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On symplectic and contact regular r-cubic configurations

(Dedicated to Professor Takuo Fukuda on his sixtieth birthday)

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Abstract. We investigate a necessary and sufficient condition for a family of 2^r Lagrangian and 2^r Legendrian submanifolds to be a symplectic and contact regular r-cubic configuration respectively.

Key words: Lagrangian singularities, Legendrian singularities, symplectic manifolds, contact manifolds, regular r-cubic configurations.

1. Introduction

Lagrangian and Legendrian singularities can be found in many problems of differential geometry, calculus of variations and mathematical physics. One of the most successful their applications is the study of singularity of caustics and wavefronts. For example, the light rays incident along geodesics from a smooth hypersurface in a Riemannian manifold M to conormal directions define a Lagrangian submanifold at a point in T^*M and the pairs of light rays and lengths of geodesics define a Legendrian submanifold at a point in $T^*M \times \mathbb{R}$.

In [8] and [9], we investigate the case when the initial hypersurface has an *r-corner.* In this case the light rays incident along geodesics from each edges of the hypersurface define a *symplectic regular r-cubic configuration* at a point in T^*M and the pairs of light rays and lengths of geodesics define a *contact regular r-cubic configuration* at a point in $T^*M \times \mathbb{R}$. Symplectic and contact regular r-cubic configurations are consist of 2^r Lagrangian and 2^r Legendrian submanifolds respectively. In these papers, we study the stabilities and classifications of symplectic and contact regular r-cubic configuration.

In this paper we consider the following problem: when does a family of 2^r Lagrangian (2^r Legendrian) submanifolds become symplectic (contact) regular r-cubic configurations? We give the answer of the problem by using data of intersections between each Lagrangian (Legendrian) submanifolds.

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The main results are Theorem 4.1 and Theorem 5.7 which will be given in §4 and §5. To prove these theorem, the notion of *generating family* which is defined in §2 plays very important parts. In §6, we give some examples illustrating the main results.

As corollaries of Theorem 4.1 and Theorem 5.7, we have the following theorems:

Let r be a non negative integer and $I_r = \{1, 2, \ldots, r\}$. We remark that $I_0 = \emptyset$.

Reticular Darboux theorem 1 *Let E^{2n} be a symplectic manifold and $\{L_\sigma\}_{\sigma \subset I_r}$ be 2^r Lagrangian submanifold germ at $x \in E$ such that L_σ has an $(r - |\sigma|)$-corner for $\sigma \subset I_r$. Then there exist symplectic coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ of E at x such that*

$$L_\sigma = \{(q,p) \in (E,x) \mid q_\sigma = p_{I_r - \sigma} = q_{r+1} = \cdots = q_n = 0,\ q_{I_r - \sigma} \geq 0\}$$

for each $\sigma \subset I_r$ if and only if the following conditions hold:

(1) *$L_\sigma \cap L_\tau$ is a submanifold of (E,x) with the codimension $n + |\sigma \cup \tau| - |\sigma \cap \tau|$,*

(2) *$T(L_\sigma \cap L_\tau) = TL_\sigma \cap TL_\tau$ for any $\sigma, \tau \subset I_r$,*

(3) *$\partial L_\sigma \cap L_\tau = L_\sigma \cap L_\tau$ for $\sigma \subset \tau \subset I_r$ $(|\tau - \sigma| = 1)$.*

Reticular Darboux theorem 2 *Let E^{2n+1} be a contact manifold and $\{\tilde{L}_\sigma\}_{\sigma \subset I_r}$ be 2^r Legendrian submanifold germ at $y \in E$ such that $\tilde{L}_\sigma$ has an $(r - |\sigma|)$-corner for $\sigma \subset I_r$. Then there exist coordinates $(q_1, \ldots, q_n, z, p_1, \ldots, p_n)$ of E at y such that the contact structure of E around y is given by the canonical 1-form $dz - pdq$ and*

$$\tilde{L}_\sigma = \{(q,z,p) \in (E,y) \mid q_\sigma = p_{I_r - \sigma} = q_{r+1} = \cdots = q_n = z = 0,\ q_{I_r - \sigma} \geq 0\}$$

for each $\sigma \subset I_r$ if and only if the following conditions hold:

(1) *$\tilde{L}_\sigma \cap \tilde{L}_\tau$ is a submanifold of (E,y) with the codimension $n + 1 + |\sigma \cup \tau| - |\sigma \cap \tau|$ for all $\sigma, \tau \subset I_r$,*

(2) *$T(\tilde{L}_\sigma \cap \tilde{L}_\tau) = T\tilde{L}_\sigma \cap T\tilde{L}_\tau$ for $\sigma, \tau \subset I_r$,*

(3) *$\partial \tilde{L}_\sigma \cap \tilde{L}_\tau = \tilde{L}_\sigma \cap \tilde{L}_\tau$ for $\sigma \subset \tau \subset I_r$ $(|\tau - \sigma| = 1)$.*

In complex analytic category, the theory of symplectic regular r-cubic configurations has been developed by Nguyen Huu Duc, Nguyen Tien Dai and F. Pham and they give the answer of the problem in complex analytic category (cf. [3]). But their proof does not work well for C^∞-category. Hence we prove Theorem 3.3 by another method. There seem to be no literatures mentioning complex analytic counterparts of Theorem 4.1 and Theorem 5.7.

All manifold and maps considered here are of class C^∞ unless otherwise stated.

2. Preliminaries

Here we shall define basic notations and give an important result which is proved in [8].

Let (q,p) be canonical coordinates of $(T^*\mathbb{R}^n, 0)$ equipped with the symplectic structure $dp \wedge dq$ and $\pi : (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be the cotangent bundle. We define

$$L_\sigma^0 = \{(q,p) \in (T^*\mathbb{R}^n, 0) \mid q_\sigma = p_{I_r-\sigma} = q_{r+1} = \cdots = q_n = 0, \\ q_{I_r-\sigma} \geq 0\},$$

$$L_\sigma'^0 = \{(q,p) \in (T^*\mathbb{R}^n, 0) \mid q_\sigma = p_{I_r-\sigma} = q_{r+1} = \cdots = q_n = 0\}$$

for each $\sigma \subset I_r = \{1, \ldots, r\}$.

