x_{pn_p} = g_{pn_p-1} + pg_{pn_p}$,

and
  (1) $(K/N)_p = \bigoplus_{i=1}^{n_p-1} \langle g'_{pi} + M \rangle \oplus D^{(p)}/N$, where $o(g'_{pi} + M) = p^{c_{pi}}$ for $1 \leqq i \leqq n_p - 1$ and $D^{(p)}/M \cong \mathbf{Z}[p^\infty]$ such that

$$D^{(p)}/M = \langle g'_{pi} + M \mid i \geqq n_p,\ pg'_{pi+1} = g'_{pi},\ p^{t_{pn_p}+1} g'_{pn_p} \in M \rangle;$$

  (2) $K_p = \bigoplus_{i=1}^{n_p} \langle x'_{pi} \rangle$, where $o(x'_{pi}) = p^{t_{pi}}$ for $1 \leqq i \leqq n_p$;
  (3) $x'_{p1} = v_p + p^{c_{p1}-t_{p1}} g'_{p1}$, $x'_{pi} = g'_{pi-1} + p^{c_{pi}-t_{pi}} g'_{pi}$ for $1 \leqq i \leqq n_p - 1$ and $x'_{pn_p} = g'_{pn_p-1} + pg'_{pn_p}$.

To extend this isomorphism to H, define $\phi(g_{pi}) = g'_{pi}$ for all $1 \leqq i \leqq n_p$ if $c_{pn_p} < \infty$, $\phi(g_{pi}) = g'_{pi}$ for all $i \geqq 1$ if $c_{pn_p} = \infty$, and

$$\phi\Big(\sum_{r=1}^{n}\Big(\sum_{i=1}^{n_{p_r}}\alpha_{p_ri}g_{p_ri}\Big)+\sum_{s=1}^{m}\beta_{q_ss}g_{q_sn_{q_s}+l_s}+u'\Big)$$
$$=\sum_{k=1}^{n}\Big(\sum_{i=1}^{n_{p_r}}\alpha_{p_ri}g'_{p_ki}\Big)+\sum_{s=1}^{m}\beta_{q_ss}g'_{q_sn_{q_s}+l_s}+\phi(u'),$$

where every α_{p_ri} and every β_{q_ss} is integer for $1\leqq r\leqq n$, $1\leqq i\leqq n_{p_r}$ and $1\leqq s\leqq m$ and $u'\in M$.

Let $h\in H$ such that

$$h=\sum_{r=1}^{n}\Big(\sum_{i=1}^{n_{p_r}}\alpha_{p_ri}g_{p_ri}\Big)+\sum_{s=1}^{m}\beta_{q_ss}g_{q_sn_{q_s}+l_s}+\delta u,$$

where every α_{p_ri} and every β_{q_ss} is integer for $1\leqq r\leqq n$, $1\leqq i\leqq n_{p_r}$ and $1\leqq s\leqq m$ and $\delta\in\mathbf{Q}$. Suppose that $h=0$. Since $\{g'_{p_ri}+N, g'_{q_sn_{q_s}+l_s}+N\mid 1\leqq r\leqq n,\ 1\leqq i\leqq n_{p_r},\ 1\leqq s\leqq m\}$ is independent in K/N, $p_r^{c_{p_ri}}$ divides α_{p_ri} and $q_s^{t_{q_sn_{q_s}}+l_s+1}$ divides β_{q_ss}. Hence we write $\alpha_{p_ri}=p_r^{c_{p_ri}}\alpha'_{p_ri}$ and $\beta_{q_ss}=q_s^{t_{q_sn_{q_s}}+l_s+1}\beta'_{q_ss}$ for some integers α'_{p_ri} and β'_{q_s}. Let

$$e_p=\begin{cases}t_{p1} & \text{if } n_p=1,\\ t_{p1}+\sum_{j=2}^{n_p}(t_{pj}-c_{pj-1}) & \text{if } n_p>1.\end{cases}$$

By (2.11.1),

$$h=\Big(\sum_{r=1}^{n}\Big(\sum_{i=1}^{n_{p_r}}\alpha'_{p_ri}(-1)^{n_{p_r}}p_r^{e_{p_r n_{p_r}}-m_{p_r}}\Big)$$
$$+\sum_{s=1}^{m}\beta'_{q_ss}(-1)^{n_{q_s}}q_s^{e_{q_sn_{q_s}}-m_{q_s}}+\delta\Big)u=0.$$

