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On kernels of purifiability in arbitrary abelian groups

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Abstract. We determine the structure of a p-vertical[vertical] subgroup A of an arbitrary abelian group G such that every neat hull of A is p-purifiable[purifiable] in G.

Key words: kernel of purifiability, purifiable subgroup, almost-dense subgroup, p-vertical subgroup, kernel of purity.

Introduction

A subgroup A of an arbitrary abelian group G is said to be *purifiable* in G if there exists a pure subgroup H of G containing A which is minimal among the pure subgroups of G that contain A. Such a subgroup H is said to be a *pure hull* of A in G. In general, it is well-known that there exist non-purifiable subgroups of some p-group, but all subgroups of an arbitrary abelian group G have neat hulls in G.

First, P. Hill and C. Megibben [6] determined the structure of pure hulls of p-groups and gave a characterization of p-groups for which every subgroup is purifiable. Next, K. Benabdallah and J. Irwin [2] introduced the concept of *almost-dense* subgroups of p-groups and used this concept to give a refinement of the structure of pure hulls in p-groups. Furthermore, K. Benabdallah and T. Okuyama [3] introduced new invariants, the so-called *n-th overhangs* of a subgroup of a p-group, which are related to the n-th relative Ulm-Kaplansky invariants. They used these invariants to give a necessary condition for a subgroup of a p-group to be purifiable. K. Benabdallah, B. Charles, and A. Mader [1] introduced the concept of maximal vertical subgroups supported by a given subsocle of a p-group and characterized a p-group for which the necessary condition on a subgroup of a p-group to be purifiable given in [3] is also sufficient.

Recently, we extended the concept of purifiable subgroups of p-groups to arbitrary abelian groups in [10]. Let p be a prime. A subgroup A of an arbitrary abelian group G is said to be *p-purifiable* in G if there exists a p-pure subgroup H of G containing A which is minimal among the p-pure

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subgroups of G that contain A. Such a subgroup H is said to be a *p-pure hull* of A in G. In [10], we showed that a subgroup A is purifiable in G if and only if A is p-purifiable in G for every prime p. Thus it suffices to study the p-purifiability in G. However, to our knowledge, a complete characterization of p-purifiable subgroups is still to be found.

C. Megibben introduced the concept of kernels of purity in [7]. A subgroup A of an arbitrary abelian group G is said to be a *kernel of purity* in G if every neat hull of A in G is pure in G. In [7], we find a complete characterization of such a subgroup A in an arbitrary abelian group G. In [8], we characterized a kernel of purity of a p-group using n-th defects of a given subgroup. It is immediate that a kernel of purity is purifiable.

Now we weaken the concept of kernels of purity. A subgroup A of an arbitrary abelian group G is said to be a *kernel of p-purity* in G if every neat hull of A in G is p-pure in G. Moreover, A is called a *kernel of p-purifiability* in G if every neat hull of A in G is p-purifiable in G. Studying these subgroups plays an important role in studying purifiable subgroups of arbitrary abelian groups.

Suppose that a subgroup A of an arbitrary abelian group G is a kernel of p-purifiability in G. If A is neat in G, then we may assume A is not p-vertical in G, because, if A is p-vertical in G, then A is p-pure in G. Then there exists a non-negative integer m such that $A \cap p^m G$ is p-vertical in $p^m G$. Hence we must consider the characterization of p-purifiability without any condition. Giving an answer to this problem is as difficult as giving a complete characterization of purifiable subgroups. To make progress on this problem, we consider the case that A is p-vertical in G.

In this article, in Section 2, we give a complete characterization of kernels of p-purity in arbitrary abelian groups. In Section 3, we give a necessary and sufficient condition for a p-vertical subgroup of an arbitrary abelian group to be a kernel of p-purifiability in a given group.

We gave a definition of strongly purifiable subgroups of p-groups in [9]. Extending it, a subgroup A of an arbitrary abelian group G is said to be *strongly p-purifiable* in G if A is eventually p-vertical in G, i.e. $V_{p,n}(G, A) = 0$ for all $n \geqq m$, and all maximal p-vertical essential extensions of $A \cap p^m G$ in $p^m G$ are pure in $p^m G$ for some non-negative integer m. We show that a kernel of p-purifiability is strongly p-purifiable in G.

All groups considered are arbitrary abelian groups. The terminologies and notations not expressly introduced here follow the usage of [4].