Let $\{L_\sigma\}_{\sigma\subset I_r}$ and $\{L'_\sigma\}_{\sigma\subset I_r}$ be families of 2^r Lagrangian submanifold germs of $(T^*\mathbb{R}^n, 0)$. Then $\{L_\sigma\}_{\sigma\subset I_r}$ is called *a symplectic regular r-cubic configuration* if there exists a symplectomorphism S on $(T^*\mathbb{R}^n, 0)$ such that $L_\sigma = S(L_\sigma^0)$ for all $\sigma \subset I_r$ and $\{L'_\sigma\}_{\sigma\subset I_r}$ is called *a symplectic regular r-cubic configuration without boundary* if there exists a symplectomorphism T on $(T^*\mathbb{R}^n, 0)$ such that $L_\sigma = T(L_\sigma'^0)$ for all $\sigma \subset I_r$.

Equivalence relations: Let $\{L_\sigma^1\}_{\sigma\subset I_r}$ and $\{L_\sigma^2\}_{\sigma\subset I_r}$ be two symplectic regular r-cubic configurations. We say that $\{L_\sigma^1\}_{\sigma\subset I_r}$ and $\{L_\sigma^2\}_{\sigma\subset I_r}$ are *Lagrangian equivalent* if there exists a Lagrangian equivalence Θ on π such that $L_\sigma^2 = \Theta(L_\sigma^1)$ for all $\sigma \subset I_r$.

The equivalence relation among symplectic regular r-cubic configuration without boundary is defined analogously.

Generating families: Let $\mathbb{H}^r = \{(x_1, \ldots, x_r) \in \mathbb{R}^r \mid x_1 \geq 0, \ldots, x_r \geq 0\}$ be an r-corner. Let $\mathcal{E}(r;l)$ be the set of smooth function germs on $(\mathbb{H}^r \times \mathbb{R}^l, 0)$ and $\mathfrak{M}(r;l) = \{f \in \mathcal{E}(r;l) \mid f(0) = 0\}$ be its maximal ideal. We denote simply $\mathcal{E}(l)$ for $\mathcal{E}(0;l)$ and $\mathfrak{M}(l)$ for $\mathfrak{M}(0;l)$ and denote $\mathcal{B}(r;l)$ the set of diffeomorphism germs on $(\mathbb{H}^r \times \mathbb{R}^l, 0)$ which preserve $(\mathbb{H}^r \cap \{x_\sigma = 0\}) \times \mathbb{R}^l$

for all $\sigma \subset I_r$. We denote $\mathcal{B}(r;l)'$ the set of diffeomorphism germs on $(\mathbb{R}^{r+l}, 0)$ which preserve $(\mathbb{R}^r \cap \{x_\sigma = 0\}) \times \mathbb{R}^l$ for all $\sigma \subset I_r$. We remark that a diffeomorphism germ ϕ on $(\mathbb{H}^r \times \mathbb{R}^l, 0)$ is an element of $\mathcal{B}(r;l)$ if and only if ϕ is written in the form:

$$\phi(x,y) = (x_1 a_1(x,y), \dots, x_r a_r(x,y), b_1(x,y), \dots, b_l(x,y)) \quad \text{for } (x,y) \in (\mathbb{H}^r \times \mathbb{R}^l, 0),$$

where $a_1, \dots, a_r, b_1, \dots, b_r \in \mathcal{E}(r;l)$ and $a_1(0) > 0, \dots, a_r(0) > 0$ and a diffeomorphism germ ϕ' on $(\mathbb{R}^{r+l}, 0)$ is an element of $\mathcal{B}(r;l)'$ if and only if ϕ' is written in the form:

$$\phi'(x,y) = (x_1 a_1(x,y), \dots, x_r a_r(x,y), b_1(x,y), \dots, b_l(x,y)) \quad \text{for } (x,y) \in (\mathbb{R}^{r+l}, 0),$$

where $a_1, \dots, a_r, b_1, \dots, b_r \in \mathcal{E}(r+l)$.

We say that function germs $F(x,y,u), G(x,y,u) \in \mathfrak{M}(r;k+n)$, where $x \in \mathbb{H}^r$, $y \in \mathbb{R}^k$ and $u \in \mathbb{R}^n$, are *reticular R^+-equivalent* (as n-dimensional unfoldings) if there exist $\Phi \in \mathcal{B}(r;k+n)$ and $\alpha \in \mathfrak{M}(n)$ satisfying the following:

(1) $\Phi = (\phi, \psi)$, where $\phi : (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0) \to (\mathbb{H}^r \times \mathbb{R}^k, 0)$ and $\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$.

(2) $G(x,y,u) = F(\phi(x,y,u), \psi(u)) + \alpha(u)$ for $(x,y,u) \in (\mathbb{H}^r \times \mathbb{R}^{k+n}, 0)$.

If $\alpha = 0$ we say that F and G are reticular R-equivalent.

We say that function germs $F(x,y,u), G(x,y,u) \in \mathfrak{M}(r+k+n)$, where $x \in \mathbb{R}^r$, $y \in \mathbb{R}^k$ and $u \in \mathbb{R}^n$, are *reticular R^+-equivalent* (as n-dimensional unfoldings) if there exist $\Phi \in \mathcal{B}(r;k+n)'$ and $\alpha \in \mathfrak{M}(n)$ satisfying the following:

(1) $\Phi = (\phi, \psi)$, where $\phi : (\mathbb{R}^{r+k+n}, 0) \to (\mathbb{R}^{r+k}, 0)$ and $\psi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$.

(2) $G(x,y,u) = F(\phi(x,y,u), \psi(u)) + \alpha(u)$ for $(x,y,u) \in (\mathbb{R}^{r+k+n}, 0)$.

If $\alpha = 0$ we say that F and G are reticular R-equivalent.

We say that function germs $F(x, y_1, \dots, y_{k_1}, u) \in \mathfrak{M}(r; k_1 + n)$ and $F(x, y_1, \dots, y_{k_2}, u) \in \mathfrak{M}(r; k_2 + n)$ are *stably reticular R^+-equivalent* if F and G are reticular R^+-equivalent after additions of nondegenerate quadratic forms in the variables y.

We define the stably reticular R^+-equivalence for the functions in $\mathfrak{M}(r+l+n)$ analogously.

A function germ $F(x,y,u) \in \mathfrak{M}^2(r;k+n)$ is called *nondegenerate* if

$$x_1,\ldots,x_r,\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_r},\frac{\partial F}{\partial y_1},\ldots,\frac{\partial F}{\partial y_k}$$

are independent on $(\mathbb{H}^k \times \mathbb{R}^{k+n},0)$, that is

$$\operatorname{rank}\begin{pmatrix} \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial x \partial u} \\ \dfrac{\partial^2 F}{\partial y \partial y} & \dfrac{\partial^2 F}{\partial y \partial u} \end{pmatrix}_0 = r+k.$$

We remark that $F(x,y,u) \in \mathfrak{M}^2(r;k+n)$ is nondegenerate only if $r \leq n$.