Let
$\lambda=\sum_{r=1}^{n}\big(\sum_{i=1}^{n_{p_r}}\alpha'_{p_ri}(-1)^{n_{p_r}}p_r^{e_{p_rn_{p_r}}-m_{p_r}}\big)+\sum_{s=1}^{m}\beta'_{q_ss}(-1)^{n_{q_s}}q_s^{e_{q_sn_{q_s}}-m_{q_s}}+\delta$. Then $\lambda=0$. It is immediate that if $\lambda=0$, then $h=0$. Therefore we proved that

$$h=\sum_{r=1}^{n}\Big(\sum_{i=1}^{n_{p_r}}\alpha_{p_ri}g_{p_ri}\Big)+\sum_{s=1}^{m}\beta_{q_ss}g_{q_sn_{q_s}+l_s}+\delta u=0$$

if and only if $\lambda = 0$, where $\alpha_{p_r i} = p_r^{c_{p_r i}} \alpha'_{p_r i}$, $\beta_{q_s s} = q_s^{t_{q_s n_{q_s}} + l_s + 1} \beta'_{q_s s}$ for some integers $\alpha'_{p_r i}$ and $\beta'_{q_s s}$ and

$$\lambda = \sum_{r=1}^{n} \Big(\sum_{i=1}^{n_{p_r}} \alpha'_{p_r i} (-1)^{n_{p_r}} p_r^{e_{p_r n_{p_r}} - m_{p_r}} \Big) + \sum_{s=1}^{m} \beta'_{q_s s} (-1)^{n_{q_s}} q_s^{e_{q_s n_{q_s}} - m_{q_s}} + \delta.$$

Hence ϕ is well-defined and a monomorphism. It is immediate that ϕ is an epimorphism and $\phi|_{T(H)}$ is an isomorphism onto $T(K)$. Therefore ϕ is an isomorphism. □

4. Purifiability of T-high subgroups

In this section, we consider T-high subgroups of arbitrary abelian groups that are purifiable in given groups.

Theorem 4.1 *Let G be an abelian group and A a T-high subgroup of G. Suppose that A is purifiable in G. For every pure hull H of A in G, then H is an ADE-group H with A as a moho subgroup and there exists a subgroup T_1 of T such that*

$$G = H \oplus T_1.$$

Moreover,

(1) *if H and K are pure hulls of A in G, then $H \cong K$;*

(2) *there exists a subgroup T' of T such that $G = H \oplus T'$ for every pure hull H of A in G.*

Proof. By Proposition 2.2 (3), H_p is bounded pure in G_p for every prime p. Hence H_p is a direct summand of G_p and there exists a subgroup T_1 of T such that $T = T(H) \oplus T_1$. We prove that $H \oplus T_1$ is pure in G. Let $ng \in H \oplus T_1$ with $g \in G$ and $n \in \mathbf{Z}$. Then there exist $h \in H$ and $t \in T_1$ such that $ng = h + t$. Moreover, we have $mng \in H$ for some integer m. Since H is pure in G, $mng \in H \cap mnG = mnH$. Hence there exists $h' \in H$ such that $mng = mnh'$. Since $ng - nh' \in T \cap nG = nT = n(T(H) \oplus T_1)$, there exist $h_1 \in T(H)$ and $t_1 \in T_1$ such that $ng - nh' = n(h_1 + t_1)$. Hence $ng = n(h' + h_1 + t_1) \in n(H \oplus T_1)$. Since $H \oplus T_1$ is essential in G, $G = H \oplus T_1$. By Proposition 2.2 (1), it follows that A is almost-dense in H. Hence H is an ADE-group with A as a moho subgroup.

Fix a prime p and recall notations as follows:

$$A_G^n(p) = (A + p^{n+1}G) \cap p^n G[p] = ((A \cap p^n G) + p^{n+1}G)[p]$$

and

$$A_n^G(p) = (A \cap p^n G)[p] + p^{n+1}G[p] = p^{n+1}G[p].$$

By Proposition 1.10 and Proposition 1.11, there exists the least integer m such that $A_G^n(p) = A_n^G(p)$ for all $n \geqq m$. Then $p^m G[p] = p^m T_1[p]$.

