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C_λ-groups and λ-basic subgroups in modular group rings

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Abstract. This paper is concerned with the investigation of two closely related questions. The first question is: What is the λ-basic subgroup of the group of normed units $V(RG)$ in an abelian group ring RG with identity of prime characteristic p if B is a λ-basic subgroup of the p-primary abelian group G? In this way it is shown that if B is a λ-basic subgroup of the p-torsion G and R is perfect, then $1+I(RG;B)$ is a λ-basic subgroup of $V(RG)$, where λ is a countable limit ordinal. Moreover, B is a direct factor of $1+I(RG;B)$ provided that it is λ-basic. This generalizes results due to Nachev (1996) and to the author (1995). The second question is the following: What is the criterion illustrated $V(RG)$ to be a C_λ-group when G is an abelian p-group and R is an unitary commutative ring with prime characteristic p? In this direction it is proved that $V(RG)$ is a p-primary C_λ-group if and only if G is a p-primary C_λ-group, provided R is perfect and $\lambda \leq \Omega$. Besides, if R is perfect and G is a p-group which is a C_λ-group, then the same holds for $V(RG)/G$, provided $\lambda \leq \Omega$. Moreover, if G is a p-torsion C_λ-group of countable length λ and R is perfect without nilpotents, then $V(RG)/G$ is totally projective and so G is a direct factor of $V(RG)$. The last extends in some aspect a result of May (1979, 1988).

Key words: C_λ-groups, λ-basic subgroups, group rings, normed units, direct factors.

1. Preliminaries

In this section, we assemble the basic concepts which are crucial in the following development. For pertinent results related to these concepts, we refer the reader to [12], [15] and [20, 23, 25].

Let G be an abelian group with p-component G_p and let R be a commutative ring with unity of prime characteristic p. For RG a group ring of G over R, $V(RG)$ will denote the group of normalized units (i.e. of augmentation 1 units) in RG, and $S(RG)$ is its p-component (i.e. its Sylow p-subgroup). For a subgroup H of G, we let $I(RG;H)$ denote the relative fundamental ideal of RG with respect to H, generated by the elements $1-h$ when h varies in H. It is not difficult to verify that if H is p-primary, then $1+I(RG;H)$ is a multiplicative p-group. Besides, define

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$V(RG;H) = 1 + I(RG;H)$.

A subgroup H of the p-group G is said to be a p^α-pure subgroup if $H \to G \to G/H$ represents an element of $\mathrm{Ext}^{p^\alpha}(G/H;H)$. This definition going from Nunke is very important and it is a generalization of the ordinary purity (i.e. p^ω-purity) in the classical theory of abelian p-groups. A subgroup H of the p-primary group G is called isotype in G if $H \cap G^{p^\alpha} = H^{p^\alpha}$ for every ordinal α. If β is an ordinal, we shall say that H is weakly p^β-pure in G if $H \cap G^{p^\alpha} = H^{p^\alpha}$ for all $\alpha \leq \beta$. It is well-documented [15, 11] that if H is a p^α-pure subgroup of G, then H is a weakly p^α-pure subgroup of G; the weak p^α-purity is equivalent to p^α-purity for $\alpha \leq \omega$, i.e. to ordinary purity.

For a more precise information and a further application, we mention that the following dependences are true. From [15] and [23, 25], it follows that if H is a p^α-pure in G, then $(G/H)^{p^\beta}[p] = G^{p^\beta}[p]H/H$ for all $\beta < \alpha$. Moreover, if G/H is divisible, where H is p^α-pure in G and α is a limit ordinal, then $G^{p^\beta}H = G$ for all $\beta < \alpha$. Besides from [12], if H is a neat subgroup in G (i.e. $H \cap G^p = H^p$) and $G[p] = G^{p^\beta}[p]H[p]$ for each $\beta < \alpha$, then H is p^α-pure in G and G/H is divisible. For more details see also [20].