Throughout this article, p denotes a prime integer and G_p the p-primary subgroup of the abelian group G.

1. Definitions and basic facts

K. Benabdallah and J. Irwin introduced the concept of almost-dense subgroups of p-groups and characterized them in [2]. We extended that concept from p-groups to arbitrary abelian groups in [10]. First, we give the definition of p-almost-dense and almost-dense subgroups of G.

Throughout this section, let G be an arbitrary abelian group and A a subgroup of G.

Definition 1.1 A is said to be *p-almost-dense* in G if the torsion part of G/K is p-divisible for every p-pure subgroup K of G containing A. Moreover, A is said to be *almost-dense* in G if A is p-almost-dense in G for every prime p.

We recall characterizations of p-almost-dense and almost-dense subgroups.

Proposition 1.2 [10, Proposition 1.3] *A is p-almost-dense in G if and only if, for all integers $n \geqq 0$, $A + p^{n+1}G \supseteq p^nG[p]$.*

Proposition 1.3 [10, Proposition 1.4] *The following properties are equivalent:*

(1) *A is almost-dense in G;*
(2) *for all integers $n \geqq 0$ and all primes p, $A + p^{n+1}G \supseteq p^nG[p]$;*
(3) *for every pure subgroup K of G containing A, $T(G/K)$ is divisible.*

Next, we give the definition of p-purifiable and purifiable subgroups.

Definition 1.4 A is said to be *p-purifiable*[*purifiable*] in G if, among the p-pure[pure] subgroups of G containing A, there exists a minimal one. Such a minimal p-pure[pure] subgroup is called a *p-pure*[*pure*] *hull* of A.

We state characterizations of a p-pure hull and a pure hull and an important relation between p-purifiability and purifiability.

Proposition 1.5 [10, Theorem 1.8] *There exists no proper p-pure subgroup of G containing A if and only if the following three conditions hold:*

(1) *A is p-almost-dense in G.*

(2) *G/A is a p-group.*
(3) *there exists a non-negative integer m such that $p^m G[p] \subseteq A$.*

Proposition 1.6 [10, Theorem 1.11] *There exists no proper pure subgroup of G containing A if and only if the following three conditions hold:*
(1) *A is almost-dense in G.*
(2) *G/A is torsion.*
(3) *for every prime p, there exists a non-negative integer m_p such that*

$$p^{m_p} G[p] \subseteq A.$$

Proposition 1.7 [10, Theorem 1.12] *A is purifiable in G if and only if A is p-purifiable in G for every prime p.*

The following result is frequently used in this article.

Proposition 1.8 [10, Lemma 4.5] *Let H be a p-pure subgroup of G containing A such that $p^m H[p] \subseteq A$ for some $m \geqq 0$ and H/A is torsion. Then A is p-purifiable in G.* □

In G, for every subgroup A of G, there exist neat hulls of A in G. We recall the properties of neat hulls in an arbitrary abelian groups.

Proposition 1.9 *Let N be a neat hull of A in G. Then we have:*
(1) *N is neat in G;*
(2) *N/A is torsion;*
(3) *$N[p] = A[p]$ for every prime p.*

However, in general, not all neat hulls of A are pure in G. C. Megibben in [7] defined kernels of purity and characterized them as follows:

Definition 1.10 A is said to be a *kernel of purity* in G if all neat hulls of A in G are pure in G.

Theorem 1.11 [7, Theorem 2] *A is a kernel of purity in G if and only if for each prime p, A satisfies the following condition $(*)$ for all positive integers n.*

$(*)$ *If $p^{n+1} g \in A$, then either $p^n g + z \in A \cap p^n G$ for some $z \in G[p]$ or else $\frac{G[p]}{A[p]} \subseteq p^n(\frac{G}{A[p]})$.* □

2. Kernels of p-purity

In this section, we consider kernels of p-purity in arbitrary abelian groups. First, we give several definitions and properties.

Definition 2.1 A subgroup A of an arbitrary abelian group G is said to be a *kernel of p-purity* in G if all neat hulls of A in G are p-pure in G. If all neat hulls of A in G are p-purifiable in G, A is called a *kernel of p-purifiability* in G.

