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On conformal transformations in tangent bundles

Kazunari YAMAUCHI

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Abstract. Let M be a complete, simply connected Riemannian manifold with positive constant scalar curvature, and TM its tangent bundle with the complete lift metric. Assume that TM admits an essential infinitesimal conformal transformation, then M is isometric to the standard sphere.

Key words: infinitesimal conformal transformation, infinitesimal projective transformation, Lie derivation.

1. Introduction

In the present paper everything will be always discussed in the C^{∞} category, and Riemannian manifolds will be assumed to be connected and dimension > 1. Let M be a Riemannian manifold, and let ϕ be a transformation of M. Then ϕ is called a projective transformation of M, if it preserves the geodesics, where each geodesic should be confounded with a subset of M by neglecting its affine parameter. Furtheremore ϕ is called an affine transformation, if it preserves the Riemannian connection. We then remark that a affine transformation may be characterized as a projective transformation which preserves the affine parameter together with the geodesics. Let V be a vector field on M, and let us consider the local one-parameter group $\{\phi_t\}$ of local transformations of M generated by V. Then V is called an infinitesimal projective transformation on M, if each ϕ_t is a local projective transformation of M. Clearly an affine transformation is a projective transformation, the converse is not true in general. Indeed consider the n-dimensional real projective space $P^n(R)$ with the standard Riemannian metric, which is the standard projectively flat Riemannian manifold, and is a space of positive constant curvature. As is well known, $P^n(R)$ admits a non-affine infinitesimal projective transformation. As a converse problem, we know the following.

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Problem Let M be a complete, simply connected Riemannian manifold with positive constant scalar curvature. Assume that M admits a non-affine projective transformation, then is it isometric to the standard sphere?

We know there are many affirmative answers for this problem under some additional conditions. For examples:

Theorem A ([4]) *Let M be a compact, simply connected Riemannian manifold with constant scalar curvature. Assume that M admits a non-affine infinitesimal projective transformation, then it is isometric to the standard sphere.*

Theorem B ([3], [5]) *Let M be a complete, simply connected Riemannian manifold with harmonic curvature. Assume that M admits a non-affine projective transformation, then it is isometric to the standard sphere.*

Let $T(M)$ be a tangent bundle over M with the complete lift metric $\overline{g}$ and X a vector field in $T(M)$. Then X is called an infinitesimal conformal transformation in $T(M)$, if there exists a scalar function ρ in $T(M)$ such that $\pounds_X \overline{g} = 2\rho\overline{g}$, where $\pounds_X$ denotes the Lie derivation with respect to X, and further it is called essential if ρ depends on (y^i) essentially, where (x^i, y^i) the induced coordinates in $T(M)$.

The purpose of the present paper is to investigate some relations between the infinitesimal conformal transformations in $T(M)$ and the infinitesimal projective transformations on M, and to prove the following theorem.

Theorem *Let M be a complete, simply connected Riemannian manifold with positive constant scalar curvature and $T(M)$ its tangent bundle with the complete lift metric. Assume that $T(M)$ admits an essential infinitesimal conformal transformation X, then we have*

(1) *X induces an infinitesimal projective transformation on M, and furthermore M is isometric to the standard sphere;*

(2) *The Weyl's conformal curvature tensor of $T(M)$ vanishes, that is, $T(M)$ is conformally flat.*

This fact seems to support the evidence that the problem has an affirmative answer.

Let $(S^n; \lambda)$ be a standard sphere of radius $\frac{1}{\sqrt{\lambda}}$ and Δ the Laplacian acting on $(S^n; \lambda)$. The first eigenvalue of Δ is $n\lambda$ and the eigenfunction f

satisfies the differential equation $\nabla_i f_j + \lambda f g_{ij} = 0$. The gradient$f$ defines an infinitesimal conformal transformation on S^n. Conversely, we have the following Obata's theorem.

Theorem C ([1]) *Let M be a complete Riemannian manifold. In order that M admits a non-constant scalar function f on M satisfying*

$$\nabla_i f_j + \lambda f g_{ij} = 0,$$

for some positive constant λ, it is necessary and sufficient that M is isometric to the standard sphere of radius $\frac{1}{\sqrt{\lambda}}$.

Next, the second eigenvalue of Δ is $2(n+1)\lambda$ and the eigenfunction$f$ satisfies the differential equation $\nabla_i \nabla_j f_k + \lambda(2 f_i g_{jk} + f_j g_{ki} + f_k g_{ij}) = 0$. The gradient f defines an infinitesimal projective transformation on S^n. Conversely, we have the following Tanno's theorem.

