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On the areas of geodesic triangles on a surface

Eiji KANEDA and Kazuhiro KISO

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Abstract. This paper treats geodesic triangles on two-dimensional orientable Riemannian manifolds M. Fixing two vertices A and B, we can consider the area and the interior angles of the geodesic triangle $\triangle PAB$ as smooth functions of P. Applying the Laplace operator to these functions, we obtain formulas for the area and interior angles of $\triangle PAB$. It is shown that if M is of constant curvature, the area and interior angles of geodesic triangles are harmonic.

Key words: Riemannian geometry, area of geodesic triangle, Laplace operator.

1. Notations and results

In the paper [1] they proved by direct calculations that the area function of geodesic triangles on the two dimensional sphere S^2 is harmonic. However, this is not true for general surfaces M. In this paper, we will show formulas for the area and interior angles of geodesic triangles in M (Theorem 1), which extend the result in [1].

Let (M, g) be a two-dimensional orientable Riemannian manifold. Let A, B be two points of M. We assume that there is a geodesically convex open set D in M containing A and B. (Recall that an open set D in M is called *geodesically convex* if any two points P, Q of D can be joined by a unique geodesic segment PQ in D, which gives the distance between P and Q.) Let $P \in D$. We denote by $\triangle PAB$ the compact, simply connected subset of D bounded by geodesic segments PA, AB and BP. Such $\triangle PAB$ is called a *small* geodesic triangle.

Fixing A and B, we can consider the area of $\triangle PAB$ as a function of $P \in D$, which is denoted by $S(P)$. Similarly, the angle $\angle APB$ may also be considered as a function of $P \in D$, which is denoted by $\Omega(P)$. We show in Theorem 1 the formula obtained by applying the Laplace operator to S and Ω.

To state Theorem 1 we prepare some notations concerning the geodesic polar coordinates around A and B. Let $\{U, (r, \theta)\}$ (resp. $\{V, (s, \varphi)\}$) be the

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geodesic polar coordinate centered at A (resp. B). Since D is geodesically convex, both U and V cover D, i.e., $D \subset U \cap V$. We assume that the arguments θ and φ are normalized in the following manner: (1) the 2-vectors $\frac{\partial}{\partial r} \wedge \frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial s} \wedge \frac{\partial}{\partial \varphi}$ define the same orientation on D, (2) $\theta(B) = 0$, $\varphi(A) = \pi$.

Let $J = J(r, \theta)$ (resp. $H = H(s, \varphi)$) be the norm of the vector field $\frac{\partial}{\partial \theta}$ (resp. $\frac{\partial}{\partial \varphi}$), i.e.,

$$J(r,\theta) = \sqrt{g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right)}, \quad H(s,\varphi) = \sqrt{g\left(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi}\right)}.$$

Then Theorem 1 can be stated as follows:

Theorem 1 *Let (M, g) be a two-dimensional orientable Riemannian manifold. Let D be a geodesically convex open set in M and let $\triangle PAB$ be a small geodesic triangle in D. Set $\varepsilon(P) = 1$ if $0 < \theta(P) < \pi$ and $\varepsilon(P) = -1$ if $\pi < \theta(P) < 2\pi$. Then it holds*

$$\text{(1)} \quad \Delta S(P) = \varepsilon(P) \left\{ \frac{1}{J} \frac{\partial}{\partial \theta} \left(\frac{1}{J} L_A(J) \right)(P) - \frac{1}{H} \frac{\partial}{\partial \varphi} \left(\frac{1}{H} L_B(H) \right)(P) \right\}.$$

$$\text{(2)} \quad \Delta \Omega(P) = -\varepsilon(P) \left(\frac{1}{J} \frac{\partial^2 \log J}{\partial r \, \partial \theta}(P) - \frac{1}{H} \frac{\partial^2 \log H}{\partial s \, \partial \varphi}(P) \right).$$

In the formulas in Theorem 1, for a differentiable function f on D we mean by $L_A(f)$ and $L_B(f)$ the functions defined by

$$L_A(f)(Q) = \int_{AQ} f \, dr = \int_0^{r(Q)} f(r, \theta(Q)) \, dr,$$

$$L_B(f)(Q) = \int_{BQ} f \, ds = \int_0^{s(Q)} f(s, \varphi(Q)) \, ds,$$

where Q is a point of D and AQ (resp. BQ) is the geodesic segment joining A (resp. B) to Q.