Definition 2.2 For every non-negative integer n, we define the *n-th p-overhang* of a subgroup A of an arbitrary abelian group G to be the vector space

$$V_{p,n}(G,A) = \frac{(A + p^{n+1}G) \cap p^n G[p]}{(A \cap p^n G)[p] + p^{n+1}G[p]}.$$

Moreover, A is said to be *p-vertical* in G if $V_{p,n}(G,A) = 0$ for all $n \geqq 0$.

It is convenient to use the following notations for the numerator and the denominator of $V_{p,n}(G,A)$:

$$A_G^n(p) = (A + p^{n+1}G) \cap p^n G[p] = ((A \cap p^n G) + p^{n+1}G)[p]$$

and

$$A_n^G(p) = (A \cap p^n G)[p] + p^{n+1}G[p].$$

Proposition 2.3 [10, Proposition 2.2] *For every p-pure subgroup K of an arbitrary abelian group G containing a subgroup A of G,*

$$V_{p,n}(G,A) \cong V_{p,n}(K,A)$$

for all $n \geqq 0$.

Proposition 2.4 [10, Theorem 2.3] *If a subgroup A is p-purifiable in an arbitrary abelian group G, then there exists a non-negative integer m such that $V_{n,p}(G,A) = 0$ for all $n \geqq m$.*

For convenience, A as in Proposition 2.4 is called an *eventually p-vertical* subgroup if there exists a non-negative integer m such that $V_{p,n}(G,A) = 0$ for all $n \geqq m$.

Definition 2.5 For every non-negative integer n, we define the *n-th p-defect* of a subgroup A of an arbitrary abelian group G to be the vector space

$$D_{p,n}(G,A) = \frac{p^n(G/A)[p]}{(p^nG[p]+A)/A}.$$

Lemma 2.6 *Let H be a proper p-pure subgroup of an arbitrary abelian group G containing a subgroup A of G. Then, for all $n \geqq 0$,*

(1) $p^n(G/A)[p] = p^n(H/A)[p] + \frac{p^nG[p]+A}{A}$ *and*

(2) $p^n(H/A)[p] \cap \frac{p^nG[p]+A}{A} = \frac{p^nH[p]+A}{A}$.

In particular, if A is p-vertical in G and if H is a p-pure hull of A in G, then we have

$$p^n(G/A)[p] = p^n(H/A)[p] \oplus \frac{p^nG[p]+A}{A}.$$

for all $n \geqq 0$.

Proof. (1) Let $p^ng+A \in p^n(G/A)[p]$. Then $p^{n+1}g \in A \cap p^{n+1}G \subseteq p^{n+1}H$. Hence there exists $h \in H$ such that $p^ng - p^nh \in p^nG[p]$ and so $p^ng + A \in p^n(H/A)[p] + \frac{p^nG[p]+A}{A}$.

(2) Let $x + A \in p^n(H/A)[p] \cap \frac{p^nG[p]+A}{A}$. Then there exist $h \in H$ and $p^ng \in p^nG[p]$ such that $p^{n+1}h \in A$ and $x + A = p^nh + A = p^ng + A$. Since $p^n(h-g) \in H \cap p^nG = p^nH$, we have $p^ng = p^n(h-h_0)$ for some $h_0 \in H$. Hence $x + A = p^n(h-h_0) + A \in \frac{p^nH[p]+A}{A}$.

Suppose that A is p-vertical in G and H is a pure hull of A in G. By Proposition 2.3, A is p-vertical in H. Hence $H[p] = A[p]$. By (2), the assertion holds. □

Using Lemma 2.6(1) and the Dedekind short exact sequence, we have:

Proposition 2.7 *Let H be a proper p-pure subgroup of an arbitrary abelian group G containing a subgroup A of G. Then*

$$D_{p,n}(G,A) \cong D_{p,n}(H,A)$$

for all $n \geqq 0$.

Proof. We have $p^n(G/A)[p] \cap H/A = p^n(H/A)[p]$ and $\frac{p^nG[p]+A}{A} \cap H/A =$

$\frac{p^n H[p]+A}{A}$. By Lemma 2.6 and the Dedekind short exact sequence, we have

$$D_{p,n}(G, A) \cong D_{p,n}(H, A).$$

□

Using the concept of n-th p-defect, we have a characterization of p-pure subgroups as follows:

Proposition 2.8 [5, Proposition 3.2] *A subgroup A is a p-pure subgroup of an arbitrary abelian group G if and only if $D_{p,n}(G, A) = 0$ for all $n \geqq 0$.*