Theorem D ([2]) *Let M be a complete, simply connected Riemannian manifold. In order that M admits a non-constant scalar function f on M satisfying*

$$\nabla_i \nabla_j f_k + \lambda(2 f_i g_{jk} + f_j g_{ki} + f_k g_{ij}) = 0, \qquad (*)$$

for some positive constant λ, it is necessary and sufficient that M is isometric to the standard sphere of radius $\frac{1}{\sqrt{\lambda}}$.

For the study of projective transformation groups, the differential equation $(*)$ plays an important role. Indeed, Theorem A and Theorem B were proved by using $(*)$, and it will be used in the proof of Theorem.

2. Preliminaries

Let $\Gamma_{i\,j}^{\,h}$ be the coefficients of the Riemannian connection of M, then $y^a \Gamma_{a\,j}^{\,h}$ can be regarded as coefficients of a non-linear connection of $T(M)$, where (x^h, y^h) are the induced coordinates in $T(M)$. The indices $a, b, c, \ldots, h, i, j, \ldots,$ run over the range $\{1, 2, \ldots, n\}$ and the indices $\bar{a}, \bar{b}, \bar{c}, \ldots, \bar{h}, \bar{i}, \bar{j}, \ldots,$ run over the range $\{\bar{1}, \bar{2}, \ldots, \bar{n}\}$. The summation convention will be used in relation to this system of indices. By using $y^a \Gamma_{a\,j}^{\,h}$, we can define a local basis $\{X_h, X_{\bar{h}}\}$ of $T(M)$ as follows:

$$X_h = \frac{\partial}{\partial x^h} - y^a \Gamma_{a\,h}^{\,m} \frac{\partial}{\partial y^m} \quad \text{and} \quad X_{\bar{h}} = \frac{\partial}{\partial y^h},$$

which is called the adapted frame of $T(M)$. We denote $\{dx^h, \delta y^h\}$ the dual basis of the adapted frame. By the straightforward caluculations, we have the following lemma.

Lemma 1 *The Lie brackets of the adapted frame of $T(M)$ satisfy the following*:

$$[X_i, X_j] = y^a K_{jia}{}^m X_{\overline{m}}, \tag{1}$$

$$[X_i, X_{\overline{j}}] = \Gamma_j{}^m{}_i X_{\overline{m}}, \tag{2}$$

$$[X_{\overline{i}}, X_{\overline{j}}] = 0, \tag{3}$$

where $K_{jia}{}^m$ denote the components of the curvature tensor of M.

Let X be a vector field in $T(M)$ and $(v^h, v^{\overline{h}})$ the components of X with respect to the adapted frame. The components v^h and $v^{\overline{h}}$ are said to be the horizontal components and the vertical components of X, respectively. Let $\pounds_X$ be the Lie derivation with respect to X. By using Lemma 1, we can easily prove the following lemma for the Lie derivatives of the adapted frame and the dual basis.

Lemma 2 *The Lie derivatives of the adapted frame and the dual basis are given as follows*:

$$\pounds_X X_h = -X_h(v^m)X_m - \{y^r v^a K_{ahr}{}^m + v^{\overline{a}}\Gamma_a{}^m{}_h + X_h(v^{\overline{m}})\}X_{\overline{m}}, \tag{1}$$

$$\pounds_X X_{\overline{h}} = -X_{\overline{h}}(v^m)X_m + \{v^a\Gamma_h{}^m{}_a - X_{\overline{h}}(v^{\overline{m}})\}X_{\overline{m}}, \tag{2}$$

$$\pounds_X dx^h = X_m(v^h)dx^m + X_{\overline{m}}(v^h)\delta y^m, \tag{3}$$

$$\begin{aligned}\pounds_X \delta y^h = \{&y^r v^a K_{amr}{}^h + v^{\overline{a}}\Gamma_a{}^h{}_m + X_m(v^{\overline{h}})\}dx^m \\ &- \{v^a\Gamma_m{}^h{}_a - X_{\overline{m}}(v^{\overline{h}})\}\delta y^m.\end{aligned} \tag{4}$$

3. Infinitesimal conformal transformations in $T(M)$

Let $g = g_{ij}dx^i dx^j$ be a Riemannian metric of M. The complete lift metric $\overline{g}$ of $T(M)$ is defined by $\overline{g} = 2g_{ij}dx^i\delta y^j$. By means of (3) and (4) of Lemma 2, we have the following lemma.

Lemma 3 *The Lie derivative of $\overline{g}$ is given as follows*:

$$\begin{aligned}\pounds_X\overline{g} = 2g_{im}&\{y^r v^a K_{ajr}{}^m + v^{\overline{a}}\Gamma_a{}^m{}_j + X_j(v^{\overline{m}})\}dx^i dx^j \\ &+ 2\{v^a\partial_a g_{ij} + g_{mj}X_i(v^m) - g_{im}(v^a\Gamma_j{}^m{}_a - X_{\overline{j}}(v^{\overline{m}})\}dx^i\delta y^j\end{aligned}$$

$$+ 2g_{mj}X_{\bar{i}}(v^m)\delta y^i \delta y^j.$$

Let X be an infinitesimal conformal transformation in $T(M)$ with the complete lift metric $\bar{g}$, that is, there exists a scalar function ρ in $T(M)$ such that $\pounds_X \bar{g} = 2\rho\bar{g}$. Then, from Lemma 3, we have the following lemma.