For integer $n \geqq 0$, let $p^n g + A \in p^n(G/A)[p]$. Since $p^{n+1}g \in H \cap p^{n+1}G = p^{n+1}H$, there exists $h \in H$ such that $p^{n+1}g = p^{n+1}h$. Since $p^n g - p^n h \in p^n G[p]$, we have $p^n(G/A)[p] = p^n(H/A)[p] + \frac{p^n G[p]+A}{A}$. Let $x \in A_G^n(p)$. Then we can write $x = a + p^{n+1}g'$ for some $a \in A$ and $g' \in G$. Since $x + A \in p^{n+1}(G/A)[p] = p^{n+1}(H/A)[p] + \frac{p^{n+1}G[p]+A}{A}$, there exist $a' \in A$, $h' \in H$, and $p^{n+1}g_0 \in p^{n+1}G[p]$ such that $x = a + p^{n+1}g' = a' + p^{n+1}h' + p^{n+1}g_0$. Since $h_p(a) \geqq n$, also $h_p(a') \geqq n$. Hence $A_G^n(p) = A_H^n(p) + A_n^G(p)$. By Proposition 2.2 (1), $p^n H[p] \subseteq A + p^{n+1}H$ for all $n \geqq 0$. Hence, for all $n \geqq 0$, there exist subsocles S_n and H_n of G such that

$$\begin{aligned} p^n G[p] &= A_G^n(p) \oplus S_n = (p^n H[p] + p^{n+1}G[p]) \oplus S_n \\ &= H_n \oplus p^{n+1}G[p] \oplus S_n. \end{aligned}$$

Similarly, since $A_G^n(p) = A_K^n(p) + A_n^G(p)$ and $p^n K[p] \subseteq A + p^{n+1}K$ for all $n \geqq 0$, for all $n \geqq 0$, there exist subsocles K_n of G such that

$$\begin{aligned} p^n G[p] &= A_G^n(p) \oplus S_n = (p^n K[p] + p^{n+1}G[p]) \oplus S_n \\ &= K_n \oplus p^{n+1}G[p] \oplus S_n. \end{aligned}$$

Let $S = \bigoplus_{i=1}^{m-1} S_i$. Then

$$G[p] = H[p] \oplus p^m T_1[p] \oplus S = K[p] \oplus p^m T_1[p] \oplus S.$$

Hence, for every prime p, there exist a nonnegative integer m_p and a subsocle S_p of G such that

$$G[p] = H[p] \oplus p^{m_p} T_1[p] \oplus S_p = K[p] \oplus p^{m_p} T_1[p] \oplus S_p.$$

Since $(S_p \oplus p^{m_p}T_1[p]) \cap p^{m_p}G_p = (S_p \cap p^{m_p}G) \oplus p^{m_p}T_1[p] = p^{m_p}T_1[p]$, $(p^{m_p}T_1)_p$ is pure in $p^{m_p}G_p$ and so $p^{m_p}T_1[p]$ is purifiable in $p^{m_p}G_p$. By Proposition 1.13, $(S_p \oplus p^{m_p}T_1[p])$ is purifiable in G_p. Then there exists a pure hull L_p of $(S_p \oplus p^{m_p}T_1[p])$ in G_p.

Let $h \in H[p]$ and $x \in L_p[p]$. Then we have $h_p(h+x) = \min\{h_p(h), h_p(x)\}$. Hence, by [8, Theorem 2], $G_p = H_p \oplus L_p$. Similarly, $G_p = K_p \oplus L_p$. Let

$T' = \bigoplus_p L_p$. By the above proof, we have

$$G = H \oplus T' = K \oplus T'.$$

□

Definition 4.2 An abelian group G is said to be a *strongly ADE decomposable group* if there exists a purifiable T-high subgroup of G.

Let G be an abelian group such that, for every prime p, G_p is the direct sum of a bounded and a divisible subgroup. By [13, Theorem 5.2], G is a strongly ADE decomposable group. Let G, H and A be groups as in Theorem 4.1. Then H is a minimal direct summand of G containing A.