Let G be a p-group and A a subgroup of G. Then $A[p]$ is said to be dense in $G[p]$ if $G[p] = A[p]+p^n G[p]$ for all $n \geqq 0$. Now, a subsocle $A[p]$ of an arbitrary abelian group G is said to be p-dense in $G[p]$ if $G[p] = A[p]+p^n G[p]$ for all $n \geqq 0$. For a p-dense subsocle $A[p]$ of $G[p]$, we have:

Lemma 2.9 *Let G be an arbitrary abelian group and A a subgroup of G. If $A[p]$ is p-dense in G, then A is a kernel of p-purity in G.*

Proof. Let N be a neat hull of A in G. We show that N is p-vertical in G. Since

$$p^n G[p] \subseteq (A \cap p^n G[p]) + p^{n+1} G[p] \subseteq (N \cap p^n G[p]) + p^{n+1} G[p]$$

and

$$p^n G[p] \subseteq A + p^{n+1} G \subseteq N + p^{n+1} G,$$

we have $N_G^n(p) = N_n^G(p)$ for all $n \geqq 0$. Hence N is p-vertical in G. By [10, Proposition 2.6], N is p-pure in G. □

Such a neat hull N of A in G in the proof of Lemma 2.9 is not necessarily a p-pure hull of A. If N/A is not a p-group, then there exists a smaller one than N. However, Proposition 1.8 guarantees that A is p-purifiable in G.

Now we determine a subgroup A of an arbitrary abelian group G when a subgroup A is a kernel of p-purity in G. Before we do this, we need four lemmas.

Lemma 2.10 *Let G be an arbitrary abelian group and A a subgroup of G. $A[p]$ is p-dense in G if and only if*

$$\frac{p^n G[p] + A}{A} = \frac{p^{n+1} G[p] + A}{A}$$

for all $n \geqq 0$.

Lemma 2.11 *Let G be an arbitrary abelian group and A a subgroup of G. For an integer $n \geqq 0$, $V_{p,n}(G,A) = 0$ if and only if*

$$\frac{p^n G[p] + A}{A} \cap p^{n+1}(G/A)[p] = \frac{p^{n+1}G[p] + A}{A}.$$

Lemma 2.12 *Let G be an arbitrary abelian group and A a subgroup of G. For non-negative integers m and t, we have*

$$D_{p,m+t}(G,A) \cong D_{p,t}(p^m G, A \cap p^m G).$$

Proof. Let $p_G^{-1}A = \{g \in G \mid pg \in A\}$. Note that

$$D_{p,m+t}(G,A) \cong \frac{p_G^{-1}A \cap (p^{m+t}G + A)}{p^{m+t}G[p] + A}$$

and

$$D_{p,t}(p^m G, p^m G \cap A) \cong \frac{p_G^{-1}A \cap (p^{m+t}G + A) \cap p^m G}{(p^{m+t}G[p] + A) \cap p^m G}.$$

Note that $(p_G^{-1}A \cap (p^{m+t}G + A) \cap p^m G) + (p^{m+t}G[p] + A) = p_G^{-1}A \cap (p^{m+t}G + A)$. By the Dedekind short exact sequence, we have

$$D_{p,m+t}(G,A) \cong D_{p,t}(p^m G, A \cap p^m G).$$

□

Lemma 2.13 *Let G be an arbitrary abelian group and A a subgroup of G. Let B be a subgroup of G such that A is essential in B. If A is a kernel of p-purifiability in G, then B is p-purifiable in G. In particular, if A is a kernel of p-purity in G, then B is p-vertical in G.*

Proof. Let N be a neat hull of B in G. Then N is a neat hull of A in G. By hypothesis, there exists a p-pure hull H of N in G. Then we have $p^m H[p] \subseteq N[p] = A[p]$ for some integer $m \geqq 0$. Since H/B is torsion, by Proposition 1.8, B is p-purifiable in G. If A is a kernel of p-purity, then N is p-pure in G. Hence B is p-vertical in G. □

Theorem 2.14 *Let G be an arbitrary abelian group and A a subgroup of G. A is a kernel of p-purity in G if and only if, either*

(1) *$A[p]$ is p-dense in $G[p]$, or*

(2) *there exists a non-negative integer k such that* $\frac{G[p]+A}{A} = \frac{p^k G[p]+A}{A} \neq \frac{p^{k+1}G[p]+A}{A}$ *and $D_{p,n}(G,A) = 0$ for all $n > k$.*