The totally projective groups as introduced by R. Nunke provide a generalization of the concept of direct sums of countable reduced groups [23]. A p-group G is p^α-projective if $\mathrm{Ext}^{p^\alpha}(G,C) = 1$ for all groups C. A reduced p-group G is totally projective if G/G^{p^α} is p^α-projective for every ordinal α. The concepts of C_λ-groups and λ-basic subgroups (extending to an arbitrary limit ordinal) are introduced by Ch. Megibben and K. Wallace, respectively in [21] and [25]. For a fixed but arbitrary limit ordinal λ, C_λ shall designate that class of all p-groups G such that G/G^{p^α} is totally projective for all $\alpha < \lambda$. Groups in the class C_λ will be referred to as C_λ-groups. The subgroup B of an abelian p-group G is said to be a λ-basic subgroup of G if

(1) B is totally projective of length at most λ,
(2) B is a p^λ-pure subgroup of G, and
(3) G/B is divisible.

If $\lambda \leq \omega$, then it is straightforward to see that B is λ-basic if and only if B is basic in G. It is well-known that every two basic subgroups in G are isomorphic [10]. Moreover the following is valid [25].

Theorem 1.1 *If B and B' are λ-basic subgroups of G then $B \cong B'$.*

Besides any abelian p-group contains basic subgroups [10]. The following excellent result due to Wallace [25] extends the above fact and the main statements in [21].

Theorem 1.2 (WALLACE) *The reduced abelian primary group G contains a proper λ-basic subgroup if and only if G is a C_λ-group and λ is confinal with ω.*

We continue with paragraph entitled

2. Criteria for total projectivity

Since all direct sums of cyclics are totally projective and every totally projective group of length confinal with ω is σ-summable [2], it is naturally to exists a criterion (identical to this of Kulikov [10]) which characterizes the total projectivity in the terms of the cited group classes. This criterion states as follows (see [20, 16, 13]; [2, 3, 4]):

Theorem 2.1 (L-M-H Criterion) *Suppose λ is a countable limit ordinal. Then G is a p-primary C_λ-group, respective C_Ω-group if and only if precisely one of the following holds*

(a) *G/G^{p^α} is summable (Megibben,* 1969)

(b) *G/G^{p^α} is σ-summable (Linton-Megibben,* 1977) *and (Hill,* 1981), *for each limit $\alpha < \lambda$, respective $\alpha < \Omega$.*

In particular G is totally projective of length λ if and only if (a) *or* (b) *is valid for each limit $\alpha \leq \lambda$.*

More recently, Hill and Ullery [14] have obtained a new simple characterization of primary totally projective groups with countable lengths. Namely, the following is fulfilled:

Theorem 2.2 (HILL-ULLERY) *The p-torsion abelian group G of countable length is totally projective if and only if $G = \bigcup_{n<\omega} G_n$, where $G_n \subseteq G_{n+1}$ and all G_n are height-finite, i.e. they have a finite height-spectrum computed in G.*

The following note is well to be documented

Remark It is a simple matter to observe that Theorem 2.2 actually is similar to the classical Honda's criterion for primary summable groups of countable lengths (cf. [11, 20]).

We now come to the main paragraph that contains the central results.

3. Main results

Here we select two sections starting with

I. A construction of λ-basic subgroup of $V(RG)$ and $V(RG)/G$.

Before proving the major statements that motivate this section, we need one useful

Lemma 3.1 *Suppose B is weakly p^α-pure in G where α is limit. Then $(BG^{p^\beta})[p] = B[p].\ G^{p^\beta}[p]$ for each $\beta < \alpha$.*

Proof. Take x in the left hand-side. Hence $x = bg$, where $b \in B$, $g \in G^{p^\beta}$ and $b^p g^p = 1$. Thus $b^p \in B \cap G^{p^{\beta+1}} = B^{p^{\beta+1}}$ $(\beta + 1 < \alpha)$ and thus $b \in B^{p^\beta} B[p]$. Observe that $x \in G^{p^\beta} B[p]$, whence $x \in G^{p^\beta}[p].\ B[p]$, concluding the proof. □

Now we are in position to formulate the following restation of the definition for a λ-basic subgroup, namely

Definition We shall say that the subgroup B of a p-primary abelian group G is λ-basic for a limit ordinal λ if

(1) B is totally projective of length at most λ,

(2′) B is a neat subgroup of G and $G[p] = B[p]G^{p^\beta}[p]$ for all $\beta < \lambda$, or equivalently B is weakly p^λ-pure in G and $G = BG^{p^\beta}$ for all $\beta < \lambda$.

