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Invariant projectively flat affine connections on Lie groups

(Dedicated to the memory of Eulalia García Rus (Lali))

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Abstract. The simple Lie groups with left invariant projectively flat affine connections are classified. This is accomplished by proving that some of the possibilities that appear in the previous work by Urakawa [U, Theorem C] do not actually occur.

Key words: invariant projectively flat affine connections, simple Lie groups.

1. Introduction

Recently, Urakawa [U], based on previous work by Shima [S], classified the irreducible simply connected Riemannian symmetric spaces with invariant projectively flat affine connections and studied the simple Lie groups with left invariant projectively flat affine connections. For the latter he obtained the next result:

Theorem 1 ([U, Theorem C]) *Let G be a real simple Lie group. If G admits a left invariant projectively flat affine connection, then its Lie algebra $\mathfrak{g}$ is one of the following:*
(a) $\mathfrak{o}(3) \cong \mathfrak{su}(2)$,
(b) $\mathfrak{sl}(n+1, \mathbb{R})$, $n \geq 1$,
(c) $\mathfrak{su}^*(2n)$, $n \geq 2$,
(d) $\mathfrak{su}(r,s)$ ($r+s =$ *even*, $r+s \geq 4$); $\mathfrak{o}(3,4)$; $\mathfrak{o}(1,9)$, $\mathfrak{o}(5,5)$; $\mathfrak{o}(3,11)$, $\mathfrak{o}(7,7)$.

It was remarked too in [U] that in the cases (a)–(c) above, being G simply connected, it admits a left invariant projectively flat affine connection, but for the case (d) it is not known wether or not G admits such a connection.

The purpose of this note is to show that none of the possibilities in

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case (d) above allow a left invariant projectively flat affine connection on the Lie group. The proof will heavily rely on the results in [U].

Also, the theoretical characterization of the simply connected homogeneous spaces $M = G/K$ admitting a G-invariant projectively flat affine connection given in [U, Theorem 1.3] (and based on [S, Theorem 1.1]) can be proved, in case $M = G$, in a different and more algebraic way, which does not require the group being simply connected. This will be our Theorem 2.

As an immediate consequence, the following classification result emerges, where we use $\mathfrak{sl}(n, \mathbb{H})$ instead of $\mathfrak{su}^*(2n)$, and since $\mathfrak{sl}(1, \mathbb{H}) \cong \mathfrak{su}(2)$, cases (a) and (c) can be put together:

Main Theorem *Let G be a real simple Lie group. Then G admits a left invariant projectively flat affine connection if and only if its Lie algebra $\mathfrak{g}$ is one of the following*:

(a) $\mathfrak{sl}(n + 1, \mathbb{R})$, $n \geq 1$,

(b) $\mathfrak{sl}(n, \mathbb{H})$, $n \geq 1$.

By [NP, Theorem 4], a connected semisimple Lie group admits a bi-invariant projectively flat affine connection if and only if its Lie algebra is either $\mathfrak{sl}(n+1, \mathbb{R})$ or $\mathfrak{sl}(n, \mathbb{H})$, ($n \geq 1$). In particular, it is simple. Therefore, no new connected simple Lie groups are obtained when the bi-invariant restriction is reduced to just left invariant. However, the restriction in our Main Theorem on G being simple (and not just semisimple) is necessary. For instance, the semisimple Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$ has a representation satisfying the requirements of our Theorem 2 below, namely $\tilde{V} = \mathrm{Mat}_{2\times 3}(\mathbb{R}) \oplus \mathrm{Mat}_{2\times 3}(\mathbb{R})$ with the natural action of $\mathfrak{g}$ on it. Therefore, the semisimple Lie group $SL(2, \mathbb{R}) \times SL(3, \mathbb{R})$ admits a left invariant projectively flat affine connection.