It is remarkable that $\Delta S(P)$ can be calculated by the local properties of (M, g) around geodesic segments AP and BP. More strongly, $\Delta \Omega(P)$ is calculated by only the local property of (M, g) around P.

As a corollary of Theorem 1 we have

Corollary *Assume that $\frac{\partial J}{\partial \theta} = 0$ holds on a neighborhood of AP and that $\frac{\partial H}{\partial \varphi} = 0$ holds on a neighborhood of BP. Then, it holds $\Delta S = \Delta \Omega = 0$ on* *a neighborhood of P. In particular, if (M,g) is of constant curvature, then S and Ω are harmonic.*

Theorem 1 is also valid for somewhat large geodesic triangles not contained in any geodesically convex set in M. We will prove this fact in §4.

2. Bi-angular coordinates for geodesic triangles

We now introduce a new coordinate called the bi-angular coordinate, which is, in a sense, suitable to parametrize geodesic triangles $\triangle PAB$.

Let D_+ and D_- be the domains in D given by

$$\begin{aligned} D_+ &= \{(r,\theta) \in D \mid 0 < \theta < \pi\}, \\ D_- &= \{(r,\theta) \in D \mid \pi < \theta < 2\pi\}. \end{aligned}$$

We define a locally constant function ε on $D_+ \cup D_-$ by setting $\varepsilon(P) = 1$ if $P \in D_+$, $\varepsilon(P) = -1$ if $P \in D_-$. Let $P \in D_+ \cup D_-$. We denote by $\xi(P)$ and $\eta(P)$ the interior angles at the vertexes A and B, respectively, i.e., $\xi(P) = \angle PAB$ and $\eta(P) = \angle PBA$. Since D is geodesically convex, it can be easily shown that the angle $\Omega(P) = \angle APB$ defined in §1 satisfies $0 < \Omega(P) < \pi$.

We first prove

Lemma 2 *On the domain $D_+ \cup D_-$, it holds*

(1) $$\frac{\partial}{\partial s} = \cos\Omega \frac{\partial}{\partial r} + \varepsilon \frac{\sin\Omega}{J}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial \varphi} = H\left(-\varepsilon \sin\Omega \frac{\partial}{\partial r} + \frac{\cos\Omega}{J}\frac{\partial}{\partial\theta}\right).$$

(2) $$ds = \cos\Omega\, dr + \varepsilon J \sin\Omega\, d\theta, \quad d\varphi = \frac{1}{H}(-\varepsilon \sin\Omega\, dr + J\cos\Omega\, d\theta).$$

Proof. We note that the Riemannian metric g can be written in the form

$$g = dr^2 + J^2\, d\theta^2 = ds^2 + H^2\, d\varphi^2.$$

Let $P \in D_+ \cup D_-$. Since the angle between the vectors $\left(\frac{\partial}{\partial r}\right)_P$ and $\left(\frac{\partial}{\partial s}\right)_P$ equals $\Omega(P)$, we have $g\left(\left(\frac{\partial}{\partial r}\right)_P, \left(\frac{\partial}{\partial s}\right)_P\right) = \cos\Omega(P)$. Moreover, since the angle between the vectors $\left(\frac{\partial}{\partial \theta}\right)_P$ and $\left(\frac{\partial}{\partial s}\right)_P$ equals $\Omega(P) - \frac{\varepsilon(P)}{2}\pi$, we have $g\left(\left(\frac{\partial}{\partial \theta}\right)_P, \left(\frac{\partial}{\partial s}\right)_P\right) = \varepsilon(P)\, J(P) \sin\Omega(P)$. This proves the first equality of (1). Similarly, we can show the second equality of (1).

The assertion (2) is just the dual version of (1) and hence can be immediately obtained by (1). □

As is easily seen, any point P of the domain D_+ (or D_-) can be completely parametrized by two angles $(\xi(P), \eta(P))$, which is called the *geodesic bi-angular coordinate* of D_+ (or D_-) with respect to the pair (A, B). Concerning this coordinate (ξ, η), the vector fields $\frac{\partial}{\partial\xi}$, $\frac{\partial}{\partial\eta}$, the Riemannian metric g and the area element dA are written as follows:

Proposition 3 (1) $\dfrac{\partial}{\partial\xi} = J\cot\Omega\dfrac{\partial}{\partial r} + \varepsilon\dfrac{\partial}{\partial\theta} = \dfrac{J}{\sin\Omega}\dfrac{\partial}{\partial s}$,
$\dfrac{\partial}{\partial\eta} = \dfrac{H}{\sin\Omega}\dfrac{\partial}{\partial r} = H\cot\Omega\dfrac{\partial}{\partial s} - \varepsilon\dfrac{\partial}{\partial\varphi}$.