Lemma 4 *Let X be an infinitesimal conformal transformation in $T(M)$ with the complete lift metric. We have the following equations:*

$$g_{im}\{y^r v^a K_{ajr}{}^m + v^{\bar{a}}\Gamma_{a}{}^{m}_{j} + X_j(v^{\overline{m}})\} + g_{jm}\{y^r v^a K_{air}{}^m + v^{\bar{a}}\Gamma_{a}{}^{m}_{i} + X_i(v^{\overline{m}})\} = 0, \tag{1}$$

$$v^a \partial_a g_{ij} + g_{mj}X_i(v^m) - g_{im}\{v^a \Gamma_{j}{}^{m}_{a} - X_{\bar{j}}(v^{\overline{m}})\} = 2\rho g_{ij}, \tag{2}$$

$$g_{mj}X_{\bar{i}}(v^m) + g_{mi}X_{\bar{j}}(v^m) = 0. \tag{3}$$

Applying $X_{\bar{k}}$ to (3) of Lemma 4, we get

$$\begin{aligned} &g_{mj}X_{\bar{k}}X_{\bar{i}}(v^m) \\ &= -g_{mi}X_{\bar{k}}X_{\bar{j}}(v^m) = -g_{mi}X_{\bar{j}}X_{\bar{k}}(v^m) = g_{mk}X_{\bar{j}}X_{\bar{i}}(v^m) = g_{mk}X_{\bar{i}}X_{\bar{j}}(v^m) \\ &= -g_{mj}X_{\bar{i}}X_{\bar{k}}(v^m) = -g_{mj}X_{\bar{k}}X_{\bar{i}}(v^m), \quad \text{from which, } X_{\bar{k}}X_{\bar{i}}(v^m) = 0. \end{aligned}$$

This shows that the horizontal components (v^h) of X can be written in the form $v^h = y^a A_a{}^h + B^h$, where $A_a{}^h$ and B^h are depend only on variables (x^h). The coordinate transformation rule implies that $A_a{}^h$ and B^h are the components of a certain $(1,1)$ tensor field A and of a certain contravariant vector field B on M, respectively. Substituting $v^h = y^a A_a{}^h + B^h$ into (3) of Lemma 4, we get $A_{ij} + A_{ji} = 0$, where $A_{ij} = g_{mj}A_i{}^m$. Thus we have

Lemma 5 *The horizontal components (v^h) of X are written in the following form:*

$$v^h = y^a A_a{}^h + B^h, \tag{1}$$

where $A_a{}^h$ and B^h are the components of a certain $(1,1)$ tensor field A and of a certain contravariant vector field B on M, respectively. And the components $A_a{}^h$ satisfy the following:

$$A_{ij} + A_{ji} = 0, \tag{2}$$

where $A_{ij} = g_{mj}A_i{}^m$.

Substituting (1) of Lemma 5 into (2) of Lemma 4, we have

$$(y^r A_r{}^a + B^a)\partial_a g_{ij} + g_{mj}(y^r \partial_i A_r{}^m + \partial_i B^m) - g_{mj} y^r \Gamma_r{}^s{}_i A_s{}^m$$
$$- g_{im}(y^r A_r{}^a + B^a)\Gamma_j{}^m{}_a + g_{im} X_{\bar{j}}(v^{\bar{m}}) = 2\rho g_{ij},$$

it follows that

$$\pounds_B g_{ij} - \nabla_j B_i + y^r \nabla_i A_{rj} + g_{im} X_{\bar{j}}(v^{\bar{m}}) = 2\rho g_{ij}, \tag{3.1}$$

where we put $B_i = g_{ia} B^a$, and $\pounds_B g_{ij}, \nabla_j B_i$ and $\nabla_i A_{rj}$ denote the components of the Lie derivative of g with respect to B and of the covariant derivatives of B and A, respectively. Applying $X_{\bar{k}}$ to (3.1), we get

$$\nabla_i A_{kj} + g_{im} X_{\bar{k}} X_{\bar{j}}(v^{\bar{m}}) = 2 X_{\bar{k}}(\rho) g_{ij}. \tag{3.2}$$

Interchanging k and j in (3.2) and using (2) of Lemma 5, we obtain

$$g_{im} X_{\bar{k}} X_{\bar{j}}(v^{\bar{m}}) = X_{\bar{k}}(\rho) g_{ij} + X_{\bar{j}}(\rho) g_{ik}. \tag{3.3}$$

The equations (3.2) and (3.3) imply $\nabla_i A_{kj} = X_{\bar{k}}(\rho) g_{ij} - X_{\bar{j}}(\rho) g_{ik}$. Putting $\nabla_a A_i{}^a = (n-1)\varphi_i$, we get $X_{\bar{k}}(\rho) = \varphi_k$. Thus we have

Lemma 6 *The components A_{ij} satisfy:*

$$\nabla_i A_{kj} = \varphi_k g_{ij} - \varphi_j g_{ik}, \tag{1}$$

and the scalar function ρ is written in the following form:

$$\rho = y^r \varphi_r + \psi, \tag{2}$$

where ψ is a certain function on M.