Remark G/B is divisible follows from (2′).

But more convenient for us are the following conditions. Before stating this modification, we give

Definition A subgroup N of an abelian p-group G is called weakly p^α-nice if $(G/N)^{p^\beta} = G^{p^\beta}N/N$ for each $\beta < \alpha$.

And so, B is λ-basic in a p-group G for limit λ if and only if

(1) B is totally projective of length at most λ,

(2″) B is weakly p^λ-pure and weakly p^λ-nice in G,

(3) G/B is divisible.

Really, (1) and (2′) obviously yield (1), (2) and (3) owing to our discussion if §1. The converse follows by the same discussion along with the simple fact that $(G/H)[p] = G[p]H/H$ when H is neat in G. The conditions in (2′) are equivalent by making use of Lemma 3.1 plus the presented in §1 facts. Besides, it is a simple exercise to verify that (2′) holds if and only if (2″) together with (3) hold.

Begin now with some other very needed for our presentation preliminary technical assertions, namely

Proposition 3.1 *Given $A \leq G$, $H \leq G$ and $1 \in L \leq R$. Then*

$$GV(RG;H) \cap V(LA) \subseteq AV(LA; A \cap H).$$

Proof. Given x in the left hand-side. Therefore $x = \sum_{a\in A} \alpha_a a = b \sum_{g\in G} r_g g$, where

$$\alpha_a \in L, \quad b \in G, \quad r_g \in R \quad \text{and} \quad \sum_{g\in \overline{g}H} r_g = \begin{cases} 1, & \overline{g} \in H \\ 0, & \overline{g} \notin H \end{cases}$$

for every $\overline{g} \in G$. According to the canonical forms we derive $\alpha_a = r_g$ and $a = bg$. Because $\sum_{g\in G} r_g g \in V(RG;H)$, there is $g' \in H$ with $r_{g'} \neq 0$. Choose arbitrary $\overline{a} \in A$ and $a' = bg'$. Furthermore $x = a' \sum_{a\in A} \alpha_a a a'^{-1} \in AV(LA; A \cap H)$ because

$$\sum_{g\in g'\overline{a}H} \alpha_a = \sum_{aa'^{-1}\in \overline{a}H} \alpha_a a'^{-1} = \begin{cases} 1, & \overline{a} \in H \\ 0, & \overline{a} \notin H \end{cases}.$$

But $\overline{a}H \cap A = \overline{a}(H \cap A)$, deducing the proof. □

Proposition 3.2 *For every ordinal α is fulfilled*

$$V^{p^\alpha}(RG) = V(R^{p^\alpha} G^{p^\alpha})$$
$$[GV(RG;H)]^{p^\alpha} = G^{p^\alpha} V(R^{p^\alpha} G^{p^\alpha}; H^{p^\alpha})$$

when H is p-isotype and p-torsion in G.

Proof. The first dependence is proved in [1] (cf. [2, 6] for example, too). Clearly, in order to show the second, we can restrict on limit α. And so, choosing x in the left hand-side we write $x \in \bigcap_{\beta<\alpha} [GV(RG;H)]^{p^\beta} = \bigcap_{\beta<\alpha} [G^{p^\beta} V(R^{p^\beta} G^{p^\beta}; H^{p^\beta})]$ using the induction hypothesis and hence $x = g_\beta \sum_{a_\beta \in G^{p^\beta}} r_{a_\beta} a_\beta = g_\delta \sum_{a_\delta \in G^{p^\delta}} r_{a_\delta} a_\delta = \cdots$ for $g_\beta \in G^{p^\beta}$, $g_\delta \in G^{p^\delta}$ and arbitrary $\beta < \delta \leq \alpha$, where