(2) $g = \left(\dfrac{1}{\sin\Omega}\right)^2 (J^2\,d\xi^2 + 2\,JH\cos\Omega\,d\xi\,d\eta + H^2\,d\eta^2)$.

(3) $dA = -\varepsilon\,\dfrac{JH}{\sin\Omega}\,d\xi\wedge d\eta$.

Proof. Let $P = (r, \theta) = (s, \varphi) \in D_+ \cup D_-$. Then we have $\xi = \varepsilon(\theta - \pi) + \pi$, $\eta = \varepsilon(\pi - \varphi)$ and hence $d\xi = \varepsilon\,d\theta$, $d\eta = -\varepsilon\,d\varphi$. By (2) of Lemma 2, we easily have $dr = J\cot\Omega\,d\theta + \frac{H}{\sin\Omega}\,d\eta$. Putting this into the formulas $g = dr^2 + J^2\,d\theta^2$, $dA = J\,dr\wedge d\theta$, we easily get (2) and (3).

Finally, we prove the assertion (1). Since $\frac{\partial\theta}{\partial\xi} = \varepsilon\frac{\partial\xi}{\partial\xi} = \varepsilon$, $\frac{\partial\varphi}{\partial\xi} = -\varepsilon\frac{\partial\eta}{\partial\xi} = 0$, it follows from (2) of Lemma 2 that $\frac{\partial r}{\partial\xi} = J\cot\Omega$, $\frac{\partial s}{\partial\xi} = \frac{H}{\sin\Omega}$. This gives the expression of $\frac{\partial}{\partial\xi}$. Similarly, we can get the expression of $\frac{\partial}{\partial\eta}$. □

We now represent the Laplace operator Δ in terms of the bi-angular coordinate (ξ, η).

Theorem 4 *Let F be a differentiable function on $D_+ \cup D_-$. Then:*

$$\Delta F = \varepsilon\left\{\frac{1}{J}\frac{\partial}{\partial\theta}\left(\frac{1}{J}\frac{\partial F}{\partial\xi}\right) - \frac{1}{H}\frac{\partial}{\partial\varphi}\left(\frac{1}{H}\frac{\partial F}{\partial\eta}\right)\right\}.$$

Proof. Let Δ_0 be the differential operator given in the right side of the above equality. By Proposition 3, we have $\frac{\partial}{\partial\theta} = \varepsilon\left(\frac{\partial}{\partial\xi} - \frac{J}{H}\cos\Omega\frac{\partial}{\partial\eta}\right)$, $\frac{\partial}{\partial\varphi} = \varepsilon\left(-\frac{\partial}{\partial\eta} + \frac{H}{J}\cos\Omega\frac{\partial}{\partial\xi}\right)$. Putting these equalities into $\Delta_0 F$, we have

$$\Delta_0 F = \frac{1}{J^2H^2}\left(H^2\frac{\partial^2 F}{\partial\xi^2} - 2\,JH\cos\Omega\frac{\partial^2 F}{\partial\xi\,\partial\eta} + J^2\frac{\partial^2 F}{\partial\eta^2}\right)$$
$$+ \text{(lower order derivatives)}.$$

By the definition of the Laplace operator Δ, we know that the part of second order derivatives of ΔF is just the same form stated above (see [4] and (2) of Proposition 3). Therefore, if we set $\Delta_1 = \Delta - \Delta_0$, then Δ_1 is a first order differential operator. It may be written in the form

$$\Delta_1 F = u(\xi,\eta)\frac{\partial F}{\partial \xi} + v(\xi,\eta)\frac{\partial F}{\partial \eta} + w(\xi,\eta)F,$$

where $u(\xi,\eta) = \Delta_1 \xi$, $v(\xi,\eta) = \Delta_1 \eta$, $w(\xi,\eta) = \Delta_1 \mathbf{1}$; $\mathbf{1}$ denotes the function identically equals 1.