Corollary 4.3 *Let G be an abelian group of torsion-free rank* 1 *such that, for every prime p, G_p is the direct sum of a bounded and a divisible subgroup. Let A be a subgroup of G such that $A \not\subseteq T$. Then there exists a minimal direct summand of G containing A.*

Proof. By [13, Theorem 5.2], A is purifiable in G. Let H be a pure hull of A in G. Then every $T(H)$-high subgroup of H is a T-high subgroup of G. By the proof of Theorem 4.1, there exists a subgroup T_1 of G such that $G = H \oplus T_1$. Since H is a pure hull of A in G, H is a minimal direct summand of G containing A. □

5. Strongly ADE Decomposable groups of torsion-free rank 1

In this section, we consider ADE decomposable groups of torsion-free rank 1. First we exhibit a strongly ADE decomposable group G of torsion-free rank 1 for which not all T-high subgroups are purifiable in G.

The existence of the following groups H and G_p are guaranteed by [12, Theorem 2.8] and [6, Vol. 1, Example, p.150], respectively.

Example 5.1. Let q be a fixed prime and for every prime $p \neq q$, let t_p and c_p be positive integers such that $t_p < c_p$. Let H be an ADE group with A as a moho subgroup A such that
(1) $r(A)$=1 and $A = qA$;
(2) for every prime $p \neq q$, $H_p = \langle y_p \rangle$, where $o(y_p) = p^{t_p}$ and $H_q = 0$;
(3) $H/A = \bigoplus_{p \neq q} \langle h_p + A \rangle$, where $h_p \in H$ and $o(h_p + A) = c_p$.
Let $G_q = \langle x_n \mid n \geqq 0 \rangle$ be defined by the defining relations

$$qx_0 = 0 \quad \text{and} \quad q^k x_k = x_0 \ \text{ for } \ k \geqq 1.$$

Let $a \in A$, $b = a + x_1$, $G = H \oplus G_q$, and N a $T(G)$-high subgroup of G containing b. Then N is not purifiable in G.

Proof. Note that $h_q(b) = 0$ and $h_q(qb) = \omega$. Hence the q-indicator of b in the height-matrix $\mathbb{H}(b)$ is

$$(0, \omega, \infty, \dots).$$

By Theorem 3.2 (3), N is not purifiable in G. □

Now we characterize the abelian groups G of torsion-free rank 1 for which all T-high subgroups are purifiable in G.

Theorem 5.2 *Let G be an abelian group of torsion-free rank 1. Then all T-high subgroups of G are purifiable in G if and only if, for every prime p and every $g \in G \setminus T$, one of the following conditions holds:*

1. *there exists an integer r_p such that $h_p(p^{r_p}g) < \omega$ and $h_p(p^{r_p+i}g) = h_p(p^{r_p}g) + i$ for all $i \geqq 0$;*
2. *there exists an integer r_p such that $h_p(p^{r_p}g) = \infty$ and if $r_p > 0$, then $h_p(p^{r_p-1}g) < \omega$.*

Proof. ($\Rightarrow$) Let $g \in G \backslash T$ and A a T-high subgroup of G containing g. By hypothesis, A is purifiable in G. Hence, by Theorem 3.2, (1) or (2) holds.

($\Leftarrow$) Let A be any T-high subgroup of G. By hypothesis and Theorem 3.2 (3), A is purifiable in G. □

We can rephrase Theorem 5.2 in terms of height matrices as follows. All T-high subgroups of an arbitrary abelian group G of torsion-free rank 1 are purifiable in G if and only if, for every $g \in G \setminus T$ and all $n \geqq 1$, the p_n-indicator of g in the height matrix $\mathbb{H}(g)$ is one of the following two types:

1. there exists a nonnegative integer r_n such that $\mathbb{H}_{n,r_n+i}(g) < \omega$ and $\mathbb{H}_{n,r_n+i}(g) = \mathbb{H}_{n,r_n}(g) + i$ for all $i \geqq 0$;
2. there exists a nonnegative integer r_n such that $\mathbb{H}_{n,r_n}(g) = \infty$ and if $r_n > 0$, then $\mathbb{H}_{n,r_n-1}(g) < \omega$.

From Theorem 3.2 and Theorem 5.2, the following is immediate:

Corollary 5.3 *Let G be an abelian group of torsion-free rank 1. Then all T-high subgroups of G are purifiable in G if and only if all torsion-free subgroups are purifiable in G.*

Now we give a characterization of an arbitrary abelian group of torsion-free rank 1 that is a strongly ADE decomposable group. Before doing it, we give a useful lemma.