We define nondegenerateness for the functions in $\mathfrak{M}(r+k+n)$ analogously.

Let Λ be a family consisting of certain subsets of I_r and $\{L_\sigma\}_{\sigma\in\Lambda}$ be a family of Lagrangian submanifolds of $(T^*\mathbb{R}^n,0)$. Then we say that $F \in \mathfrak{M}^2(r+k+n)$ is a *generating family* of $\{L_\sigma\}_{\sigma\in\Lambda}$ if F is nondegenerate and $F|_{x_\sigma=0}$ is a generating family of L_σ for all $\sigma \in \Lambda$.

Theorem 2.1 ([8], P.577, Theorem 3.2) (1) *For any symplectic regular r-cubic configuration $\{L_\sigma\}_{\sigma\subset I_r}$, there exists a function germ $F \in \mathfrak{M}(r;k+n)^2$ which is a generating family of $\{L_\sigma\}_{\sigma\subset I_r}$.*

(2) *For any nondegenerate function germ $F \in \mathfrak{M}(r;k+n)^2$, there exists a symplectic regular r-cubic configuration of which F is a generating family.*

(3) *Two symplectic regular r-cubic configurations are Lagrangian equivalent if and only if their generating families are stably reticular R^+-equivalent.*

By the almost same proof of this theorem, we have the following lemma.

Lemma 2.2 (1) *For any symplectic regular r-cubic configuration without boundary $\{L'_\sigma\}_{\sigma\subset I_r}$, there exists a function germ $F' \in \mathfrak{M}(r+k+n)^2$ which is a generating family of $\{L'_\sigma\}_{\sigma\subset I_r}$.*

(2) *For any nondegenerate function germ $F' \in \mathfrak{M}(r+k+n)^2$, there exists a symplectic regular r-cubic configuration without boundary of which F' is a generating family.*

(3) *Two symplectic regular r-cubic configurations without boundaries are Lagrangian equivalent if and only if their generating families are stably reticular R^+-equivalent.*

3. Symplectic regular r-cubic configurations without boundaries

In this section we prove Theorem 3.3 which will be used in the next section.

Lemma 3.1 *Let $\{L_\sigma\}_{\sigma\subset I_r}$ be a symplectic regular r-cubic configurations without boundaries. If $G(x,y,q)\in\mathfrak{M}^2(r+k+n)$ is a generating family of $L_\emptyset$ and $G(0,y,q)$ is a generating family of L_{I_r} then G is nondegenerate.*

Proof. We denote $\frac{\partial^2 G}{\partial x\partial y}(0)$ by G_{xy} and denote other notations analogously. For every vector v in $T_0L_\emptyset\cap T_0L_{I_r}$ there exists $(b,c)\in\mathbb{R}^{k+n}$ such that

$$\begin{pmatrix} G_{xx} & G_{xy} & G_{xq} \\ G_{yx} & G_{yy} & G_{yq} \end{pmatrix}\begin{pmatrix} 0 \\ b \\ c \end{pmatrix} = 0$$

and

$$v = c\frac{\partial}{\partial q} + (G_{qx}0 + G_{qy}b + G_{qq}c)\frac{\partial}{\partial p}.$$

Since $\dim T_0L_{I_r}\cap T_0L_\emptyset = n-r$ we have that

$$\operatorname{rank}\begin{pmatrix} G_{xy} & G_{xq} \\ G_{yy} & G_{yq} \end{pmatrix} = r+k.$$

Hence G is nondegenerate. □

Lemma 3.2 *Let L^1, L^2 be Lagrangian submanifolds of $(T^*\mathbb{R}^n,0)$ without boundaries. Suppose the following conditions hold:*
(1) $L^1\cap L^2$ is a submanifolds of $(T^*\mathbb{R}^n,0)$ with the codimension $n+r$.
(2) $T(L^1\cap L^2) = TL^1\cap TL^2$.
Then there exists an nondegenerate function germ $F(x,y,q)\in\mathfrak{M}^2(r+k+n)$ for some k such that $F(x,y,q)$ is a generating family of L^1 and $F(0,y,q)$ is a generating family of L^2.

Proof. By considering some Lagrangian equivalence of L^1 and L^2, we may assume that there exist function germs $S_1, S_2\in\mathfrak{M}^2(n)$ such that

$$L^i = \left\{\left(-\frac{\partial S_i}{\partial p}(p), p\right)\right\} \quad (i=1,2).$$

Define $\phi\in\mathfrak{M}^2(n)$ by $\phi(p) = S_2(p) - S_1(p)$. By the splitting lemma there exist coordinates $(z_1, \dots, z_n)$ of $(\mathbb{R}^n, 0)$ in which ϕ is written in the form:

$$\pm z_1^2 \pm \cdots \pm z_{r'}^2 + \phi_0(z_{r'+1}, \dots, z_n) \quad (\phi_0 \in \mathfrak{M}^3(n - r')).$$

Then it follows from (2) that $r' = r$ and from (1) that $\phi_0 \equiv 0$. Define $F \in \mathfrak{M}^2(r + n + n)$ by

$$F(x, y, q) = \pm(x_1 + z_1(y))^2 \pm \cdots \pm (x_r + z_r(y))^2 + S_1(y) + \sum_{i=1}^{n} y_i q_i,$$

then F satisfies the all conditions we need. □

Theorem 3.3 *A family $\{L_\sigma\}_{\sigma \subset I_r}$ of 2^r Lagrangian submanifolds without boundaries is a symplectic regular r-cubic configuration without boundary if and only if the following conditions hold:*

(1) *$L_\sigma \cap L_\tau$ is a submanifold of $(T^*\mathbb{R}^n, 0)$ with the codimension $n + |\sigma \cup \tau| - |\sigma \cap \tau|$,*

(2) *$T_x(L_\sigma \cap L_\tau) = T_x L_\sigma \cap T_x L_\tau$ for any $x \in L_\sigma \cap L_\tau$.*

Proof. It is enough to prove 'if'. We use induction on r. For the case $r = 0$ is reduced to [1, p.300 Theorem] and the case $r = 1$ is reduced to the case $r = 1$ of Lemma 3.2 and Lemma 2.2(2). Therefore for $r \geq 2$ we suppose that there exist generating families $f_i(x_1, \dots, \check{x}_i, \dots, x_r, y, q) \in \mathfrak{M}^2(r-1+k+n)$ of the regular $(r-1)$-cubic configuration without boundary $\{L_\sigma\}_{i \in \sigma \subset I_r}$ for each $i = 1, \dots, r$ and prove the existence of a generating family of $\{L_\sigma\}_{\sigma \subset I_r}$. Indeed by the fact that $f_i(0, y, q)$ $(i = 1, \dots, r)$ are generating families of L_{I_r} and the splitting lemma, we may assume that the number of arguments y are all equal.