To show the theorem it suffices to prove $u(\xi,\eta) = v(\xi,\eta) = w(\xi,\eta) = 0$. By use of the expression of Δ in the geodesic polar coordinate (r,θ), we have

$$\begin{aligned}\Delta \xi &= \frac{1}{J}\left\{\frac{\partial}{\partial r}\left(J\frac{\partial \xi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{J}\frac{\partial \xi}{\partial \theta}\right)\right\} \\ &= \frac{1}{J}\left\{\frac{\partial}{\partial r}\left(J\frac{\partial(\varepsilon\theta)}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{J}\frac{\partial(\varepsilon\theta)}{\partial \theta}\right)\right\} \\ &= \frac{\varepsilon}{J}\frac{\partial}{\partial \theta}\left(\frac{1}{J}\right).\end{aligned}$$

On the other hand, we can easily have $\Delta_0 \xi = \frac{\varepsilon}{J}\frac{\partial}{\partial \theta}\left(\frac{1}{J}\right)$. This proves $u(\xi,\eta) = \Delta_1 \xi = 0$. Similarly, we can also verify $v(\xi,\eta) = w(\xi,\eta) = 0$. This completes the proof of the theorem. □

3. Multiple integration on geodesic triangles

Before proceeding to the proof of Theorem 1 we make some criteria concerning the multiple integration on geodesic triangles from a somewhat general viewpoint.

Let f be a differentiable function on $D_+ \cup D_-$. Let us denote by $V(f)$ the function given by integrating f over $\triangle PAB$, i.e.,

$$V(f)(P) = \iint_{\triangle PAB} f\, dA, \quad P \in D_+ \cup D_-.$$

We first prove

Proposition 5 *Let $P \in D_+ \cup D_-$. Then*

$$V(f)(P) = \int_0^{\xi(P)} L_A(Jf)(\xi,\eta(P))\, d\xi$$

$$= \int_0^{\eta(P)} L_B(Hf)(\xi(P),\eta)\,d\eta.$$

To prove the proposition we show

Lemma 6 *Let $Q \in D_+ \cup D_-$. Then*

$$L_A(f)(Q) = \int_0^{\eta(Q)} \left(\frac{Hf}{\sin\Omega}\right)(\xi(Q),\eta)\,d\eta,$$
$$L_B(f)(Q) = \int_0^{\xi(Q)} \left(\frac{Jf}{\sin\Omega}\right)(\xi,\eta(Q))\,d\xi.$$

Proof. Let us show the first equality. We note that on the geodesic segment AQ, ξ identically equals $\xi(Q)$. Therefore, by (2) of Lemma 2 we have $dr = \frac{H}{\sin\Omega}\,d\eta$ on AQ. Hence, we immediately have

$$L_A(f)(Q) = \int_{AQ} f\,dr = \int_0^{\eta(Q)} \left(\frac{Hf}{\sin\Omega}\right)(\xi(Q),\eta)\,d\eta$$

The second equality can be similarly dealt with. □

Proof of Proposition 5. In the bi-angular coordinate (ξ,η), $\triangle PAB$ is denoted by the subset $\{(\xi,\eta) \mid 0 \le \xi \le \xi(P),\ 0 \le \eta \le \eta(P)\}$. Hence by (3) of Proposition 3, we have

$$V(f)(P) = \int_0^{\xi(P)} \left(\int_0^{\eta(P)} \left(\frac{JH}{\sin\Omega} f\right)(\xi,\eta)\,d\eta\right) d\xi$$
$$= \int_0^{\eta(P)} \left(\int_0^{\xi(P)} \left(\frac{JH}{\sin\Omega} f\right)(\xi,\eta)\,d\xi\right) d\eta.$$

Consequently, our proposition follows from Lemma 6. □

We now start the proof of Theorem 1.

Proof of Theorem 1. It is easily seen that the area function S is given by $S = V(\mathbf{1})$, where $\mathbf{1}$ is the function identically equals 1. By Proposition 5 we have

$$\frac{\partial V(\mathbf{1})}{\partial\xi} = L_A(J), \quad \frac{\partial V(\mathbf{1})}{\partial\eta} = L_B(H).$$

Therefore, by Theorem 4 we obtain (1) of Theorem 1.