As in [S, U], all the connections considered will be assumed to be torsion-free and Ricci-symmetric. It is well-known (see [NS]) that the affine connection ∇ is projectively flat if and only if for any vector fields X, Y, Z:

i) its curvature tensor R satisfies

$$R(X, Y)Z = \frac{1}{n-1}\Big(\mathrm{Ric}(Y, Z)X - \mathrm{Ric}(X, Z)Y\Big), \tag{1}$$

where Ric is the Ricci tensor and n is the dimension of the manifold, and

ii) Ric satisfies the Codazzi equation:

$$(\nabla_X \operatorname{Ric})(Y, Z) = (\nabla_Y \operatorname{Ric})(X, Z). \tag{2}$$

2. Left invariant projectively flat affine connections on Lie groups

This section is devoted to characterize the Lie groups, not necessarily simply connected, admitting a left invariant projectively flat affine connection.

First notice that any left invariant affine connection ∇ on a Lie group G gives rise to a bilinear multiplication $\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, given by $\lambda(X, Y) = \nabla_X Y$ ($\mathfrak{g}$ will always be considered as the Lie algebra of the left invariant vector fields on G), and conversely, any such λ determines a unique left invariant affine connection ∇. The fact that ∇ is torsion-free is equivalent to the condition

$$\lambda(X, Y) - \lambda(Y, X) = [X, Y], \tag{3}$$

for any $X, Y \in \mathfrak{g}$ (that is, $(\mathfrak{g}, \lambda)$ is a Lie-admissible algebra). Besides, the curvature tensor R is determined by the trilinear map $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X, Y, Z) \mapsto R(X, Y)Z$, which in terms of the multiplication λ is given by

$$R(X, Y)Z = \lambda(X, \lambda(Y, Z)) - \lambda(Y, \lambda(X, Z)) - \lambda([X, Y], Z), \tag{4}$$

for any $X, Y, Z \in \mathfrak{g}$. In the same way, by left invariance $\operatorname{Ric}(X, Y)$ is constant for any $X, Y \in \mathfrak{g}$, and hence Ric is determined by the bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $(X, Y) \mapsto \operatorname{Ric}(X, Y)$. For simplicity, as in [S, U], we will consider the bilinear map $\gamma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $\gamma(X, Y) = \frac{1}{n-1} \operatorname{Ric}(X, Y)$, n being the dimension of G.

Now, the conditions (1) and (2) for a connection to be projectively flat become, because of the left invariance,

$$R(X, Y)Z = \gamma(Y, Z)X - \gamma(X, Z)Y, \tag{1'}$$

$$\gamma(\lambda(X, Y), Z) + \gamma(Y, \lambda(X, Z)) = \gamma(\lambda(Y, X), Z) + \gamma(X, \lambda(Y, Z)), \tag{2'}$$

for any $X, Y, Z \in \mathfrak{g}$.

Theorem 2 *A real Lie group G admits a left invariant projectively flat affine connection if and only if there exists a real representation $\tilde{\lambda} : \mathfrak{g} \longrightarrow \operatorname{End}_{\mathbb{R}}(\tilde{V})$ of $\mathfrak{g}$ and a vector $v_0 \in \tilde{V}$ such that $\dim \tilde{V} = 1 + \dim \mathfrak{g}$ and $\tilde{V} =$*

$\tilde{\lambda}(\mathfrak{g})v_0 \oplus \mathbb{R}v_0$.