$$\sum_{a_\beta \in \overline{g}_\beta \in H^{p^\beta}} r_{a_\beta} = \begin{cases} 1, & \overline{g}_\beta \in H^{p^\beta} \\ 0, & \overline{g}_\beta \notin H^{p^\beta} \end{cases} \quad \text{and} \quad \sum_{a_\delta \in \overline{g}_\delta \in H^{p^\delta}} r_{a_\delta} = \begin{cases} 1, & \overline{g}_\delta \in H^{p^\delta} \\ 0, & \overline{g}_\delta \notin H^{p^\delta} \end{cases}$$

for every $\overline{g}_\beta \in G^{p^\beta}$ and $\overline{g}_\delta \in G^{p^\delta}$. The canonical forms yield $r_{a_\beta} = r_{a_\delta}$

and $g_\beta a_\beta = g_\delta a_\delta$. Fix $a'_\beta \in H^{p^\beta}$ such that $g_\beta a'_\beta = g_\delta a'_\delta$ with $a'_\delta \in H^{p^\delta}$. Consequently $x = g_\beta a'_\beta \sum_{a_\beta \in G^{p^\beta}} r_{a_\beta} a_\beta {a'_\beta}^{-1} \in G^{p^\delta} V(R^{p^\delta} G^{p^\delta}; H^{p^\delta})$, because

$$\sum_{a_\beta {a'_\beta}^{-1} \in \overline{g}_\delta H^{p^\delta}} r_{a_\beta} = \sum_{a_\delta \in \overline{g}_\delta {a'_\delta}^{-1} H^{p^\delta}} r_{a_\delta} = \sum_{a_\delta \in \overline{g}_\delta H^{p^\delta}} r_{a_\delta} = \begin{cases} 1, & \overline{g}_\delta \in H^{p^\delta} \\ 0, & \overline{g}_\delta \notin H^{p^\delta} \end{cases}.$$

Therefore, the right hand-side contains the left hand-side. The converse follows owing to the formula in [1], $V(R^{p^\alpha} G^{p^\alpha}; H^{p^\alpha}) = V^{p^\alpha}(RG; H)$. The proof is finished. □

A matter of these two facts is the next valuable

Proposition 3.3 *Assume H is a subgroup of G with p-torsion. If H is isotype in G, then $GV(RG;H)$ is isotype in $V(RG)$ and $GV(RG;H)/G$ is isotype in $V(RG)/G$.*

Proof. In fact, $H \cap G^{p^\alpha} = H^{p^\alpha}$ and Propositions 3.1 plus 3.2 lead us to this that $[GV(RG;H)] \cap V^{p^\alpha}(RG) = GV(RG;H) \cap V(R^{p^\alpha} G^{p^\alpha}) = G^{p^\alpha} V(R^{p^\alpha} G^{p^\alpha}; H^{p^\alpha}) = [GV(RG;H)]^{p^\alpha}$.

We now recall the standard group theoretic fact that if C is isotype in p-torsion G and $N \leq C$ is nice in G, then C/N is isotype in G/N. And so, by what we have shown above and the fact that G is nice in $V(RG)$ [18, 19, 4] we derive the claim. This concludes the proof. □

Proposition 3.4 *Given $G^p \neq 1$. Then $V(RG)$ is a divisible p-group if and only if G is a divisible p-group and R is perfect.*

Proof. Really, $V^p(RG) = V(R^p G^p) = V(RG)$ is equivalent to $R = R^p$ and $G = G^p$ elementarily. This finishes the proof. □

A part of the formulated below main assertions, however, is announced in [1]. So, we can attack

Theorem 3.1 *Presume that G is a p-group and λ is countable limit. Then $V(RG;B)$ is a proper λ-basic subgroup of $V(RG)$ if and only if B is a proper λ-basic subgroup of G and R is perfect. Moreover, if B is λ-basic in G and R is perfect, then B is a direct factor of $V(RG;B)$ and $V(RG;B)G/G$ is λ-basic in $V(RG)/G$.*