Proof. ($\Rightarrow$) If $A[p]$ is not p-dense in $G[p]$, then, by Lemma 2.10, there exists a non-negative integer k such that

$$\frac{G[p]+A}{A} = \frac{p^kG[p]+A}{A} \neq \frac{p^{k+1}G[p]+A}{A}.$$

Suppose that $D_{p,k+1}(G,A) \neq 0$. Then there exists $x \in G[p]$ such that $x \notin p^{k+1}G[p] + A[p]$ and there exists $p^{k+1}g + A \in p^{k+1}(G/A)[p]$ such that $g \in G$ and $p^{k+1}g + A \notin \frac{p^{k+1}G[p]+A}{A}$.

Let $K = \langle p^{k+1}g + x, A\rangle$. We show that $K[p] = A[p]$. Let $t(p^{k+1}g+x) + a \in K[p]$ such that $a \in A$ and t is an integer. Without loss of generality, we may assume that $(t,p) = 1$ and $p^{k+1}g + A \in \frac{G[p]+A}{A}$. By hypothesis, we have $p^{k+1}g + A \in \frac{p^kG[p]+A}{A}$. By Lemma 2.11, we have $p^{k+1}g + A \in \frac{p^{k+1}G[p]+A}{A}$. This is a contradiction. Hence $K[p] = A[p]$. Since K/A is a p-group, we have $K[q] = A[q]$ for every prime q. Let N be a neat hull of A in G containing K. Note that $p^{k+1}g + N \in p^{k+1}(G/N)[p]$. On the other hand, if $p^{k+1}g + N \in \frac{p^{k+1}G[p]+N}{N}$, then $p^{k+1}g = p^{k+1}g_0 + y$ for some $p^{k+1}g_0 \in p^{k+1}G[p]$ and $y \in N$. Since $x = (p^{k+1}g + x) - (p^{k+1}g_0 + y)$, we have $p^{k+1}g + x - y \in N[p] = A[p]$. Hence $x \in p^{k+1}G[p] + A[p]$. This is a contradiction. Hence $p^{k+1}g + N \notin \frac{p^{k+1}G[p]+N}{N}$. By Proposition 2.8, N is not p-pure in G. Therefore $D_{p,k+1}(G,A) = 0$. Moreover, By Lemma 2.13, A is p-vertical in G. Hence, by Lemma 2.11, we have $D_{p,n}(G,A) = 0$ for all $n > k$.

($\Leftarrow$) If $A[p]$ is p-dense in $G[p]$, then A is a kernel of p-purity by Lemma 2.9. Suppose that the condition (2) holds. Let N be a neat hull of A in G. By Lemma 2.12, $A \cap p^{k+1}G$ is p-pure in $p^{k+1}G$. Since $A \cap p^{k+1}G$ is essential in $N \cap p^{k+1}G$, $\frac{N\cap p^{k+1}G}{A\cap p^{k+1}G}[p] = 0$ and so $D_{p,n}(G,N) = 0$ for all $n > k$. By Lemma 2.11, $V_{p,n}(G,N) = 0$ for all $n \geqq k$. Moreover, since $\frac{G[p]+N}{N} = \frac{p^kG[p]+N}{N}$, we have $V_{p,n}(G,N) = 0$ for all $n \geqq 0$. By [10, Proposition 2.6], N is p-pure in G. □

By Megibben's result Theorem 1.11 and Theorem 2.14, we state that a subgroup A of an arbitrary abelian group G is a kernel of purity in G if and only if, for every prime p, A is kernel of p-purity in G. By Lemma 2.13, kernels of p-purity are p-purifiable. The condition for a subgroup to be a kernel of purity is stronger than the condition for it to be p-purifiable. Under a weaker condition, we have the following results.

Corollary 2.15 *Let m be a non-negative integer. Suppose that either*

(1) $\frac{p^nG[p]+A}{A} = \frac{p^{n+1}G[p]+A}{A}$ *for all $n \geqq m$, or*

(2) $D_{p,n}(G, A) = 0$ *for all $n \geqq m$.*

Then A is p-purifiable in G.

Proof. (1) By hypothesis, $p^nG[p] \subseteq A[p] + p^{n+1}G[p]$ for all $n \geqq m$. Hence $A[p] \cap p^mG$ is p-dense in $p^mG[p]$. By Lemma 2.9, $A[p] \cap p^mG$ is p-purifiable in p^mG. By [10, Theorem 4.1], A is p-purifiable in G.