We now prove (2). Let K be the Gaussian curvature of g. Applying the Gauss-Bonnet formula to the geodesic triangle $\triangle QAB$ ($Q \in D_+ \cup D_-$), we have

$$\Omega(Q) = V(K)(Q) - \xi(Q) - \eta(Q) + \pi.$$

Applying Δ to both sides, we get

$$\begin{aligned}\Delta\,\Omega = \;& \varepsilon\left\{\frac{1}{J}\frac{\partial}{\partial\theta}\left(\frac{1}{J}L_A(JK)\right) - \frac{1}{H}\frac{\partial}{\partial\varphi}\left(\frac{1}{H}L_B(HK)\right)\right\} \\ & - \varepsilon\left\{\frac{1}{J}\frac{\partial}{\partial\theta}\left(\frac{1}{J}\right) - \frac{1}{H}\frac{\partial}{\partial\varphi}\left(\frac{1}{H}\right)\right\}.\end{aligned}$$

Since $K = -\frac{1}{J}\frac{\partial^2 J}{\partial r^2} = -\frac{1}{H}\frac{\partial^2 H}{\partial s^2}$ and $\frac{\partial J}{\partial r}(A) = \frac{\partial H}{\partial s}(B) = 1$ (see [4]), we have

$$\begin{aligned} L_A(JK)(Q) &= -\int_0^{r(Q)} \frac{\partial^2 J}{\partial r^2}(r, \theta(Q))\, dr = -\frac{\partial J}{\partial r}(Q) + 1, \\ L_B(HK)(Q) &= -\int_0^{s(Q)} \frac{\partial^2 H}{\partial s^2}(s, \varphi(Q))\, ds = -\frac{\partial H}{\partial s}(Q) + 1. \end{aligned}$$

Therefore we have

$$\Delta\,\Omega = -\varepsilon\left\{\frac{1}{J}\frac{\partial}{\partial\theta}\left(\frac{1}{J}\frac{\partial J}{\partial r}\right) - \frac{1}{H}\frac{\partial}{\partial\varphi}\left(\frac{1}{H}\frac{\partial H}{\partial s}\right)\right\}.$$

This proves (2) of Theorem 1.

Finally we prove Corollary of Theorem 1. If $\frac{\partial J}{\partial\theta} = 0$ on a neighborhood of AP, then it is clear that $\frac{\partial L_A(J)}{\partial\theta} = 0$ holds around P. Similarly, if $\frac{\partial H}{\partial\varphi} = 0$ on a neighborhood of BP, $\frac{\partial L_B(H)}{\partial\varphi} = 0$ holds on a neighborhood of P. Then by Theorem 1 we have $\Delta\,S = \Delta\,\Omega = 0$ around P. If (M, g) is of constant curvature, then by the symmetry around A (resp. B) we know that J (resp. H) does not depend on the argument θ (resp. φ). Consequently, we have $\frac{\partial J}{\partial\theta} = \frac{\partial H}{\partial\varphi} = 0$. This shows the corollary. Moreover, it is easily seen that under the same condition the interior angles ξ and η satisfy $\Delta\,\xi = \Delta\,\eta = 0$ (see the formulas in the proof of Theorem 4). □

4. Somewhat large geodesic triangles

We now consider geodesic triangles not contained in any geodesically convex open set in M.

Let A, B and P be three points of M such that there is no geodesically convex open set of M containing all A, B and P simultaneously. We assume that A and B are not so distant, i.e., A and B are joined by a geodesic segment AB; and the geodesic polar coordinates $\{U, (r, \theta)\}$ and $\{V, (s, \varphi)\}$ centered at A and B have a non-trivial intersection and $P \in U \cap V$. We join P to A (resp. B) by the geodesic segment AP (resp. BP) originated at A (resp. B). The geodesic triangle bounded by AP, BP and AB is called a *somewhat large* geodesic triangle, which may contain more complicated figures than those considered in §2.

We say that our geodesic triangle defined above is in *good condition* if it satisfies the following: (1) The curve composed of AP, BP and AB divide M into two distinct domains and at least one of them is compact. One of the compact domains of this division is denoted by $\triangle PAB$. (In case both sides are compact, then any side can be selected as $\triangle PAB$ as desired. The interior angles $\xi(P)$, $\eta(P)$ and $\Omega(P)$ may exceed π.) (2) $\Omega(P) \neq 0, \pi$.

Let Q be a point of $U \cap V$ sufficiently close to P. Then $\triangle QAB$ is defined in the same manner as above. We promise that $\triangle QAB$ is synchronized with $\triangle PAB$, i.e., $\triangle QAB$ can be continuously deformed to $\triangle PAB$ and is homeomorphic to $\triangle PAB$. Accordingly, the area of $\triangle QAB$ and the interior angle $\Omega(Q)$ can be considered as continuous functions of Q.