A direct consequence is the classical

Corollary 3.1 [1, 6, 22] *$V(RG;B)$ is proper basic in the p-group $V(RG)$ if and only if B is so in G and R is perfect. Besides, if B is basic in a p-group G and R is perfect, then B is a direct factor of $V(RG;B)$ and $GV(RG;B)/G$ is basic in $V(RG)/G$.*

These statements are established in conjuction with the next (stated with no proof in [1])

Theorem 3.2 *Let H be an isotype p-subgroup of G with countable limit length and R perfect. Then $V(RG;H)$ is totally projective if and only if H is. Besides, if H is totally projective, then so is $V(RG;H)/H$ and thus H is a direct factor of $V(RG;H)$ with totally projective complement.*

Immediate consequences are the following points (see, for instance, [17] and [14], respectively)

Corollary 3.2 [1] *Assume that R is perfect and*

(*) *G is totally projective p-torsion of countable length. Then $V(RG)/G$ is totally projective.*

(**) *G_p is totally projective with countable limit length and R is with no nilpotents. Then $S(RG)/G_p$ is totally projective. In particular, if G_p is the maximal torsion subgroup of G which is totally projective in countable limit length and R is a field, $V(RG)/G$ is a totally projective p-group.*

Now we come to the

Proof of Theorem 3.2. First and foremost, suppose $V(RG;H)$ is totally projective. Then since clearly H is isotype in $V(RG;H)$, we can apply the main result in [12] (that, however, follows directly from Theorem 2.2) to finish the proof in this direction.

The converse claim can be proved by virtue of the significant technique raised by us in [7] that gives more strong results, but now for simpleness and compactness of the article, we shall follow essentially the idea in [14, 8]. Well, according to the cited above in §2 criterion, $H = \bigcup_{n<\omega} H_n$, where $H_n \subseteq H_{n+1}$ and H_n are height-finite in H, hence in G since H is an isotype subgroup. It is a simple matter to show that $H = \bigcup_{\alpha<\mu} N_\alpha$, where each N_α has a finite height-spectrum in G and $|N_{\alpha+1}/N_\alpha|$ is finite for all α. But N_α are nice in G, whence $V(RG;H) = \bigcup_{\alpha<\mu} V(RG;N_\alpha)$ where all $V(RG;N_\alpha)$ are nice in $V(RG)$ [18], and so are nice in the isotype by Proposition 3.3 subgroup $GV(RG;H)$. By means of a claim of May (see [18] or Lemma 3.1

in [14]), the chain

$$1 = V(RG; N_0) \subseteq \cdots \subseteq V(RG; N_\alpha) \subseteq \cdots \qquad (\alpha < \mu)$$

can be refined to a nice composition series for $V(RG; H)$ where each member of the sequence is p-nice in $V(RG)$. If T is a generic subgroup in this refinement, then GT is p-nice in $V(RG)$, whence in $GV(RG; H)$. Therefore, $GV(RG; H)/G \cong V(RG; H)/H$ is totally projective, as desired. But it is well-known and documented that H is balanced in $V(RG; H)$ [3] (see also [19] but when R is a field) and so the direct factor property is guaranteed from [11]. The proof is completed after all. □

Proof of Corollary 3.2. Follows at the substitution $H = G$ or $H = G_p$ and the formula in [2], $S(RG) = S(RG; G_p)$. Besides, $V(RG) = GS(RG)$ [19] when R is a field, and so all is proved. □

Remark The second half of (**) appears in [14].

Now we are ready to give

Proof of Theorem 3.1. We shall consider some steps:

Step 1. B is λ-basic in G and R is perfect; or $V(RG; B)$ is λ-basic in $V(RG)$.

We shall show that the three conditions from the definition for a λ-basic subgroup listed above are satisfied. Indeed:

(1) follows by means of Theorem 3.2 observing that B is isotype in G.