(2) By hypothesis and Lemma 2.12, $A[p] \cap p^mG$ is p-pure in p^mG. We also use [10, Theorem 4.1] to prove that A is p-purifiable in G. □

3. Kernels of Purifiability

First of all, we state the relation between kernels of p-purifiability and kernels of purifiability.

Proposition 3.1 *Let G be an arbitrary abelian group and A a subgroup of G. A is a kernel of purifiability in G if and only if, for every prime p, A is a kernel of p-purifiability in G.*

Proof. ($\Rightarrow$) Let N be a neat hull of A in G and H a pure hull of N in G. Then H is p-pure in G. Moreover, by Proposition 1.6, H/N is torsion and there exists a non-negative integer m such that $p^mH[p] \subseteq N[p]$. By Proposition 1.8, N is p-purifiable in G.

($\Leftarrow$) By [10, Theorem 1.12], it is immediate. □

From Proposition 3.1, it suffices to characterize kernels of p-purifiability. Suppose that a subgroup A of an arbitrary abelian group G is a kernel of p-purifiability in G. If A is neat in G, then we may assume A is not p-vertical in G, otherwise A would be p-pure in G. Then there exists a non-negative integer m such that $A \cap p^mG$ is p-vertical in p^mG. Hence we must consider the characterization of p-purifiability with no condition. Giving an answer to this problem is as difficult as giving a complete characterization of purifiable subgroups. To make progress on this problem, we consider the case where A is p-vertical in G. Before we give the main Theorem of this article, we establish various properties.

Proposition 3.2 *Let G be an arbitrary abelian group and A a subgroup of G. For a p-vertical subgroup A of G, we have:*

(1) *if A is a kernel of p-purifiability in G, then $A \cap p^n G$ is a kernel of p-purifiability in $p^n G$ for all $n \geqq 0$;*

(2) *if $A \cap p^m G$ is a kernel of p-purifiability in $p^m G$ for some integer $m \geqq 0$, then A is a kernel of p-purifiability in G.*

Proof. (1) Let L_n be a neat hull of $A \cap p^n G$ in $p^n G$. Let $L = L_n + A$. Then $L \cap p^n G = (L_n + A) \cap p^n G = L_n + (A \cap p^n G) = L_n$.

We prove that $L[q] = A[q]$ for every prime q. Let $x \in L[p]$. Then we can write $x = a + p^n g$, where $a \in A$ and $p^n g \in L_n$ with $g \in G$. Since A is p-vertical in G, we have $x = a + p^n g \in (A + p^n G)[p] = A[p] + p^n G[p]$ by [10, Proposition 2.7]. Then $x = a_0 + p^n g_0$ for some $a_0 \in A[p]$ and $p^n g_0 \in p^n G[p]$. Since $p^n g_0 = x - a_0 \in (L \cap p^n G)[p] = L_n[p] = (A \cap p^n G)[p] \subseteq A[p]$, it follows that $x \in A[p]$. For a prime $q \neq p$, let $y_q \in L[q]$. Then we have $y_q = a_q + z_q$, where $a_q \in A$ and $z_q \in L_n$. Since $qa_q = -qz_q \in p^n G$ and $G/p^n G$ is a p-group, we have $a_q \in p^n G$. Hence $y_q \in L \cap p^n G[q] = L_p[q] \subseteq A[q]$. Therefore $L[q] = A[q]$ for every prime q. Since $L/A = \frac{L_n + A}{A} \cong \frac{L_n}{A \cap L_n}$ and $\frac{L_n}{A \cap p^n G}$ is torsion, L/A is torsion.

Let M be a neat hull of L in G. By Proposition 1.9, M becomes a neat hull of A in G. By hypothesis, there exists a p-pure hull H of M in G. Since H/L is torsion and $p^m H[p] \subseteq M[p] = L[p]$ for some integer $m \geqq 0$, L is p-purifiablity in G by Proposition 1.8. By [10, Theorem 4.1], $L_n = L \cap p^n G$ is p-purifiable in $p^n G$. Hence $A \cap p^n G$ is a kernel of p-purifiable in $p^n G$.