Proof. Let ∇ be a left invariant projectively flat affine connection on the Lie group G and λ the associated bilinear multiplication on $\mathfrak{g}$. As in [S, U] we enlarge $\mathfrak{g}$ to $\tilde{V} = \mathfrak{g} \oplus \mathbb{R}e$ and define $\tilde{\lambda} : \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(\tilde{V})$ by $\tilde{\lambda}_X e = X$ and $\tilde{\lambda}_X Y = \lambda(X,Y) - \gamma(X,Y)e$, for any $X, Y \in \mathfrak{g}$. Then for any $X, Y, Z \in \mathfrak{g}$:

$$\begin{aligned}
[\tilde{\lambda}_X, \tilde{\lambda}_Y]e &= \lambda(X,Y) - \gamma(X,Y)e - \lambda(Y,X) + \gamma(Y,X)e \\
&= [X,Y] = \tilde{\lambda}_{[X,Y]}e,
\end{aligned}$$

because of (3) and the Ricci-symmetry of ∇, and

$$\begin{aligned}
[\tilde{\lambda}_X, \tilde{\lambda}_Y]Z &= \tilde{\lambda}_X(\lambda(Y,Z) - \gamma(Y,Z)e) - \tilde{\lambda}_Y(\lambda(X,Z) - \gamma(X,Z)e) \\
&= \lambda(X, \lambda(Y,Z)) - \gamma(X, \lambda(Y,Z))e - \gamma(Y,Z)X \\
&\qquad - \lambda(Y, \lambda(X,Z)) + \gamma(Y, \lambda(X,Z))e + \gamma(X,Z)Y \\
&= \lambda([X,Y],Z) - \gamma([X,Y],Z)e \\
&= \tilde{\lambda}_{[X,Y]}Z,
\end{aligned}$$

where we have used (2′), (3) and (4).

Therefore, $\tilde{\lambda}$ is a representation of $\mathfrak{g}$ and, with $v_0 = e$, it satisfies $\tilde{V} = \tilde{\lambda}(\mathfrak{g})v_0 \oplus \mathbb{R}v_0$.

Conversely, given such a representation $\tilde{\lambda} : \mathfrak{g} \to \mathrm{End}_{\mathbb{R}}(\tilde{V})$, for any $X, Y \in \mathfrak{g}$, $\tilde{\lambda}_X(\tilde{\lambda}_Y v_0) \in \tilde{V} = \tilde{\lambda}(\mathfrak{g})v_0 \oplus \mathbb{R}v_0$, and hence there exist bilinear maps $\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and $\gamma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ such that

$$\tilde{\lambda}_X(\tilde{\lambda}_Y v_0) = \tilde{\lambda}_{\lambda(X,Y)}v_0 - \gamma(X,Y)v_0. \tag{5}$$

Since $(\tilde{\lambda}_X\tilde{\lambda}_Y - \tilde{\lambda}_Y\tilde{\lambda}_X)v_0 = \tilde{\lambda}_{[X,Y]}v_0$, equation (5) implies that γ is symmetric and λ satisfies equation (3). Therefore, there is a unique left invariant torsion-free affine connection ∇ on G with $\nabla_X Y = \lambda(X,Y)$ for any $X, Y \in \mathfrak{g}$. Again, since $\tilde{\lambda}$ is a representation, for any $X, Y, Z \in \mathfrak{g}$,

$$[\tilde{\lambda}_X, \tilde{\lambda}_Y]\tilde{\lambda}_Z v_0 = \tilde{\lambda}_{[X,Y]}\tilde{\lambda}_Z v_0. \tag{6}$$

The left hand side of (6) gives

$$\begin{aligned}
&\tilde{\lambda}_X\left(\tilde{\lambda}_{\lambda(Y,Z)}v_0 - \gamma(Y,Z)v_0\right) - \tilde{\lambda}_Y\left(\tilde{\lambda}_{\lambda(X,Z)}v_0 - \gamma(X,Z)v_0\right) \\
&\quad = \tilde{\lambda}_{\lambda(X,\lambda(Y,Z))}v_0 - \gamma(X, \lambda(Y,Z))v_0 - \gamma(Y,Z)\tilde{\lambda}_X v_0 \\
&\qquad - \tilde{\lambda}_{\lambda(Y,\lambda(X,Z))}v_0 + \gamma(Y, \lambda(X,Z))v_0 + \gamma(X,Z)\tilde{\lambda}_Y v_0
\end{aligned}$$