A. *For each $i = 1, \dots, r$ there exists a generating family $F_i(x_1, \dots, x_r, y, q) \in \mathfrak{M}^2(r + k + n)$ of $\{L_\sigma\}_{i \in \sigma \subset I_r, \sigma = \emptyset}$.*

It is enough to prove this for $i = 1$. Since the case $r = 1$, there exists a generating family $f_1'(x_1, z_1, \dots, z_{k+r-1}, q) \in \mathfrak{M}^2(1 + (k + r - 1) + n)$ of L_1 and $L_\emptyset$ (that is f_1' is non-degenerate as a generating family of a symplectic regular 1-cubic configuration without boundary and f_1' is a generating family of $L_\emptyset$ and $f_1'|_{x_1=0}$ is a generating family of L_1). Since $f_1(x_2, \dots, x_r, y, q)$ and $f_1'(0, z, q)$ are generating families of L_1, there exists a right equivalence from f_1 to $f_1'|_{x_1=0}$ of the form:

$$f_1(x_2, \dots, x_r, y, q) = f_1'(0, \phi(x_2, \dots, x_r, y), q).$$

Define $F_1 \in \mathfrak{M}^2(r+k+n)$ by $F_1(x,y,q) = f_1'(x_1, \phi(x_2,\dots,x_r,y),q)$. Then F_1 is a generating family of L_σ for $i \in \sigma \subset I_r$, $\sigma = \emptyset$ and hence F_1 is non-degenerate by Lemma 3.1.

B. *For each $i,j = 1,\dots,r$ $(i \neq j)$, there exits a generating family $F_{i,j} \in \mathfrak{M}^2(r+k+n)$ of $\{L_\sigma\}_{i\in\sigma\subset I_r,\ j\in\sigma\subset I_r,\sigma=\emptyset}$.*

It is enough to prove this for $i=1$, $j=2$. Since $F_1|_{x_1=x_2=0}$ and $F_2|_{x_1=x_2=0}$ are generating families of $\{L_\sigma\}_{1,2\in\sigma\subset I_r}$, we may assume that $F_1|_{x_1=x_2=0} = F_2|_{x_1=x_2=0}$ by Lemma 2.2(3). By using analogous methods of the proof (D).(a)~(d) of [1, p.304 Theorem] we may assume that $F_1, F_2 \in \mathfrak{M}^2(r+k+n)$ and

$$\frac{\partial^2 F_i}{\partial y^2}(0) = 0,\ \frac{\partial^2 F_i}{\partial y \partial q_J}(0) = E_k\ (J \subset I_r, |J| = k,\ i = 1,2).$$

On the other hand, since F_1 and F_2 are generating families of $L_\emptyset$, there exists $\phi : (\mathbb{R}^{r+k+n},0) \to (\mathbb{R}^{r+k},0)$ such that $F_1(x,y,q) = F_2(\phi(x,y,q),q)$. By using analogous methods of the proof of Theorem 2.1(3) in [8], there exists a right equivalence of F_1 and F_2 of the following form:

$$\begin{aligned} F_1(x,y,q) &= F_2(a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2, \dots, \\ &\qquad c_s x_s + a_{s1}x_1 + a_{s2}x_2, \dots, h, q), \end{aligned}$$

where $\frac{\partial h}{\partial y}(0) = E_k$, $\frac{\partial h}{\partial q}(0) = 0$.

Let $\{L_\sigma^i\}_{\sigma\subset I_r}$ be the symplectic regular r-cubic configuration without boundary defined by F_i for each $i = 1,2$. Then the following hold:

1 $a_{22}(0) = 0 \Longrightarrow T_0L_\emptyset \cap T_0L_1 = T_0L_\emptyset \cap T_0L_2$,

2 $a_{11}(0) = 0 \Longrightarrow T_0L_\emptyset \cap T_0L_2^1 = T_0L_\emptyset \cap T_0L_1^2$.

It is enough to prove the assertion 1. Set

$$A = \begin{pmatrix} a_{11} & a_{12} & & & 0 & \\ a_{21} & a_{22}(=0) & & & & \\ a_{31} & a_{32} & & c_3 & & 0 \\ \vdots & \vdots & & & \ddots & \\ a_{r1} & a_{r2} & & 0 & & c_n \end{pmatrix}_0,$$

$$B = \begin{pmatrix} A & 0 & 0 \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial q} \\ 0 & 0 & E_n \end{pmatrix}_0.$$

Then

$$\begin{pmatrix} F^1_{xx} & F^1_{xy} & F^1_{xq} \\ F^1_{yx} & F^1_{yy} & F^1_{yq} \end{pmatrix} = \begin{pmatrix} A^t & 0 \\ 0 & E_k \end{pmatrix} \begin{pmatrix} F^2_{xx} & F^2_{xy} & F^2_{xq} \\ F^2_{yx} & F^2_{yy} & F^2_{yq} \end{pmatrix} B$$

$$(F^1_{qx}, F^1_{qy}, F^1_{qq}) = (F^2_{qx}, F^2_{qy}, F^2_{qq})B,$$

For every vector v in $T_0L_\emptyset \cap T_0L_1$ there exists $(a,b,c) \in \mathbb{R}^{r+k+n}$ such that

$$\begin{pmatrix} F^1_{xx} & F^1_{xy} & F^1_{xq} \\ F^1_{yx} & F^1_{yy} & F^1_{yq} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0, \ a_1 = 0$$

and

$$v = c\frac{\partial}{\partial q} + \left(F^1_{qx}a + F^1_{qy}b + F^1_{qq}c\right)\frac{\partial}{\partial p}.$$

On the other hand, Set

$$\begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = B \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} F^2_{xx} & F^2_{xy} & F^2_{xq} \\ F^2_{yx} & F^2_{yy} & F^2_{yq} \end{pmatrix} \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = 0, \ a'_2 = 0$$

and

$$v = c'\frac{\partial}{\partial q} + \left(F^2_{qx}a' + F^2_{qy}b' + F^2_{qq}c'\right)\frac{\partial}{\partial p}.$$

But this implies $v \in T_0L_\emptyset \cap T_0L_2$. Therefore the assertion 1 is proved.