(2) Let N be a neat hull of A in G. Then, for every prime q, we have $(N \cap p^m G)[q] = N[q] \cap p^m G = A[q] \cap p^m G = (A \cap p^m G)[q]$. Let L' be a neat hull of $N \cap p^m G$ in $p^m G$. Then L' becomes a neat hull of $A \cap p^m G$. By hypothesis, there exists a p-pure hull H of L in $p^m G$. Then $\frac{H}{N \cap p^m G}$ is torsion and $p^r H[p] \subseteq L'[p] = (N \cap p^m G)[p]$ for some integer $r \geqq 0$. By [10, Theorem 4.1] and Proposition 1.8, N is p-purifiable in G. Hence A is a kernel of p-purifiability in G. □

Lemma 3.3 *Let G be an arbitrary abelian group, A a subgroup of G, and H a p-pure subgroup of G containing A such that $p^k G[p] \subseteq H$ for some integer $k \geqq 0$. If G/A is a p-group, then $p^k G \subseteq H$.*

Proof. Let $p^k g \in p^k G \setminus H$ such that $g \in G$ and $p^{k+1} g \in H$. Since $p^{k+1} g \in H \cap p^{k+1} G = p^{k+1} H$, there exists $h \in H$ such that $p^k g - p^k h \in p^k G[p] \subseteq H$. Hence $p^k g \in H$. This is a contradiction. Therefore $p^k G \subseteq H$. □

Lemma 3.4 *Let G be an arbitrary abelian group and A a subgroup of*

G. Suppose that there exists an increasingly sequence of positive integers $n_1 < n_2 < \cdots < n_i < \ldots$ *such that*

$$p^{n_i}G[p] \neq (A \cap p^{n_i}G)[p] + p^{n_i+1}G[p]$$

for all $i \geqq 1$, *and there exists a subgroup* K *of* G *containing* A *and an increasingly sequence of positive integers* $m_1 < m_2 < \cdots < m_j < \ldots$ *such that*

(1) $K[p] = A[p]$ *and*
(2) $p^{m_j}(K/A)[p] \neq p^{m_j+1}(K/A)[p]$.

Then there exists a subgroup L *of* G *containing* A *such that*

(1) $L[q] = A[q]$ *for every prime* q,
(2) L/A *is a* p*-group, and*
(3) L *is not eventually* p*-vertical in* G.

Proof. By hypothesis, we can choose an increasingly sequence of positive integers $t_1 < t_2 < \cdots < t_n < \ldots$ such that

(1) $p^{t_{2k-1}}G[p] \neq (A \cap p^{t_{2k-1}}G)[p] + p^{t_{2k-1}+1}G[p]$ and
(2) $\{x_{2k} + A \mid k = 1, 2, \ldots\}$ is linearly independent in $(K/A)[p]$ such that $h_p(x_{2k} + A) = t_{2k}$.

Let $s_{2k-1} \in p^{t_{2k-1}}G[p] \setminus ((A \cap p^{t_{2k-1}}G)[p] + p^{t_{2k-1}+1}G[p])$ for $k = 1, 2, \ldots$ and

$$L = \langle A, s_{2k-1} + x_{2k} \mid k = 1, 2, \ldots \rangle.$$

Let $y \in L[p]$. Then we can write $y = a + \Sigma_k \alpha_k (s_{2k-1} + x_{2k})$, where $a \in A$ and $\alpha_k \in \mathbf{Z}$. Since $y - \Sigma_k \alpha_k s_{2k-1} = a + \Sigma_k \alpha_k x_{2k} \in K[p] = A[p]$, we have $\Sigma_k \alpha_k x_{2k} \in A$. Therefore p divides α_k. Hence $L[p] = A[p]$. Since L/A is a p-group, $L[q] = A[q]$ for every prime q. Next, suppose that $V_{p,t_{2k-1}}(G, L) = 0$. Let $y_k = s_{2k-1} + x_{2k}$. Then we have

$$\begin{aligned} s_{2k-1} &= y_k - x_{2k} \in (L + p^{t_{2k-1}+1}G) \cap p^{t_{2k-1}}G[p] \\ &= (A \cap p^{t_{2k-1}}G)[p] + p^{t_{2k-1}+1}G[p]. \end{aligned}$$

This is a contradiction. Hence L is not eventually p-vertical in G. □

Lemma 3.5 *Let* G *be an arbitrary abelian group such that* $G = D \oplus B$, *where* D *is a divisible* p*-group of finite rank and* B *is unbounded.*

(1) *If* $p^\omega B[p] \neq 0$, *then there exists a subgroup* L *of* G_p *such that* $L[p] = D[p]$ *and* L *is not* p*-purifiable in* G.
(2) *If* $p^\omega B[p] = 0$ *and if* N *is a subgroup of* $T(G)$ *such that* $\dim N[p] < \infty$, *then there exists a* p*-pure subgroup* K *of* $T(G)$ *containing* $N + D$ *such*

that $p^k K_p \subseteq D$ for some integer $k \geqq 0$.