$$
\begin{aligned}
&= \tilde{\lambda}_{\lambda(X,\lambda(Y,Z))-\lambda(Y,\lambda(X,Z))-\gamma(Y,Z)X+\gamma(X,Z)Y}\, v_0 \\
&\quad + \Big(\gamma(Y,\lambda(X,Z)) - \gamma(X,\lambda(Y,Z))\Big)v_0,
\end{aligned}
$$

while the right hand side of (6) is

$$
\tilde{\lambda}_{\lambda([X,Y],Z)}v_0 - \gamma([X,Y],Z)v_0.
$$

Therefore:

$$
\begin{aligned}
&\lambda(X,\lambda(Y,Z)) - \lambda(Y,\lambda(X,Z)) - \lambda([X,Y],Z) \\
&\qquad = \gamma(Y,Z)X - \gamma(X,Z)Y \qquad (7)
\end{aligned}
$$

$$
\gamma([X,Y],Z) = \gamma(X,\lambda(Y,Z)) - \gamma(Y,\lambda(X,Z)) \qquad (8)
$$

But equations (4) and (7) give $R(X,Y)Z = \gamma(Y,Z)X - \gamma(X,Z)Y$ for any $X,Y,Z \in \mathfrak{g}$, so that

$$
\begin{aligned}
\mathrm{Ric}(Y,Z) &= \operatorname{tr} R(-,Y)Z = \operatorname{tr}(\gamma(Y,Z)Id - \gamma(-,Z)Y) \\
&= (n-1)\gamma(Y,Z)
\end{aligned}
$$

for any $Y,Z \in \mathfrak{g}$, thus obtaining (1′). And, using (3), equation (8) becomes (2′). Therefore, ∇ is a left invariant projectively flat affine connection on G. □

3. Proof of the Main Result

Because of Theorems 1 and 2, and the construction in [U] of the convenient representations for the Lie algebras in cases (a)–(c) of Theorem 1, to prove the Main Theorem it is enough to prove the impossibility of case (d) in Theorem 1. This will be based on the work by Urakawa with only one extra ingredient that appears in the next Lemma. It says that there is a great freedom in choosing the element v_0 in Theorem 2.

Recall that the Zariski topology on a finite dimensional vector space V over an infinite field F is the topology where the closed sets are the common zeros of families of polynomials on V (see, for instance, [H, Appendix to §23]). By a Zariski dense subset we mean a dense subset in the Zariski topology. Any nonempty open subset in the Zariski topology is dense.

Lemma 3 *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an arbitrary infinite field F, let $\tilde{\lambda} : \mathfrak{g} \to \mathrm{End}_F(\tilde{V})$ be a representation of $\mathfrak{g}$ with* $\dim \tilde{V} =$

$1 + \dim \mathfrak{g}$ *and let* $v \in \tilde{V}$ *such that* $\tilde{V} = Fv \oplus \tilde{\lambda}(\mathfrak{g})v$. *Then the set*

$$B = \{w \in \tilde{V} : \tilde{V} = Fw \oplus \tilde{\lambda}(\mathfrak{g})w\}$$

is a Zariski dense subset of $\tilde{V}$.

Proof. Let $\{g_1, \dots, g_n\}$ be a fixed basis of $\mathfrak{g}$ and $\{v_0, v_1, \dots, v_n\}$ one of $\tilde{V}$. Then a vector $w \in \tilde{V}$ is in B if and only if the elements $w, \tilde{\lambda}(g_1)w, \dots, \tilde{\lambda}(g_n)w$ are linearly independent if and only if the determinant of the matrix P such that

$$(w, \tilde{\lambda}(g_1)w, \dots, \tilde{\lambda}(g_n)w) = (v_0, v_1, \dots, v_n)P$$

is nonzero. Since this determinant is a polynomial in the coordinates of w, this gives a nonempty open (and hence dense) set in the Zariski topology of $\tilde{V}$. □

In what follows, we will write $\mathfrak{g}w$ instead of $\tilde{\lambda}(\mathfrak{g})w$.