Since $\{L_\sigma\}_{\sigma \subset I_r}$ satisfies the condition (2), $T_0L_\emptyset \cap T_0L_1 \neq T_0L_\emptyset \cap T_0L_2$, so that $a_{22}(0) \neq 0$. Define $F_3 \in \mathfrak{M}^2(r+k+n)$ by $F_3(x,y,q) = F_2(x_1 + ax_2, x_2, \dots, x_r, y, q)$ $(a \neq 0)$. Then we have $T_0L_\emptyset \cap T_0L_1^2 \neq T_0L_\emptyset \cap T_0L_1^3$ by an analogous method of the proof of 1, where L_1^3 is the Lagrangian submanifold defined by $F_3|_{x_1=0}$. Therefore by replacing F_2 by F_3 if necessary, we have $T_0L_\emptyset \cap T_0L_2^1 \neq T_0L_\emptyset \cap T_0L_1^2$, so that $a_{11}(0) \neq 0$. Define $F_{1,2} \in \mathfrak{M}^2(r+k+n)$

by

$$F_{1,2}(x,y,q) = F_2(a_{11}x_1 + a_{21}x_2, a_{22}x_2, \dots, c_s x_s + a_{s2}x_2, \dots, h, q).$$

Since $F_{1,2}|_{x_1=0} \equiv F_1|_{x_1=0}$ generates $\{L_\sigma\}_{1\in\sigma\subset I_r}$ and $F_{1,2}|_{x_2=0} \sim F_2|_{x_2=0}$ generates $\{L_\sigma\}_{2\in\sigma\subset I_r}$ and $F_{1,2}$ is a generating family of $L_\emptyset$ and L_{I_r}, we have that $F_{1,2}$ is nondegenerate (where $\sim$ means reticular R-equivalent).

C. *For each $\rho \subset I_r$ there exists a generating family F_ρ of $\{L_\sigma\}_{\rho\cap\sigma\neq\emptyset,\sigma=\emptyset}$.* Induction on $|\rho|$. Since F_{I_r} is a generating family of $\{L_\sigma\}_{\sigma\subset I_r}$, this assertion completes the proof by Lemma 2.2(2). For $|\rho| = 2$, the assertion is reduced to B. Therefore it is enough to prove the assertion for $\rho = \{1,\dots,s\}$ $(s \geq 3)$. Set $F_1 = F_{\{1,3,\dots,s\}}$ and $F_2 = F_{\{2,\dots,s\}}$. We may assume that $F_1, F_2 \in \mathfrak{M}^2(r+k+n)$ and

$$F_1(0,y,q) = F_2(0,y,q),\ \frac{\partial^2 F_i}{\partial y^2}(0) = 0,\ \frac{\partial^2 F_i}{\partial y \partial q_J}(0) = E_k\ (i = 1,2).$$

By using analogous methods of B, there exists a right equivalence of F_1 and F_2 of the following form:

$$\begin{aligned} F_1(x,y,q) &= F_2(a_1x_1 + (b_1^1x_1 + b_1^2x_2)x_3\cdots x_s, \dots, \\ &\qquad a_rx_r + (b_r^1x_1 + b_r^2x_2)x_3\cdots x_s, h, q), \end{aligned}$$

where $a_1(0) \neq 0, \dots, a_r(0) \neq 0$. Define $F_\rho \in \mathfrak{M}^2(r+k+n)$ by

$$F_\rho(x,y,q) = F_2(a_1x_1 + b_1^2x_2x_3\cdots x_s, \dots, a_rx_r + b_r^2x_2x_3\cdots x_s, h, q).$$

Then F_ρ is a generating family of $L_\emptyset$ and L_{I_r}. Hence F_ρ is non-degenerate. Other hand since $F_\rho|_{x_1=0} = F_1|_{x_1=0}$, $F_\rho|_{x_2=0} \sim F_2|_{x_2=0}$, $F_\rho|_{x_i=0} = F_1|_{x_i=0}$ $(i = 3,\dots,s)$, F_ρ is a generating family of $\{L_\sigma\}_{\rho\cap\sigma\neq\emptyset,\sigma=\emptyset}$. □

4. Symplectic regular r-cubic configurations

In this section we state the first main theorem in this paper.

Let M be a manifold and N be a submanifold germ of M around $p \in M$. Then we say that N has an l-corner if there exist coordinates $(x_1,\dots,x_l,y_1,\dots,y_i,z_1,\dots,z_j)$ of M around p such that N is defined by $x_1 \geq 0,\dots,x_l \geq 0$, $y_1 = 0,\dots,y_i = 0$.

Theorem 4.1 *Let $\{L_\sigma\}_{\sigma\subset I_r}$ be 2^r Lagrangian submanifolds such that L_σ has an $(r-|\sigma|)$-corner for $\sigma \subset I_r$. Then $\{L_\sigma\}_{\sigma\subset I_r}$ is a symplectic regular*

r-cubic configuration if and only if the following conditions hold:

(1) *$L_\sigma \cap L_\tau$ is a submanifold of $(T^*\mathbb{R}^n, 0)$ with the codimension $n + |\sigma \cup \tau| - |\sigma \cap \tau|$,*
(2) *$T_x(L_\sigma \cap L_\tau) = T_x L_\sigma \cap T_x L_\tau$ for any $x \in L_\sigma \cap L_\tau$.*
(3) *$\partial L_\sigma \cap L_\tau = L_\sigma \cap L_\tau$ for $\sigma \subset \tau \subset I_r$ ($|\tau - \sigma| = 1$), where ∂L_σ is the boundary in L_σ.*

Proof. It is enough to prove 'if'. Let $F'(x,y,q) \in \mathfrak{M}^2(r+k+n)$ be a generating family of a regular r-cubic configuration without boundary $\{\tilde{L}_\sigma\}_{\sigma \subset I_r}$ which is an extension of $\{L_\sigma\}_{\sigma \subset I_r}$. Since

$$\tilde{L}_{I_r - 1} \cap L_{I_r} = \left\{ \left(q, \frac{\partial F'}{\partial q}(x,y,q)\right) \mid x_1 = \cdots = x_r = \frac{\partial F'}{\partial x_1} = \frac{\partial F'}{\partial y} = 0 \right\},$$

we have

$$\partial L_{I_r - 1} = \left\{ \left(q, \frac{\partial F'}{\partial q}(x,y,q)\right) \mid x_1 = \cdots = x_r = \frac{\partial F'}{\partial x_1} = \frac{\partial F'}{\partial y} = 0 \right\}.$$

Therefore we have

$$L_{I_r - 1} = \left\{ \left(q, \frac{\partial F'}{\partial q}(x,y,q)\right) \mid x_2 = \cdots = x_r = \frac{\partial F'}{\partial x_1} = \frac{\partial F'}{\partial y} = 0, e_1 x_1 \geq 0 \right\} \quad (e_1 = 1 \text{ or } -1).$$