Substituting (1) and (2) of Lemma 6 into (3.1), we have

Lemma 7 *The vertical components $(v^{\bar{h}})$ of X are written in the following form:*

$$v^{\bar{h}} = y^h y^r \varphi_r + y^r (2\delta_r^h \psi - g^{hm} \nabla_m B_r) + C^h,$$

where C^h are the components of a certain contravariant vector field C on M.

By virtue of (1) of Lemma 5, Lemma 7 and (1) of Lemma 4, we obtain

$$-(\nabla_i C_j + \nabla_j C_i)$$
$$+ y^r\{\nabla_i \nabla_j B_r + K_{aijr} B^a + \nabla_j \nabla_i B_r + K_{ajir} B^a - 2\psi_j g_{ir} - 2\psi_i g_{jr}\}$$
$$+ y^r y^s \{K_{ajis} A_r{}^a + K_{aijs} A_r{}^a - g_{is} \nabla_j \varphi_r - g_{js} \nabla_i \varphi_r\} = 0.$$

Thus we have

Lemma 8 *The following equations hold:*

$$\nabla_i \nabla_j B^h + K_{aij}{}^h B^a = \delta_i^h \psi_j + \delta_j^h \psi_i, \tag{1}$$

where we put $\psi_i = \nabla_i \psi$;

$$\nabla_i C_j + \nabla_j C_i = 0, \tag{2}$$

where we put $C_i = g_{ia} C^a$, *and* $\nabla_i C_j$ *denote the components of the covariant derivative of* C.

$$K_{ajis} A_r{}^a + K_{ajir} A_s{}^a + K_{aijs} A_r{}^a + K_{aijr} A_s{}^a$$
$$= g_{is} \nabla_j \varphi_r + g_{ir} \nabla_j \varphi_s + g_{js} \nabla_i \varphi_r + g_{jr} \nabla_i \varphi_s, \tag{3}$$

where we put $K_{ajis} = g_{sm} K_{aji}{}^m$, *and* $\nabla_j \varphi_r$ *denote the components of the covariant derivative of* $\varphi = \varphi_i dx^i$.

4. Lemmas

Transvecting (3) of Lemma 8 by g^{sr} and using (2) of Lemma 5, we have

$$\nabla_i \varphi_j + \nabla_j \varphi_i = 0. \tag{4.1}$$

Applying Ricci identity for (1) of Lemma 6, we obtain

$$K_{ajis} A_r{}^a + g_{ji} \nabla_s \varphi_r - g_{js} \nabla_i \varphi_r = K_{aris} A_j{}^a + g_{ri} \nabla_s \varphi_j - g_{rs} \nabla_i \varphi_j. \tag{4.2}$$

Here, we put $Y_{jisr} = K_{ajis} A_r{}^a + g_{ji} \nabla_s \varphi_r - g_{js} \nabla_i \varphi_r$. Then the first Bianchi identity implies

$$Y_{jisr} + Y_{isjr} + Y_{sjir} = 0. \tag{4.3}$$

By means of (4.1) and (3) of Lemma 8, we get

$$Y_{jisr} + Y_{ijrs} + Y_{jirs} + Y_{ijsr} = 0. \tag{4.4}$$

By the definition of Y_{jisr} and (4.2), Y_{jisr} is symmetric in the indices j and r, and skew symmetric in the indices i and s. Thus, from (4.3), we have $Y_{jisr} = Y_{ijsr} - Y_{sjir}$. Substituting this equation into (4.4), we obtain

$$0 = (Y_{ijsr} - Y_{sjir}) + Y_{ijrs} + (Y_{ijrs} - Y_{rjis}) + Y_{ijsr}$$

$$= 2(Y_{ijsr} + Y_{ijrs} - Y_{sjir})$$
$$= 2(Y_{ijsr} + Y_{ijrs} + Y_{rijs})$$
$$= 2(Y_{ijsr} - Y_{jris})$$
$$= 2(Y_{ijsr} + Y_{jirs}),$$

from which

$$Y_{ijsr} = -Y_{jirs} = -Y_{sirj} = Y_{srij},$$

it follows that

$$Y_{jisr} = Y_{srji} = -Y_{sjri} = -Y_{risj} = -Y_{jisr}, \quad \text{hence } Y_{jisr} = 0.$$