(2″) The first half on the isotype holds owing to [1, Proposition 2]. The weakly niceness is analogous to [18, Lemma 4].

(3) The divisibility is fulfilled by Proposition 3.4, because is valid the simple fact that $V(RG)/V(RG; B) \cong V(R(G/B))$.

Step 2. The direct factor property.

It follows again from Theorem 3.2.

To conclude the proof in general it is sufficient to obtain that

Step 3. $GV(RG; B)/G$ is λ-basic in $V(RG)/G$.

Well, (1) is true according to Theorem 3.2 since as we have promised B is isotype in G and besides it is easily seen that $GV(RG; B)/G \cong V(RG; B)/B$.

(2″) The weakly isotypity may be gotten from Proposition 3.3. The weakly niceness can be deduced thus. As we have shown above, $V(RG; B)$

is weakly nice in $V(RG)$ and further we can copy the proof of Lemma 1 in [24] to establish that $GV(RG;B)$ is weakly nice in $V(RG)$. Therefore [11], $GV(RG;B)/G$ is weakly nice in $V(RG)/G$.

(3) The divisibility is standard since $V(RG)/G/GV(RG;B)/G \cong V(RG)/GV(RG;B)$ is an epimorphic image of the divisible group $V(RG)/V(RG;B) \cong V(R(G/B))$. The proof is finished. □

We will extend now the last proof, as a λ-basic subgroup of $S(RG)$ will be obtained. Well, we can formulate

Theorem 3.3 *Let G be torsion, $B \leq G_p$ and λ be countable limit. Then $V(RG;B)$ is a proper λ-basic subgroup of $S(RG)$ if and only if B is a proper λ-basic subgroup of G_p and R is perfect. Moreover, if B is λ-basic in G_p and R is perfect, then B is a direct factor of $V(RG;B)$ and $V(RG;B)G_p/G_p$ is λ-basic in $S(RG)/G_p$.*

Proof. The proof goes on the same conclusions as for the p-primary case observing that G/B must be p-divisible since $G/B/G_p/B \cong G/G_p$ is p-divisible [10], and moreover B is weakly p^λ-nice in G because G_p is balanced in G. The theorem is verified. □

Corollary 3.3 [1, 6] *Assume G is torsion and $B < G_p$. Then $V(RG;B)$ is basic in $S(RG)$ if and only if B is basic in G_p and R is perfect. Moreover, if B is basic in G_p and R is perfect, then B is a direct factor of $V(RG;B)$ and $V(RG;B)G_p/G_p$ is basic in $S(RG)/G_p$.*

Remark Probably Theorem 3.2 is absolute true and for uncountable lengths, whence Theorem 3.1 will be immediately valid for such lengths, i.e. in the general case.

Other our major results are selected in the next section.

II. Criteria for C_λ-groups.

A part of the central results stated here, however, is announced in [5] and [9].

A direct consequence in one way to Theorem 3.3 is the following

Theorem 3.4 *Let G be a torsion group whose G_p is of limit length $< \Omega$ and let R be perfect. Then $S(RG)$ is a C_λ-group if and only if G_p is a C_λ-group. Besides, G_p a C_λ-group yields that the same is $S(RG)/G_p$.*

Before proving the claim in general, we need

Lemma 3.2 *An isotype subgroup of a primary C_λ-group is a C_λ-group when $\lambda \leq \Omega$.*

Proof. Take C to be isotype in p-primary G that is a C_λ-group. Hence, because C/C^{p^α} is isotype (isomorphic) in G/G^{p^α}, i.e. in other words $C/C^{p^\alpha} \cong CG^{p^\alpha}/G^{p^\alpha}$ is isotype in G/G^{p^α}; $\alpha < \lambda$, we can apply the main result in [12] to get the assertion. □

Thus we are in position to attack

Proof of Theorem 3.4. The necessity follows by virtue of the above lemma plus the simple fact that G is isotype in $V(RG)$.