Take similarly $e_2, \ldots, e_r$ satisfying

$$L_{I_r - i} = \left\{ \left(q, \frac{\partial F'}{\partial q}(x,y,q)\right) \mid x_1 = \cdots = \check{x}_i = \cdots = x_r = \frac{\partial F'}{\partial x_i} = \frac{\partial F'}{\partial y} = 0, e_i x_i \geq 0 \right\}.$$

We now prove that $F(x,y,q) := F'(e_1 x_1, \ldots, e_r x_r, y, q) \in \mathfrak{M}^2(r; k+n)$ is generating family of $\{L_\sigma\}_{\sigma \subset I_r}$. We use induction on $r - |\sigma|$. The case $r - |\sigma| = 1$ is proved as above. We suppose that the following holds in the case $r - |\sigma| < i$:

$$L_\sigma = \left\{ \left(q, \frac{\partial F}{\partial q}(x,y,q)\right) \mid x_\sigma = \frac{\partial F}{\partial x_{I_r - \sigma}} = \frac{\partial F}{\partial y} = 0 \ (x_{I_r - \sigma} \geq 0) \right\}$$

and prove the case $r - |\sigma| = i$. Set $j = r - i$ and let $r - |\sigma| = i$. We may suppose that $\sigma = \{1, \ldots, j\}$. Take $s \in \mathbf{N}$ such that $j < s \leq r$ and let

$\sigma_s = \sigma \cup \{s\}$. Then we have

$$\tilde{L}_\sigma \cap L_{\sigma_s} = \Big\{\Big(q, \frac{\partial F}{\partial q}(x,y,q)\Big) \mid x_\sigma = x_s = \frac{\partial F}{\partial x_{I_r-\sigma}} = \frac{\partial F}{\partial y} = 0 \quad (x_{I_r-\sigma_s} \geq 0)\Big\}.$$

Since $\partial L_\sigma \cap L_{\sigma_s} \subset \tilde{L}_\sigma \cap L_{\sigma_s}$ and ∂L_σ is consist of $(r-j)$'s $(r-j-1)$-corner, we have

$$\partial L_\sigma = \bigcup_{s=j+1}^{n} \Big\{\Big(q, \frac{\partial F}{\partial q}(x,y,q)\Big) \mid x_\sigma = x_s = \frac{\partial F}{\partial x_{I_r-\sigma}} = \frac{\partial F}{\partial y} = 0 \quad (x_{I_r-\sigma_s} \geq 0)\Big\}.$$

Therefore

$$L_\sigma = \Big\{\Big(q, \frac{\partial F}{\partial q}(x,y,q)\Big) \mid x_\sigma = \frac{\partial F}{\partial x_{I_r-\sigma}} = \frac{\partial F}{\partial y} = 0\Big\}.$$

Hence the case $r - |\sigma| = i$ is proved. As a result, F is a generating family of $\{L_\sigma\}_{\sigma\subset I_r}$. Therefore $\{L_\sigma\}_{\sigma\subset I_r}$ is a symplectic regular r-cubic configuration by Theorem 2.1(2). □

5. Contact regular r-cubic configurations

In this section we shall prove the second main theorem in this paper. In order to realize this, we require some lemma's which have been developed in [9].

Let $(q_1, \dots, q_n, z, p_1, \dots, p_n)$ be canonical coordinates of $J^1(\mathbb{R}^n, \mathbb{R})$ equipped with the contact structure defined by the canonical 1-form $\alpha = dz - pdq$. Let $\tilde{\pi} : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^{n+1}((q, z; p) \mapsto (q, z))$ be the canonical Legendrian bundle. Set $\tilde{L}^0_\sigma = \{(q, z, p) \in (J^1(\mathbb{R}^n, \mathbb{R}), 0) \mid q_\sigma = p_{I_r-\sigma} = q_{r+1} = \cdots = q_n = z = 0,\ q_{I_r-\sigma} \geq 0\}$ for each $\sigma \subset I_r$.

Definition 5.1 Let $\{\tilde{L}_\sigma\}_{\sigma\subset I_r}$ be a family of 2^r Legendrian submanifold germs of $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$. Then $\{\tilde{L}_\sigma\}_{\sigma\subset I_r}$ is called *a contact regular r-cubic configuration* if there exists a contact diffeomorphism C on $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ such that $\tilde{L}_\sigma = C(\tilde{L}^0_\sigma)$ for all $\sigma \subset I_r$.

Let $\{\tilde{L}^1_\sigma\}_{\sigma\subset I_r}$ and $\{\tilde{L}^2_\sigma\}_{\sigma\subset I_r}$ be two contact regular r-cubic configurations. We say that $\{\tilde{L}^1_\sigma\}_{\sigma\subset I_r}$ and $\{\tilde{L}^2_\sigma\}_{\sigma\subset I_r}$ are *Legendrian equivalent* if there exist Legendrian equivalence Θ on $\tilde{\pi}$ such that $\tilde{L^2_\sigma} = \Theta(\tilde{L^1_\sigma})$ for all $\sigma \subset I_r$.

Lemma 5.2 ([1], P.313, Proposition) *Let* $\tilde{p} : (J^1(\mathbb{R}^n, \mathbb{R}), 0) \to (T^*\mathbb{R}^n, 0)$ $((q, z, p) \mapsto (q, p))$ *be a projection. Let* $\mathcal{C}^n$ *be the set of Legendrian submanifolds of* $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ *and* $\mathcal{S}^n$ *be the set of Lagrangian submanifolds of* $(T^*\mathbb{R}^n, 0)$. *Then* $\tilde{p}$ *gives a bijection from* $\mathcal{C}^n$ *to* $\mathcal{S}^n$ *and* $\tilde{p}|_{\tilde{L}} : \tilde{L} \to \tilde{p}(\tilde{L})$ *is a diffeomorphism for each* $\tilde{L} \in \mathcal{C}^n$.