Thus we have

Lemma 9 *The following equation holds:*

$$K_{ajis}A_r{}^a + g_{ji}\nabla_s\varphi_r - g_{js}\nabla_i\varphi_r = 0.$$

Operating $g^{sm}\nabla_m$, $g^{jm}\nabla_m$ and $g^{rm}\nabla_m$ to the equation of Lemma 9, and using (1) of Lemma 6 and (4.1), we have

$$\nabla_j\nabla_i\varphi_r = (\nabla_a R_{ji} - \nabla_j R_{ai})A_r{}^a - K_{ajir}\varphi^a + R_{ji}\varphi_r - g_{ji}R_{ra}\varphi^a, \quad (4.5)$$

$$(\nabla_i R_{sa} - \nabla_s R_{ia})A_r{}^a = 0, \quad (4.6)$$

$$A^{ab}\nabla_a K_{bjis} = (n-1)K_{ajis}\varphi^a - g_{ji}R_{sa}\varphi^a + g_{js}R_{ia}\varphi^a, \quad (4.7)$$

where we put $\varphi^a = g^{ai}\varphi_i$ and $A^{ab} = g^{ai}A_i{}^b$, and $R_{ji}, \nabla_a R_{ji}$ and $\nabla_a K_{bjis}$ denote the components of the Ricci tensor of M and of the covariant derivatives of the Ricci tensor and of the curvature tensor of M, respectively.

Transvecting the equation of Lemma 9 by g^{sr}, and using (2) of Lemma 5 and (4.1), we get

$$K_{ajib}A^{ab} = \nabla_j\varphi_i. \quad (4.8)$$

Operating ∇_r to the equation (4.8) and using (1) of Lemma 6, we have

$$A^{ab}\nabla_r K_{ajib} = K_{arji}\varphi^a + \nabla_r\nabla_j\varphi_i. \quad (4.9)$$

By virtue of (4.9), the second Bianchi identity and the Ricci identity, we obtain

$$A^{ab}\nabla_a K_{bjir} = A^{ab}\nabla_a K_{rijb} = -(\nabla_r K_{iajb} + \nabla_i K_{arjb})A^{ab}$$

$$\begin{aligned}&= A^{ab}\nabla_r K_{aijb} - A^{ab}\nabla_i K_{arjb} \\ &= K_{arij}\varphi^a - K_{airj}\varphi^a + \nabla_r\nabla_i\varphi_j - \nabla_i\nabla_r\varphi_j \\ &= (K_{arij} + K_{aijr} + K_{ajri})\varphi^a = 0.\end{aligned}$$

Hence, by (4.7), we have

Lemma 10 *The following equation holds*:

$$(n-1)K_{ajis}\varphi^a = g_{ji}R_{sa}\varphi^a - g_{js}R_{ia}\varphi^a.$$

Transvecting the equation of Lemma 9 by g^{ji}, we have

$$R_{as}A_r{}^a = -(n-1)\nabla_s\varphi_r. \tag{4.10}$$

Operating ∇_j to the equation (4.10) and using (1) of Lemma 6, we have

$$(n-1)\nabla_j\nabla_s\varphi_r = -A_r{}^a\nabla_j R_{as} - R_{js}\varphi_r + g_{jr}R_{as}\varphi^a. \tag{4.11}$$

Transvecting the equation (4.11) by g^{js} and using (4.1), we obtain

Lemma 11 *The following equation holds*:

$$\frac{1}{2}A_r{}^a\nabla_a S - nG_{ra}\varphi^a = 0,$$

where S denotes the scalar curvature of M and $G_{ra} = R_{ra} - \frac{S}{n}g_{ra}$.

From (4.10) and (4.1), we have

$$R_{as}A_r{}^a + R_{ar}A_s{}^a = 0. \tag{4.12}$$

Operating ∇_j to the equation (4.12) and using (1) of Lemma 6, we obtain

$$\begin{aligned}&A_r{}^a\nabla_j R_{sa} - g_{js}G_{ar}\varphi^a + G_{js}\varphi_r \\ &\qquad = -(A_s{}^a\nabla_j R_{ra} - g_{jr}G_{as}\varphi^a + G_{jr}\varphi_s).\end{aligned} \tag{4.13}$$

Here, we put $Y_{jsr} = A_r{}^a\nabla_j R_{sa} - g_{js}G_{ar}\varphi^a + G_{js}\varphi_r$, then from (4.6) and (4.13), Y_{jsr} is symmetric in the indices j and s, and skew symmetic in the indices s and r. Thus we have $Y_{jsr} = -Y_{jrs} = -Y_{rjs} = Y_{rsj} = Y_{srj} = -Y_{sjr} = -Y_{jsr}$, hence, $Y_{jsr} = 0$. Therefore we have