Next, we treat the more difficult converse question. And so, this part holds applying Theorems 1.2 and 3.1 plus the simple fact that G is nice in $V(RG)$ ([18, 19]; [4]). This completes the proof. □

We shall generalize now the above theorem for p-primary groups with uncountable length, that is, Ω. And so, the following, announced in [5], is fulfilled:

Theorem 3.5 *Suppose G is an abelian p-group, $\lambda \leq \Omega$ and R is perfect. Then $V(RG)$ is a C_λ-group if and only if the same holds for G. Besides, G a C_λ-group implies that so is $V(RG)/G$. Moreover, if G is a C_λ-group of countable length λ and R is perfect with no nilpotents, then G is a direct factor of $V(RG)$ with a totally projective complement.*

Proof. Since Ω is not confinal with ω, we need another method for proof. The necessity in the first half is true using again Lemma 3.2.

Now we shall examine the sufficiency. Well, fix a countable ordinal α. But G/G^{p^α} is totally projective for $\alpha < \lambda$ and thus Theorem 2.2 means that $G/G^{p^\alpha} = \bigcup_{n<\omega}(G_n/G^{p^\alpha})$, where $G_n \subseteq G_{n+1}$ and G_n/G^{p^α} are height-finite in G/G^{p^α}. Hence $G = \bigcup_{n<\omega} G_n$, where G_n has elements with heights (as calculated in G) $\geq \alpha$ and elements with a finite number of heights $< \alpha$. Thus the same is true for $V(RG) = \bigcup_{n<\omega} V(RG_n)$. Further it is easy to see that

$$V(RG)/V(RG^{p^\alpha}) = \bigcup_{n<\omega} [V(RG_n)V(RG^{p^\alpha})/V(RG^{p^\alpha})]$$

and because G is nice in $V(RG)$ [18, 4],

$$V(RG)/G/(V(RG/G)^{p^\alpha} \cong V(RG)/GV(RG^{p^\alpha}) = \bigcup_{n<\omega}[V(RG_n)GV(RG^{p^\alpha})/GV(RG^{p^\alpha})].$$

Foremost, choose $1 \neq x \in V(RG_n)V(RG^{p^\alpha})/V(RG^{p^\alpha})$ whence $x = yV(RG^{p^\alpha})$, where $y \in V(RG_n) \setminus V(RG^{p^\alpha})$ and thus it has height $< \alpha$. Apparently since $V(RG^{p^\alpha})$ is nice in $V(RG)$, then height(x) = height(y). But consequently $V(RG_n)V(RG^{p^\alpha})/V(RG^{p^\alpha})$ has a finite height-spectrum in $V(RG)/V(RG^{p^\alpha})$ and Theorem 2.2 yields the fact, immediately.

Further, given $1 \neq a \in V(RG_n)GV(RG^{p^\alpha})/GV(RG^{p^\alpha})$. By the same token $a = bGV(RG^{p^\alpha})$, where $b \in V(RG_n) \setminus GV(RG^{p^\alpha})$, hence this element has height $< \alpha$. But it is no loss of generality in assuming that α is limit (cf. [20]) and so a lemma in [3] guarantees that $GV(RG^{p^\alpha})$ is nice in $V(RG)$. Furthermore height(a) = height(bg_b) for some $g_b \in G$ because $bg_b \notin V(RG^{p^\alpha})$. Write $b = \sum_{g_n \in G_n} r_{g_n} g_n$ where $r_{g_n} \in R$. It is no harm in presuming that some g_n of this sum is equal to 1, owing to the form of a. On the other hand certainly height$(b) \leq$ height(a). As a final, we observe that height$(g_b) \geq$ height$(b) = \min\{$height$(g_n)\}$ and thus this implies height(a) = height(b), where there is at least one g_n with height $< \alpha$ and so a is an "height-finite" element, finishing the conclusions.

To close the proof in general, it is enough to show only that $V(RG)/G$ is totally projective, according to Theorem 2.2 stated in paragraph 2 or to Theorem 2.1 (see as well [2]) proving also that this factor group is σ-summable since as we have just seen it is a C_λ-group.