Lemma 5.4 *Let G be an abelian group of torsion-free rank* 1 *and A a torsion-free subgroup of G. If A is purifiable in G and K is a pure hull of A in G, then there exists a subgroup T' of T such that $G = K \oplus T'$. Hence G is a strongly ADE decomposable group. Moreover, if A is p-vertical in G for every prime p, then G is splitting.*

Proof. Let N be a $T(K)$-high subgroup of K containing A. Then N is a T-high subgroup of G and K is a pure hull of N in G. By Theorem 4.1, there exists a subgroup T' of T such that $G = K \oplus T'$. Hence G is a strongly ADE decomposable group. If A is p-vertical in G for every prime p, then, by Proposition 1.10 and The comment after Definition 1.8, K is torsion-free. Hence G is splitting. □

Theorem 5.5 *Let G be an abelian group of torsion-free rank* 1. *Then G is a strongly ADE decomposable group if and only if there exists an element $g \in G \setminus T$ such that, for all $n \geqq 1$, the p_n-indicator of g in the height matrix $\mathbb{H}(g)$ is one of the following two types*:

(1) *there exists a nonnegative integer r_n such that $\mathbb{H}_{n,r_n+i}(g) < \omega$ and $\mathbb{H}_{n,r_n+i}(g) = \mathbb{H}_{n,r_n}(g) + i$ for all $i \geqq 0$;*

(2) *there exists a nonnegative integer r_n such that $\mathbb{H}_{n,r_n}(g) = \infty$ and if $r_n > 0$, then $\mathbb{H}_{n,r_n-1}(g) < \omega$.*

Proof. ($\Rightarrow$) There exists a purifiable T-high subgroup of G. By Theorem 3.2 (3), the assertion holds.

($\Leftarrow$) Let $g \in G \setminus T$ satisfying one of the above two conditions. Let $B = \langle g \rangle$. Consider an element $ng \in B$ for some integer n and the height matrices $\mathbb{H}(g)$ and $\mathbb{H}(ng)$. By hypothesis, $\mathbb{H}(g)$ and $\mathbb{H}(ng)$ are equivalent. By Theorem 3.2 and Lemma 5.4, G is a strongly ADE decomposable group. □

Lemma 5.4 and Theorem 5.5 combined lead to the splitting theorem of arbitrary abelian groups of torsion-free rank 1 established in [14] by Stratton.

Corollary 5.6 [14, Theorem] *Let G be an abelian group of torsion-free rank* 1. *Then the following properties are equivalent*:

(1) *G is splitting*;

(2) *there exists $g \in G \setminus T$ such that $\langle g \rangle$ is p-vertical in G for every prime*

p and if $h_p(g) \geqq \omega$, *then* $h_p(g) = \infty$;

(3) *G is a strongly ADE decomposable group satisfying the following condition and let A be a purifiable T-high subgroup of G and for every prime p, let t_{pn_p} be the least integer such that $V_{p,n}(G, A) = 0$ for $n \geqq t_{pn_p}$. Then, for almost all prime p and every $a \in A$,*

$$h_p(a) \geqq t_{pn_p}.$$

Proof. (1) ⇒ (2) By hypothesis, we can write $G = F \oplus T$ for some torsion-free subgroup F of G. Let $g \in F$. Since F is torsion-free, the assertion immediately holds.

(2) ⇒ (1) Let $B = \langle g \rangle$. By Theorem 3.2 (3), it is immediate that B is purifiable in G. By Lemma 5.4, G is splitting.

(1) ⇒ (3) By hypothesis, $G = F \oplus T$, where F be a torsion-free subgroup of G. Clearly, G is a strongly ADE decomposable group of torsion-free rank 1. Let A be a purifiable T-high subgroup of G and for every prime p, let t_{pn_p} be the least integer such that $V_{p,n}(G, A) = 0$ for $n \geqq t_{pn_p}$. Let $a \in A$ and r the least integer such that $ra \in F$. Suppose that there exists a prime q such that $(r, q) = 1$ and $h_q(a) < t_{qn_q}$. Then $t_{qn_q} \geqq 1$ and hence A is not q-vertical in G. For every prime p, let $m_p = h_p^A(a)$. Let H be a pure hull of A in G. Then $H_q \neq 0$ and let

$$\mathbf{QT}_q^A(G) = \begin{pmatrix} c_{q1}, c_{q2}, \ldots, c_{qn_q} \\ t_{q1}, t_{q2}, \ldots, t_{qn_q} \end{pmatrix}$$

and

$$e_q = \begin{cases} t_{q1} & \text{if } n_q = 1, \\ t_{q1} + \sum_{j=2}^{n_q} (t_{qj} - c_{qj-1}) & \text{if } n_q > 1. \end{cases}$$