Lemma 12 *The following equation holds*:

$$A_r{}^a\nabla_j R_{sa} - g_{js}G_{ar}\varphi^a + G_{js}\varphi_r = 0.$$

Substituting the equation of Lemma 12 into (4.11) and combining Lemma 10, we obtain

Lemma 13 *The following equation holds*:

$$\nabla_j \nabla_i \varphi_r = -K_{ajir}\varphi^a.$$

5. Proof of Theorem

Let M be a complete, simply connected Riemannian manifold with positive constant scalar curvature and $T(M)$ its tangent bundle with the complete lift metric, and assume that $T(M)$ admits an essential infinitesimal conformal transformation X.

Proof of (1) *in Theorem.* It is well known that a vector field $P = p^h \frac{\partial}{\partial x^h}$ on M is an infinitesimal projective transformation if and only if the components p^h satisfy the following equation:

$$\nabla_i \nabla_j p^h + K_{aij}{}^h p^a = \delta_i^{\ h} u_j + \delta_j^{\ h} u_i,$$

where u_i denote the components of a certain gradient vector field on M. Thus from (1) of Lemma 8, the induced vector field B on M is an infinitesimal projective transformation on M. □

Next we prove that M is isometric to the standard sphere. Since the scalar curvature S of M is constant, Lemma 11 implies

$$G_{ra}\varphi^a = 0. \tag{5.1}$$

Transvecting the equation (4.10) by φ^s, and using (4.1) and (5.1), we get

$$A_r{}^a \varphi_a = f_r, \tag{5.2}$$

where we put $f = \frac{n(n-1)}{2S}\varphi_s\varphi^s$ and $\nabla_r f = f_r$.

Operating ∇_j to the equation (5.2) and using (1) of Lemma 6, we obtain

$$\nabla_j f_r = \varphi_r \varphi_j - \frac{2S}{n(n-1)} g_{jr} f + A_r{}^a \nabla_j \varphi_a, \tag{5.3}$$

hence, by virtue of (1) of Lemma 6, we get

$$\nabla_l \nabla_j f_r = \varphi_j \nabla_l \varphi_r - \frac{2S}{n(n-1)} g_{jr} f_l - \frac{S}{n(n-1)} g_{lr} f_j + A_r{}^a \nabla_l \nabla_j \varphi_a. \tag{5.4}$$

Combining Lemma 10, Lemma 13 and (5.1), we have

$$\nabla_l \nabla_j \varphi_a = \frac{S}{n(n-1)}(g_{la}\varphi_j - g_{lj}\varphi_a), \tag{5.5}$$

thus by (5.2), we obtain

$$A_r{}^a \nabla_l \nabla_j \varphi_a = \frac{S}{n(n-1)} A_{rl}\varphi_j - \frac{S}{n(n-1)} g_{lj} f_r. \tag{5.6}$$

Substituting (5.6) into (5.4) and using (2) of Lemma 5, we get

$$\begin{aligned}\nabla_l \nabla_j f_r &+ \frac{S}{n(n-1)}(2f_l g_{jr} + f_j g_{lr} + f_r g_{lj}) \\ &= \Big(\nabla_l \varphi_r - \frac{S}{n(n-1)} A_{lr}\Big)\varphi_j.\end{aligned} \tag{5.7}$$

Hence, by means of (4.10), we have

$$\nabla_l \nabla_j f_r + \frac{S}{n(n-1)}(2f_l g_{jr} + f_j g_{lr} + f_r g_{lj}) = \frac{1}{n-1} G_{ar} A_l{}^a \varphi_j. \tag{5.8}$$

Here, we put $Y_{rlj} = G_{ar} A_l{}^a \varphi_j$, then by (5.8), (2) of Lemma 5 and (4.12), Y_{rlj} is symmetric in the indices r and j, and skew symmetric in the indices r and l. Hence we have $Y_{rlj} = 0$, it follows that

$$\nabla_l \nabla_j f_r + \frac{S}{n(n-1)}(2f_l g_{jr} + f_j g_{lr} + f_r g_{jl}) = 0.$$

Therefore if f is non-constant then by Theorem D, M is isometic to the standard sphere. Next we assume f is constant. Since X is essential, f is non-zero constant. From (5.2) we have

$$A_r{}^a \varphi_a = 0. \tag{5.9}$$

From (5.7) we obtain

$$\nabla_r \varphi_s = \frac{S}{n(n-1)} A_{rs}. \tag{5.10}$$

Substituting (5.10) into the equation of Lemma 9, we get

$$K_{ajis} A_r{}^a + \frac{S}{n(n-1)}(g_{ji} A_{sr} - g_{js} A_{ir}) = 0. \tag{5.11}$$

Operating ∇_l to the equation (5.11), and using (1) of Lemma 6, Lemma 10 and (5.1), we have

$$A_r{}^a \nabla_l K_{ajis} + \left\{ K_{ljis} - \frac{S}{n(n-1)}(g_{ls}g_{ji} - g_{js}g_{li}) \right\} \varphi_r = 0. \qquad (5.12)$$

Transevecting the equation (5.12) by φ^r and using (5.9), we obtain

$$K_{ljis} = \frac{S}{n(n-1)}(g_{ls}g_{ji} - g_{js}g_{li}).$$

This shows M is a space of positive constant curvature, that is, M is isometric to the standard sphere.