Now, a case by case analysis will be carried out:

i) Proof of the impossibility for $\mathfrak{g} = \mathfrak{o}(3,4)$:

According to [U, Remark 3.9], the only representation to be checked is the direct sum of two copies of the natural representation X and one copy of the spin one Y:

$$\tilde{V} = X \oplus X \oplus Y.$$

Let $b(\,,\,)$ denote the bilinear form (of signature $(3,4)$) on X such that $\mathfrak{o}(3,4) = \{g \in \mathrm{End}_{\mathbb{R}}(X) : b(gx_1, x_2) + b(x_1, gx_2) = 0 \,\forall x_1, x_2 \in X\}$. Let $B = \{w \in \tilde{V} : \tilde{V} = \mathbb{R}w \oplus \mathfrak{g}w\}$. It has to be proved that B is empty. Assuming the contrary:

Claim There is an element $v_0 = (x_1, x_2, y) \in B$ such that

$$\begin{vmatrix} b(x_1,x_1) & b(x_1,x_2) \\ b(x_2,x_1) & b(x_2,x_2) \end{vmatrix} \neq 0.$$

This is because the set

$$\left\{(x_1,x_2,y) \in \tilde{V} : \begin{vmatrix} b(x_1,x_1) & b(x_1,x_2) \\ b(x_2,x_1) & b(x_2,x_2) \end{vmatrix} \neq 0\right\}$$

is an open set in the Zariski topology.

Now, consider such a $v_0 = (x_1, x_2, y)$. Then $\hat{\mathfrak{g}} = \{g \in \mathfrak{g} : gx_1 = gx_2 = 0\}$ is the orthogonal Lie algebra of the orthogonal complement $(\mathbb{R}x_1 + \mathbb{R}x_2)^\perp$, with the induced bilinear form, hence the dimension of $\hat{\mathfrak{g}}$ is $\binom{5}{2} = 10$ and the kernel of the linear map

$$\begin{aligned} \hat{\mathfrak{g}} &\longrightarrow \tilde{V} \\ g &\mapsto gv_0 = (gx_1, gx_2, gy) = (0, 0, gy) \end{aligned}$$

has dimension $\geq \dim \hat{\mathfrak{g}} - \dim Y = 10 - 8 = 2$. Therefore we arrive at the contradiction

$$0 = \{g \in \mathfrak{g} : gv_0 = 0\} \supseteq \{g \in \hat{\mathfrak{g}} : gy = 0\} \neq 0.$$

ii) Proof of the impossibility for $\mathfrak{g} = \mathfrak{o}(3, 11)$ and $\mathfrak{g} = \mathfrak{o}(7, 7)$:

This is similar. By [U, Remark 3.11], the representation to be considered is

$$\tilde{V} = X \oplus X \oplus Y,$$

where X is the natural representation ($\dim X = 14$) and Y is a half-spin representation ($\dim Y = 64$). Assuming $B \neq \emptyset$, take $v_0 = (x_1, x_2, y)$ and $\hat{\mathfrak{g}}$ as in the previous case, so that $\dim \hat{\mathfrak{g}} = \binom{12}{2} = 66$ and the kernel of the corresponding linear map has dimension $\geq 66 - 64 = 2 > 0$, arriving at a contradiction as before.

iii) Proof of the impossibility for $\mathfrak{g} = \mathfrak{o}(1, 9)$ and $\mathfrak{g} = \mathfrak{o}(5, 5)$:

Again by [U, Remark 3.11], the representation to be considered here is

$$\tilde{V} = X \oplus X \oplus X \oplus Y,$$

where X is the natural representation ($\dim X = 10$) and Y is a half-spin representation ($\dim Y = 16$). As before, by Zariski density, if $B \neq \emptyset$ there is an element $v_0 = (x_1, x_2, x_3, y) \in B$ with $\det(b(x_i, x_j)) \neq 0$. Now consider $\hat{\mathfrak{g}} = \{g \in \mathfrak{g} : gx_1 = gx_2 = gx_3 = 0\}$, which is an orthogonal Lie algebra of dimension $\binom{7}{2} = 21$. The kernel of the linear map

$$\begin{aligned} \hat{\mathfrak{g}} &\longrightarrow \tilde{V} \\ g &\mapsto gv_0 = (gx_1, gx_2, gx_3, gy) = (0, 0, 0, gy) \end{aligned}$$

has dimension $\geq \dim \hat{\mathfrak{g}} - \dim Y = 21 - 16 = 5 > 0$, a contradiction again.

iv) Proof of the impossibility for $\mathfrak{g} = \mathfrak{su}(r,s)$, $r+s=2k\geq 4$:

Let $V = \mathbb{C}^{2k}$ with the natural action of $\mathfrak{g}$ on V; let $b : V\times V \to \mathbb{C}$ be the associated hermitian form, so that

$$\mathfrak{g} = \{g\in \mathrm{End}_{\mathbb{C}}(V) : b(gx,y)+b(x,gy)=0 \quad \forall x,y\in V \text{ and } \operatorname{tr} g = 0\}.$$

By [U, Remark 3.7], one has to consider the representation $\tilde{V} = lV\oplus(k-l)\hat{V}$, for $0\leq l\leq k$, where $\hat{V}=V$ but with the action $g\cdot x = \bar{g}x$, where $\bar{g}=-g^t$. Let $B=\{w\in\tilde{V} : \tilde{V} = \mathbb{R}w\oplus\mathfrak{g}w\}$ and assume it is not empty. As before, by Zariski density there is an element $v_0 = (x_1,\dots,x_l,y_1,\dots y_{k-l})\in B$ such that $\det\left(b(x_i,x_j)\right)\neq 0\neq \det\left(b(y_r,y_s)\right)$.

Let $W_1 = \mathbb{C}x_1+\cdots+\mathbb{C}x_l\subseteq \mathbb{C}^{2k} = V$ and $W_2 = \mathbb{C}y_1+\cdots+\mathbb{C}y_{k-l}\subseteq \mathbb{C}^{2k}=\hat{V}$. Now, for such a v_0,

$$\{g\in\mathfrak{g} : gv_0 = 0\} = \{g\in\mathfrak{g} : gW_1 = 0\}\cap\{g\in\mathfrak{g} : \bar{g}W_2=0\}.$$

But $\{g\in\mathfrak{g} : gW_1=0\}$ is the special unitary algebra $\mathfrak{su}(W_1^{\perp})$, associated to the nondegenerate subspace $W_1^{\perp}$, so its dimension is $(2k-l)^2-1$, and similarly the dimension of $\{g\in\mathfrak{g} : \bar{g}W_2 = 0\}$ is $(2k-(k-l))^2-1 = (k+l)^2-1$. Therefore we arrive at the contradiction

$$\begin{aligned}
0 &= \dim_{\mathbb{R}}\{g\in\mathfrak{g} : gv_0=0\}\\
&= \dim_{\mathbb{R}}(\{g\in\mathfrak{g} : gW_1=0\}\cap\{g\in\mathfrak{g} : \bar{g}W_2=0\})\\
&\geq \dim_{\mathbb{R}}\{g\in\mathfrak{g} : gW_1=0\}+\dim_{\mathbb{R}}\{g\in\mathfrak{g} : \bar{g}W_2=0\}-\dim_{\mathbb{R}}\mathfrak{g}\\
&= (2k-l)^2-1+(k+l)^2-1-((2k)^2-1)\\
&= k^2-2kl+2l^2-1 = (k-l)^2+l^2-1>0.
\end{aligned}$$

This finishes the proof.

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