Then there exists $a_q \in A$ such that $h_q(a_q) = 0$, $a = q^{m_q} a_q$, $h_q(q^{e_q - 1} a_q) = t_{qn_q} - 1$ and $m_q < e_q$. By Corollary 2.11, there exist $x_q \in A_G^{t_{qn_q-1}}(q) \setminus A_{t_{qn_q-1}}^G(q)$ and $g_q \in G$ such that $x_q = q^{e_q - m_q - 1} a + p^{t_{qn_q}} g_q$. Since $(r, q) = 1$ and F is q-vertical in G, we have

$$0 \neq r x_q = q^{e_q - m_q - 1} ra + q^{t_{qn_q}} r g_q \in F_G^{t_{qn_q-1}}(q) = F_{t_{qn_q-1}}^G(q) = q^{t_{qn_q}} G[q].$$

This is a contradiction. Hence (3) holds.

(3) ⇒ (1) Let H be a pure hull of A in G. By Theorem 4.1, there exists a subgroup T' of T such that $G = H \oplus T'$. It suffices to prove that

H is splitting. By hypothesis, there exists $a \in A$ such that, for all prime p, $h_p(a) \geqq t_{pn_p}$. Let $B = \langle a \rangle$. We show that, for every prime p, B is p-vertical in H. If $H_p = 0$, then, by Proposition 1.13 (2), B is p-vertical in H. Without loss of generality, we may assume that $H_p \neq 0$. Let

$$\mathbf{QT}_p^A(G) = \begin{pmatrix} c_{p1}, c_{p2}, \ldots, c_{pn_p} \\ t_{p1}, t_{p2}, \ldots, t_{pn_p} \end{pmatrix}$$

and

$$e_p = \begin{cases} t_{p1} & \text{if } n_p = 1, \\ t_{p1} + \sum_{j=2}^{n_p} (t_{pj} - c_{pj-1}) & \text{if } n_p > 1. \end{cases}$$

By Corollary 2.11, there exists $a_p \in A$ such that $h_p(a_p) = 0$, $a = p^{m_p} a_p$, and $h_p(p^{e_p - 1} a_p) = t_{pn_p} - 1$. Then $m_p \geqq e_p$.

If $c_{pn_p} < \infty$, then, by Corollary 2.11, there exists $h_{pn_p} \in H$ such that $p^{c_{pn_p}} h_{pn_p} = p^{e_p} a_p$ and $h_p(p^{c_{pn_p}+i} h_{pn_p}) = c_{pn_p} + i$ for all $i \geqq 0$. Since $p^{c_{pn_p}+m_p-e_p} h_{pn_p} = p^{m_p} a_p = a$, we have $h_p(p^i a) = c_{pn_p} + m_p - e_p + i$ for all $i \geqq 0$. By Proposition 1.13 (3), B is p-vertical in H.

If $c_{pn_p} = \infty$, then, by Corollary 2.11, there exists $h_{pn_p} \in H$ such that $h_p(h_{pn_p}) = \infty$ and $p^{t_{pn_p}+1} h_{pn_p} = p^{e_p} a_p$. Since $m_p \geqq e_p$, $h_p(a) = \infty$. Hence, by Proposition 1.13 (2), B is p-vertical in G for every prime p.

By [13, Theorem 5.2], B is purifiable in H. Let K be a pure hull of B in H. By Lemma 5.4, $H = K \oplus T(H)$. □

We can rephrase Theorem 5.6 in terms of height matrices as follows.

Corollary 5.7 *An abelian group G of torsion-free rank 1 is splitting if and only if there exists an element $g \in G \setminus T$ such that, for all $n \geqq 1$, the p_n-indicator of g in the height matrix $\mathbb{H}(g)$ is one of the following two types:*
(1) $\mathbb{H}_{n,0}(g) < \omega$ *and* $\mathbb{H}_{n,i}(g) = \mathbb{H}_{n,0}(g) + i$ *for all* $i \geqq 0$;
(2) $\mathbb{H}_{n,0}(g) = \infty$.

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