Proof of (2) *in Theorem.* Let $\overline{\nabla}$ be the Riemannian connection of $T(M)$ and $\overline{\Gamma}_B{}^A{}_C$ the coefficients of $\overline{\nabla}$, that is,

$$\overline{\nabla}_{X_i} X_j = \overline{\Gamma}_j{}^m{}_i X_m + \overline{\Gamma}_j{}^{\overline{m}}{}_i X_{\overline{m}}, \quad \overline{\nabla}_{X_i} X_{\overline{j}} = \overline{\Gamma}_{\overline{j}}{}^m{}_i X_m + \overline{\Gamma}_{\overline{j}}{}^{\overline{m}}{}_i X_{\overline{m}},$$

$$\overline{\nabla}_{X_{\overline{i}}} X_j = \overline{\Gamma}_j{}^m{}_{\overline{i}} X_m + \overline{\Gamma}_j{}^{\overline{m}}{}_{\overline{i}} X_{\overline{m}}, \quad \overline{\nabla}_{X_{\overline{i}}} X_{\overline{j}} = \overline{\Gamma}_{\overline{j}}{}^m{}_{\overline{i}} X_m + \overline{\Gamma}_{\overline{j}}{}^{\overline{m}}{}_{\overline{i}} X_{\overline{m}},$$

where the indices A, B, C run over the range $\{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$, it follows

$$\overline{\nabla}_{X_i} dx^h = -\overline{\Gamma}_m{}^h{}_i dx^m - \overline{\Gamma}_{\overline{m}}{}^h{}_i \delta y^m, \quad \overline{\nabla}_{X_i} \delta y^h = -\overline{\Gamma}_m{}^{\overline{h}}{}_i dx^m - \overline{\Gamma}_{\overline{m}}{}^{\overline{h}}{}_i \delta y^m,$$

$$\overline{\nabla}_{X_{\overline{i}}} dx^h = -\overline{\Gamma}_m{}^h{}_{\overline{i}} dx^m - \overline{\Gamma}_{\overline{m}}{}^h{}_{\overline{i}} \delta y^m, \quad \overline{\nabla}_{X_{\overline{i}}} \delta y^h = -\overline{\Gamma}_m{}^{\overline{h}}{}_{\overline{i}} dx^m - \overline{\Gamma}_{\overline{m}}{}^{\overline{h}}{}_{\overline{i}} \delta y^m.$$

□

Then we have

Lemma 14 ([6]) *The connection coefficients* $\overline{\Gamma}_B{}^A{}_C$ *of* $\overline{\nabla}$ *with the complete lift metric satisfy the following*:

(1) $\overline{\Gamma}_j{}^h{}_i = \Gamma_j^h{}_i$, (2) $\overline{\Gamma}_j{}^{\overline{h}}{}_i = y^a K_{aij}{}^h$, (3) $\overline{\Gamma}_{\overline{j}}{}^h{}_i = 0$, (4) $\overline{\Gamma}_j{}^h{}_{\overline{i}} = 0$,

(5) $\overline{\Gamma}_{\overline{j}}{}^{\overline{h}}{}_i = \Gamma_j{}^h{}_i$, (6) $\overline{\Gamma}_j{}^{\overline{h}}{}_{\overline{i}} = 0$, (7) $\overline{\Gamma}_{\overline{j}}{}^h{}_{\overline{i}} = 0$, (8) $\overline{\Gamma}_{\overline{j}}{}^{\overline{h}}{}_{\overline{i}} = 0$.