Lemma 5.3 *Let* $\tilde{L}_1$ *and* $\tilde{L}_2$ *be Legendrian submanifolds of* $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ *and* $L_1 = \tilde{p}(\tilde{L}_1)$, $L_2 = \tilde{p}(\tilde{L}_2)$ *be corresponding Lagrangian submanifolds of* $(T^*\mathbb{R}^n, 0)$. *Then* $\tilde{p}(\tilde{L}_1 \cap \tilde{L}_2) = L_1 \cap L_2$

Proof. It is enough to prove that $L_1 \cap L_2 \subset \tilde{p}(\tilde{L}_1 \cap \tilde{L}_2)$. By considering some Legendrian equivalence of $\tilde{L}_1$ and $\tilde{L}_2$, we may assume that there exist function germs $S_1, S_2 \in \mathfrak{M}^2(n)$ such that

$$\tilde{L}^i = \left\{\left(-\frac{\partial S_i}{\partial p}(p), S_i(p) + \left\langle \frac{\partial S_i}{\partial p}(p), p\right\rangle, p\right)\right\} \quad (i = 1, 2).$$

Let $(q, p) \in L_1 \cap L_2$. Let $c : [0, 1] \to L_1 \cap L_2$ be smooth pass connects 0 and (q, p). Then we have

$$\begin{aligned} S_1(p) &= \int_0^1 \frac{d}{dt} S_1(c(t))dt = \int_0^1 \frac{\partial S_1}{\partial p}(c(t))\frac{dc}{dt}(t)dt \\ &= \int_0^1 \frac{\partial S_2}{\partial p}(c(t))\frac{dc}{dt}(t)dt = \int_0^1 \frac{d}{dt} S_2(c(t))dt \\ &= S_2(p). \end{aligned}$$

Therefore $(q, p) \in \tilde{p}(\tilde{L}_1 \cap \tilde{L}_2)$ □

In order to consider contact diffeomorphism germs on $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$, we define the following notations:
$\imath : (J^1(\mathbb{R}^n, \mathbb{R}) \cap \{Z = 0\}, 0) \to (J^1(\mathbb{R}^n, \mathbb{R}), 0)$ be the inclusion map,

$$\begin{aligned} C(J^1(\mathbb{R}^n, \mathbb{R}), 0) &= \{C : (J^1(\mathbb{R}^n, \mathbb{R}), 0) \to (J^1(\mathbb{R}^n, \mathbb{R}), 0) \mid \\ &\qquad C : \text{ contact diffeomorphism}\}, \\ C^\alpha(J^1(\mathbb{R}^n, \mathbb{R}), 0) &= \{C \in C(J^1(\mathbb{R}^n, \mathbb{R}), 0) \mid \\ &\qquad C \text{ preserves the canonical 1-form}\}, \\ C_Z(J^1(\mathbb{R}^n, \mathbb{R}), 0) &= \{C \circ \imath \mid C \in C(J^1(\mathbb{R}^n, \mathbb{R}), 0)\}, \end{aligned}$$

$$C_Z^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0) = \{C \circ \imath \mid C \in C^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0)\}.$$

Lemma 5.4 ([9], P.9, Lemma 5.3) *Let $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$ be a contact regular r-cubic configuration in $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ defined by $C \in C(J^1(\mathbb{R}^n,\mathbb{R}),0)$. Then there exists $C' \in C^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0)$ which also defines $\{\tilde{L_\sigma}\}_{\sigma\subset I_r}$.*

By this lemma, we may assume that all contact regular r-cubic configuration in $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ is defined by an element in $C^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0)$ or $C_Z^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0)$.

Lemma 5.5 ([9], P.10, Lemma 5.4) *Let $S(T^*\mathbb{R}^n,0)$ be the set of symplectic diffeomorphism germs on $(T^*\mathbb{R}^n,0)$. We define the following maps:*

$$\begin{array}{ccc} C_Z^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0) & \to & S(T^*\mathbb{R}^n,0) \\ C=(q_C,z_C,p_C) & \mapsto & (S^C:(Q,P)\mapsto (q_C,p_C)(Q,P)) \end{array}$$

$$\begin{array}{ccc} S(T^*\mathbb{R}^n,0) & \to & C_Z^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0) \\ S=(q_S,p_S) & \mapsto & (C^S:(Q,P)\mapsto (q_S,f^S,p_S)(Q,P)), \end{array}$$

where $f^S(Q,P)$ is uniquely defined by the relation that $S^*(pdq) - PdQ = df^S, f^S(0,0) = 0$. Then these maps are well defined and inverse to each other.

Proposition 5.6 ([9], P.10, Proposition 5.5(1)) *Let $\mathcal{C}_r^n$ be the set of contact regular r-cubic configurations in $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ and $\mathcal{S}_r^n$ be the set of symplectic regular r-cubic configurations in $(T^*\mathbb{R}^n,0)$. We define*

$$T_S : \mathcal{C}_r^n \to \mathcal{S}_r^n \ (\{C(\tilde{L_\sigma^0})\}_{\sigma\subset I_r} \mapsto \{S^C(L_\sigma^0)\}_{\sigma\subset I_r}), \quad \text{where } C \in C_Z^\alpha(J^1(\mathbb{R}^n,\mathbb{R}),0),$$

$$T_C : \mathcal{S}_r^n \to \mathcal{C}_r^n \ (\{S(L_\sigma^0)\}_{\sigma\subset I_r} \mapsto \{C^S(\tilde{L}_\sigma^0)\}_{\sigma\subset I_r}), \quad \text{where } S \in S(T^*\mathbb{R}^n,0).$$

Then T_S and T_C are well defined and inverse to each other.

Theorem 5.7 *Let $\{\tilde{L}_\sigma\}_{\sigma\subset I_r}$ be 2^r Legendrian submanifolds of $(J^1(\mathbb{R}^n,\mathbb{R}),0)$ such that $\tilde{L}_\sigma$ has an $(r-|\sigma|)$-corner for $\sigma \subset I_r$. Then $\{\tilde{L}_\sigma\}_{\sigma\subset I_r}$ is a contact regular r-cubic configuration if and only if the following conditions hold:*

(1) $\tilde{L}_\sigma \cap \tilde{L}_\tau$ *is a submanifold of* $(J^1(\mathbb{R}^n, \mathbb{R}), 0)$ *with the codimension* $n + 1 + |\sigma \cup \tau| - |\sigma \cap \tau|$ *for all* $\sigma, \tau \subset I_r$,
(2) $T_x(\tilde{L}_\sigma \cap \tilde{L}_\tau) = T_x\tilde{L}_\sigma \cap T_x\tilde{L}_\tau$ *for* $x \in \tilde{L}_\sigma \cap \tilde{L}_\tau$ $\sigma, \tau \subset I_r$,
(3) $\partial\tilde{L}_\sigma \cap \tilde{L}_\tau = \tilde{L}_\sigma \cap \tilde{L}_\tau$ *for* $\sigma \subset \tau \subset I_r$ $(|\tau - \sigma| = 1)$, *where* $\partial\tilde{L}_\sigma$ *is the boundary in* $\tilde{L}_\sigma$.