For this aim, as above, it is a simple matter to verify that $V(RG)/G = \bigcup_{n<\omega}[V(RG_n)G/G]$. Because λ is countable, all ordinals strictly less than λ can be ordered at ω and the relation "$<$" thus:

$$\cdots < \alpha_1 < \cdots < \alpha_n < \cdots$$

Next, we shall construct a special ascending at n chain of subgroups M_n in $V(RG)$ such that they have an almost finite height-spectrum in $V(RG)$ and $V(RG)/G = \bigcup_{n<\omega}(M_nG/G)$.

In fact, choose these groups in the following manner

$$M_n = \langle r_1^{(n)} + r_2^{(n)} g_2^{(n)} + \cdots + r_{s_n}^{(n)} g_{s_n}^{(n)} \mid 0 \neq r_1^{(n)}, \ldots, r_{s_n}^{(n)} \in R,$$

$r_1^{(n)} + \cdots + r_{s_n}^{(n)} = 1$; $g_2^{(n)}, \ldots, g_{s_n}^{(n)} \in G_n$, $g_2^{(n)}, \ldots, g_{s_n}^{(n)}$ along with their nontrivial degrees and all their possible products have heights computed in G which are $< \alpha_n; s_n \in \mathbb{N}\rangle$. Clearly $M_n \subseteq M_{n+1}$ and $M_n \subseteq V(RG_n)$, and hence it is evident that all M_nG/G generate $V(RG)/G$. Every element in M_n has the form $x_1^{\varepsilon_1} \dots x_t^{\varepsilon_t}$, where x_i are of the above kind, $0 \leq \varepsilon_i \leq \text{order}(x_i)$ and $1 \leq i \leq t$, $t \in \mathbb{N}$. Apparently, by the construction, each generating element in M_n has height $< \alpha_n$.

Now, we shall show that M_n is almost height-finite. Really, using an ordinary combinatorical arguments, in the canonical form of an arbitrary degree of each generating element of M_n does exist members of the kind fg or fg^ε $(0 < \varepsilon < \text{order}(g))$ where $f \in R$ and $g \in G_n$ possesse the above properties, as is not difficult to be seen. On the other hand, it is only a technical matter to verify that every element of M_n may be written as ay_n, where $a \in G_n$ and y_n has a basis element 1 and has a basis element with height $< \alpha_n$. So, the height of y_n taken in $V(RG)$ is bounded by some ordinal α_n, whence the heights of such special elements are of a finite number, owing to the construction of G as a countable union of special selected groups and choosing $\alpha = \alpha_n$. That is why M_n are indeed almost height-finite, as we claimed.

Further, our final purpose is to obtain that GM_n/G are height-finite and so the proof will be finished after all.

In order to prove this, we take $1 \neq x_nG$ with $x_n \in M_n$. The niceness of G in $V(RG)$ means that $\text{height}(x_nG) = \text{height}(x_ng_{x_n})$ for some $g_{x_n} \in G$. But the major is that by the above construction x_n has or may be reduced to has an identity basis element, hence it is easily to check that $\text{height}(x_ng_{x_n}) = \text{height}(x_n)$ because $\text{height}(x_n) \leq \text{height}(x_ng_{x_n})$ and therefore $\text{height}(g_{x_n}) \geq \text{height}(x_n)$. As we have shown above, M_n is with a finite number of heights for such elements as x_n, furthermore so is GM_n/G. This deduces the statement. □

The next assertion is actual.

Corollary 3.4 *Suppose G is splitting such that G/G_p is p-divisible (in particular, G is torsion) and R is perfect without nilpotents. Then $S(RG)$ is a C_λ-group if and only if G_p is a C_λ-group when $\lambda \leq \Omega$.*

Proof. Write $G = G_p \times G/G_p$. Therefore, it is elementary to derive that $S(RG) \cong S(R(G/G_p)G_p)$, where $R(G/G_p)$ is perfect without nilpotents. So, we can apply the last theorem to finish the proof. □

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