The curvature tensor $\overline{K}$ of $T(M)$ is defined by

$$\overline{K}(X,Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z.$$

From Lemma 1 and Lemma 14, by the straightforward calculations, we have

Lemma 15 *The curvature tensor of* $T(M)$ *are given as follows:*

$$\overline{K}(X_i, X_j)X_k = K_{ijk}{}^m X_m + y^a\{\nabla_i K_{ajk}{}^m - \nabla_j K_{aik}{}^m\}X_{\overline{m}}, \tag{1}$$

$$\overline{K}(X_i, X_j)X_{\overline{k}} = K_{ijk}{}^m X_{\overline{m}}, \tag{2}$$

$$\overline{K}(X_{\overline{i}}, X_j)X_k = K_{ijk}{}^m X_{\overline{m}}, \tag{3}$$

$$\overline{K}(X_{\overline{i}}, X_j)X_{\overline{k}} = 0, \tag{4}$$

$$\overline{K}(X_{\overline{i}}, X_{\overline{j}})X_k = 0, \tag{5}$$

$$\overline{K}(X_{\overline{i}}, X_{\overline{j}})X_{\overline{k}} = 0. \tag{6}$$

Let $\overline{g}_{AB}$ be the components of the complete lift metric and $\overline{K}_{ABCD}$ the components of $\overline{K}$, that is, $\overline{K}_{ABCD} = \overline{g}(\overline{K}(X_A, X_B)X_C, X_D)$. The scalar curvature $\overline{S}$ of $T(M)$ is defined by $\overline{S} = \overline{g}^{AD}\overline{g}^{BC}\overline{K}_{ABCD}$, where $\overline{g}^{AB}$ denote the components of the inverse matrix of $(\overline{g}_{AB})$. The tangent bundle $T(M)$ is said to be conformally flat if the components of the curvature tensor of $T(M)$ are given as follows:

$$\overline{K}_{ABCD} = \frac{1}{2(n-1)}(\overline{g}_{AD}\overline{R}_{BC} - \overline{g}_{BD}\overline{R}_{AC} + \overline{g}_{BC}\overline{R}_{AD} - \overline{g}_{AC}\overline{R}_{BD}) - \frac{\overline{S}}{2(2n-1)(n-1)}(\overline{g}_{AD}\overline{g}_{BC} - \overline{g}_{BD}\overline{g}_{AC}),$$

where $\overline{R}_{BC}$ denote the components of the Ricci tensor of $T(M)$. It is well known that the scalar curvature $\overline{S}$ of $T(M)$ with the complete lift metric vanishes, ([7]). Thus, $T(M)$ with the complete lift metric is conformally flat if the components of the curvature tensor of $T(M)$ are given

$$\overline{K}_{ABCD} = \frac{1}{2(n-1)}(\overline{g}_{AD}\overline{R}_{BC} - \overline{g}_{BD}\overline{R}_{AC} + \overline{g}_{BC}\overline{R}_{AD} - \overline{g}_{AC}\overline{R}_{BD}). \tag{5.13}$$

Since M is a space of constant curvature, from Lemma 15, we have

$$\overline{K}(X_i, X_j)X_k = \frac{S}{n(n-1)}(\delta_i{}^m g_{jk} - \delta_j{}^m g_{ik})X_m, \tag{1}$$

$$\overline{K}(X_i, X_j)X_{\overline{k}} = \frac{S}{n(n-1)}(\delta_i{}^m g_{jk} - \delta_j{}^m g_{ik})X_{\overline{m}}, \tag{2}$$

$$\overline{K}(X_{\overline{i}}, X_j)X_k = \frac{S}{n(n-1)}(\delta_i{}^m g_{jk} - \delta_j{}^m g_{ik})X_{\overline{m}}, \tag{3}$$

$$\overline{K}(X_{\bar{i}}, X_j)X_{\bar{k}} = 0, \tag{4}$$

$$\overline{K}(X_{\bar{i}}, X_{\bar{j}})X_k = 0, \tag{5}$$

$$\overline{K}(X_{\bar{i}}, X_{\bar{j}})X_{\bar{k}} = 0. \tag{6}$$

Using these equations, we can show that (5.13) holds. This completes the proof of Theorem.

References

[1] Obata M., *Riemannian manifolds admitting a solution of a certain differential equations.* Pro. U.S-Japan Sem. in Differential Geome., Kyoto, Japan, (1965), 101–114.

[2] Tanno S., *Some differential equations on Riemannian manifolds.* J. Math. Soc. Japan **30** (1978), 509–531.

[3] Tanno S., *Projective transformations of Riemannian manifolds.* Pre-print.

[4] Yamauchi K., *On infinitesimal projective transformations of Riemannian manifolds with constant scalar curvature.* Hokkaido Math. J. **8** (1979), 167–175.

[5] Yamauchi K., *On Riemannian manifolds admitting infinitesimal projective transformations.* Hokkaido Math. J. **16** (1987), 115–125.

[6] Yamauchi K., *On Infinitesimal Projective Transformations of the Tangent Bundles with the Complete Lift Metric over Riemannian Manifolds.* Ann. Rep. Asahikawa Med. Coll. **19** (1998), 49–55.

[7] Yano K and Ishihara S., *Tangent and Cotangent Bundles.* Marcel Dekker, 1973.

Department of Mathematics
Asahikawa Medical College
Midorigaoka Higashi 2-1-1-1, Japan
E-mail: yamauchi@asahikawa-med.ac.jp