Proof. It is enough to prove 'if'. Set $L_\sigma = \tilde{p}(\tilde{L}_\sigma)$ for all $\sigma \subset I_r$. We prove $\{L_\sigma\}_{\sigma \subset I_r}$ is a symplectic regular r-cubic configuration. Then $\{\tilde{L}_\sigma\}_{\sigma \subset I_r} = T_C(\{L_\sigma\}_{\sigma \subset I_r})$ is a contact regular r-cubic configuration by Proposition 5.6.

(1) Let $\sigma, \tau \subset I_r$. Since $\tilde{p}|_{\tilde{L}_\sigma}$ is a diffeomorphism by Lemma 5.2, we have that $\tilde{p}(\tilde{L}_\sigma \cap \tilde{L}_\tau) = \tilde{p}|_{\tilde{L}_\sigma}(\tilde{L}_\sigma \cap \tilde{L}_\tau) \cong L_\sigma \cap L_\tau$, where $\cong$ means 'diffeomorphic'. By the hypothesis (1), $L_\sigma \cap L_\tau$ has codimension $n + |\sigma \cup \tau| - |\sigma \cap \tau|$.

(2) Let $\sigma, \tau \subset I_r$ and $x \in \tilde{L}_\sigma \cap \tilde{L}_\tau$. Set $y = \tilde{p}(x)$. Then we have

$$\begin{aligned} T_y(L_\sigma \cap L_\tau) &= \tilde{p}_*(T_x(\tilde{L}_\sigma \cap \tilde{L}_\tau)) = \tilde{p}_*(T_x\tilde{L}_\sigma \cap T_x\tilde{L}_\tau) \\ &= \tilde{p}_*(T_x\tilde{L}_\sigma) \cap \tilde{p}_*(T_x\tilde{L}_\tau) = T_yL_\sigma \cap T_yL_\tau. \end{aligned}$$

(3) Let $\sigma \subset \tau \subset I_r$ $(|\tau - \sigma| = 1)$. Then we have that

$$\begin{aligned} \partial L_\sigma \cap L_\tau &= \tilde{p}(\partial\tilde{L}_\sigma) \cap \tilde{p}(\tilde{L}_\tau) = \tilde{p}(\partial\tilde{L}_\sigma \cap \tilde{L}_\tau) \\ &= \tilde{p}(\tilde{L}_\sigma \cap \tilde{L}_\tau) = L_\sigma \cap L_\tau. \end{aligned}$$

□

6. Examples

Here we give some examples which are families of 2^r-Lagrangian (2^r-Legendrian) submanifolds but not Symplectic (Contact) regular r-cubic configurations.

I. $r = k = 1$: Let $L_\emptyset = L_\emptyset^0 = \{(0, p) \in (T^*\mathbb{R}, 0)\}$ and $L_1 = \{(p^2, p) \mid p \geq 0\}$. Then $\{L_\sigma\}_{\sigma=\emptyset,1}$ satisfies the condition (1), (3) in Theorem 4.1 but does not satisfy the condition (2). Therefore $\{L_\sigma\}_{\sigma=\emptyset,1}$ is not symplectic regular 1-cubic configuration.

Set $\tilde{L}_\emptyset = \{(0, 0, p) \in (J^1(\mathbb{R}, \mathbb{R}), 0)\}$ and $\tilde{L}_1 = \{(p^2, \frac{2}{3}p^3, p) \mid p \geq 0\}$. Then $\{\tilde{L}_\sigma\}_{\sigma=\emptyset,1}$ satisfies the condition (1), (3) but does not satisfy the condition (2) in Theorem 5.7. Therefore $\{\tilde{L}_\sigma\}_{\sigma=\emptyset,1}$ is not contact regular 1-cubic configuration.

II. $r = k = 2$: Let $L_\sigma = L_\sigma^0$ for $\sigma = \emptyset, 1, 2$ and set $L_{\{1,2\}} = \{(q_1, q_2, 0, 0) \in (T^*\mathbb{R}^2, 0) \mid q_1 \geq -\frac{1}{4}q_2, q_2 \geq -\frac{1}{4}q_1\}$. Then $\{L_\sigma\}_{\sigma \subset \{1,2\}}$ satisfies the condition (1), (2) in Theorem 4.1 but does not satisfy the condition (3). Hence $\{L_\sigma\}_{\sigma\subset\{1,2\}}$ is not symplectic regular 2-cubic configuration.

Let $\tilde{L}_\sigma = \tilde{L}^0_\sigma$ for $\sigma = \emptyset, 1, 2$ and set $L_{\{1,2\}} = \{(q_1, q_2, 0, 0, 0) \in (J^1(\mathbb{R}^2, \mathbb{R}), 0) \mid q_1 \geq -\frac{1}{4}q_2,\ q_2 \geq -\frac{1}{4}q_1\}$. Then $\{\tilde{L}_\sigma\}_{\sigma\subset\{1,2\}}$ satisfies the conditions (1), (2) in Theorem 5.7 but does not satisfy the condition (3). Hence $\{\tilde{L}_\sigma\}_{\sigma\subset\{1,2\}}$ is not contact regular 2-cubic configuration.

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References

[1] Arnold V.I., Gusein-Zade S.M. and Varchenko A.N., *Singularities of differential maps*, Vol.I. Birkhäuser, Basel, 1986.

[2] Bröcker Th. and Lander L., *Differentiable germs and catastrophes.* LMS-Lecture note series **17**, CUP 1975.

[3] Nguyen Huu Duc, Nguyen Tien Dai and Pham F., *Singularités non-dégénérées des systèmes de Gauss-Manin réticulés.* Memoire de la S. M. F., Nouvelle serie No.6, 1981.

[4] Nguyen Huu Duc, *Involutive singularities.* Kodai Math. J. **17** (1994), 627–635.

[5] Jänich K., *Caustics and catastrophes.* Math. Ann. **209** (1974), 161–180.

[6] Matov V.I., *Unimodal germs of functions on a manifold with a boundary.* Functional. Anal. i Prilozhen **14** (1980) No.1, 69–70.

[7] Scherbak I.G., *Boundary fronts and caustics and their metamorphosis.* LMS-Lecture note series **201**, Singularities, 363–373.

[8] Tsukada T., *Reticular Lagrangian Singularities.* The Asian J. of Math. Vol. 1, No.3 (1997), 572–622.

[9] Tsukada T., *Reticular Legendrian Singularities.* To appear in The Asian J. of Math.

[10] Wassermann G., *Stability of unfolding.* Lecture note in mathematics **393** (1974).

[11] Wassermann G., *Stability of caustics.* Math. Ann. **216** (1975), 43–50.

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