We now show that the formulas in Theorem 1 also hold for our somewhat large geodesic triangles in good condition. As in §1 we may assume that $\frac{\partial}{\partial r} \wedge \frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial s} \wedge \frac{\partial}{\partial \varphi}$ define the same orientation of M. Since $\Omega(P) \neq 0, \pi$, the bi-angular coordinate (ξ, η) is effective on a sufficiently small neighborhood W of P. It can be verified that Lemma 2 and hence Theorem 4 are valid even in the case where $\Omega(P) > \pi$. Here $\varepsilon(P)$ is determined as follows: $\varepsilon(P) = 1$ (resp. $\varepsilon(P) = -1$) if the interior angle $\xi(P)$ is measured in the positive (resp. negative) direction with respect to the argument θ. In other words, $\varepsilon(P)$ is determined by the ratio between $d\theta$ and $d\xi$ at P.

Let f be a differentiable function on a neighborhood of $\triangle PAB$. As in §3, we denote by $V(f)(Q)$ the value given by integrating f over $\triangle QAB$. Since the bi-angular coordinate (ξ, η) does not cover the whole interior of $\triangle PAB$, $V(f)$ cannot be expressed in terms of the coordinate (ξ, η), however, the partial derivatives of $V(f)$ can be calculated. In fact, we can prove the following equalities:

$$\frac{\partial V(f)}{\partial \xi} = L_A(Jf), \quad \frac{\partial V(f)}{\partial \eta} = L_B(Hf).$$

Let $Q \in W$. For a real number σ sufficiently close to 0 we define a point $Q_\sigma \in W$ by setting $\xi(Q_\sigma) = \sigma + \xi(Q)$ and $\eta(Q_\sigma) = \eta(Q)$. If $\sigma > 0$ then we easily have $\triangle Q_\sigma AB = \triangle QAB \cup \triangle Q_\sigma AQ$ and if $\sigma < 0$ we have $\triangle Q_\sigma AB = \triangle QAB \setminus \triangle Q_\sigma AQ$. Consequently, we have

$$V(f)(Q_\sigma) = V(f)(Q) + \int_0^\sigma \left(\int_0^{r(Q_\tau)} (Jf)(r, \theta(Q_\tau))\, dr \right) d\tau.$$

Therefore, we easily get

$$\frac{\partial V(f)}{\partial \xi}(Q) = \int_0^{r(Q)} (Jf)\,(r, \theta(Q))\, dr = L_A(Jf)(Q).$$

Similarly, we can get

$$\frac{\partial V(f)}{\partial \eta}(Q) = L_B(Hf)(Q).$$

Now in the almost same manner as in §3, we can get the formula (1) in Theorem 1 for our somewhat large geodesic triangles in good condition. The formula (2) can be also shown by the extended form of Gauss-Bonnet formula

$$\Omega(Q) = V(K)(Q) - \xi(Q) - \eta(Q) - 2\,\pi\,\chi(\triangle QAB) + 3\,\pi,$$

where $\chi(\triangle QAB)$ denotes the Euler characteristic of the triangle $\triangle QAB$ (see [2]). Since $\triangle QAB$ is homeomorphic to $\triangle PAB$, $\chi(\triangle QAB)$ is identically equal to $\chi(\triangle PAB)$ around P. Consequently, we get the formula (2).

We can resume the above result in the following

Theorem 7 *Let $\triangle PAB$ be a somewhat large geodesic triangle in a two-dimensional orientable Riemannian manifold (M, g). If $\triangle PAB$ is in good condition, then the formulas* (1) *and* (2) *hold for $\triangle PAB$, where $\varepsilon(P)$ is determined by the ratio of $d\theta$ and $d\xi$ at P.*

Finally, we note that Corollary in §1 is also holds for a somewhat large geodesic triangle in good condition.

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Eiji Kaneda
Department of International Studies
Osaka University of Foreign Studies
8-1-1 Aomadani-Higashi, Minoo-shi
Osaka 562-8558, Japan
E-mail: kaneda@post01.osaka-gaidai.ac.jp

Kazuhiro Kiso
Department of Mathematical Sciences
Faculty of Science
Ehime University
Bunkyou-cho, Matsuyama-shi
Ehime 790-8577, Japan
E-mail: kkiso@dpc.ehime-u.ac.jp