Proof. (1) Let $x \in p^\omega B[p]$ and $D = D' \oplus E$, where D' is a subgroup of D and $E[p] = \langle d \rangle$. Then there exists $a \in D$ such that $pa = d$. Let $L = D' \oplus \langle a + x \rangle$. Then L is a p-group and $L[p] = D[p]$. Suppose that L is p-purifiable in G. Let H be a p-pure hull of L in G. By Proposition 1.6 and [6, Theorem 2], H is a p-group and $H = M \oplus N$, where $M[p] = L[p] = D[p]$ and $p^m N = 0$ for some integer $m \geqq 0$. Since $a + x \in H^1 \subseteq M$ and $D' = p^m D' \subseteq p^m M = p^m H$, we have $L \subseteq M$ and hence $H = M$. Then H is divisible and $H = D$. Since $a \in H$, it follows that $x \in H = D$. This is a contradiction. Therefore L is not p-purifiable in G.

(2) By hypothesis, we may assume that N is a p-group. Since $\dim N[p] < \infty$, $\frac{N+D}{D}$ is finite. Moreover, since $B \cong G/D \supseteq \frac{N+D}{D}$ and $p^\omega B[p] = 0$, we have $\frac{N+D}{D} \cap p^m(G/D) = 0$ for some integer $m \geqq 0$. Hence there exists a pure hull K/D of $\frac{N+D}{D}$ in $(G/D)_p$ such that $p^t K \subseteq D$ for some integer $t \geqq 0$. Then K is p-pure in G. □

Main Theorem 3.6 *Let G be an arbitrary abelian group and A a p-vertical subgroup of G. Then A is a kernel of p-purifiability in G if and only if one of the following three conditions holds:*
(1) *$A \cap p^m G$ is p-dense in $p^m G$ for some $m \geqq 0$;*
(2) *$D_{p,m+t}(G, A) = 0$ for some intger $m \geqq 0$ and all $t \geqq 0$;*
(3) *there exist an integer $m \geqq 0$ and subgroups H, K of G such that*

$$\frac{p^m G}{A \cap p^m G} = \frac{H}{A \cap p^m G} \oplus \frac{K}{A \cap p^m G},$$

where $\frac{H}{A \cap p^m G}$ is a divisible subgroup of $(\frac{G}{A \cap p^m G})_p$ of finite rank and

$$\frac{K}{A \cap p^m G}[p] = \frac{p^m G[p] + (A \cap p^m G)}{A \cap p^m G}$$

such that $p^\omega(\frac{K}{A \cap p^m G})[p] = 0$.

Proof. ($\Rightarrow$) Suppose that both of (1) and (2) are not satisfied. Then there exists an increasingly sequence of positive integers $n_1 < n_2 < \cdots < n_k < \ldots$ such that

$$p^{n_i} G[p] \neq (A \cap p^{n_i} G)[p] + p^{n_i+1} G[p]$$

for all $i \geqq 1$. By Lemma 2.13, A is p-purifiable in G. Let H be a p-pure hull of A in G. If $p^m(H/A)[p] = 0$ for some $m \geqq 0$, we have $D_{p,m+t}(G, A) = 0$

for all $t \geqq 0$. Hence we may assume that $p^n(H/A)[p] \neq 0$ for every $n \geqq 0$.

Next, we prove that there exists a non-negative integer m such that

$$p^m(G/A) = p^m(H/A) \oplus K'/A,$$

where $p^m(H/A)$ is a divisible p-group of finite rank and K'/A is a direct summand of $p^m(G/A)$. By Lemma 2.6, for all $n \geqq 0$, we have

$$p^n(G/A)[p] = p^n(H/A)[p] \oplus \frac{p^n G[p] + A}{A}.$$

Let $r_1 < r_2 < \cdots < r_i < \dots$ be an increasingly sequence of positive integers such that $p^{r_i}(H/A)[p] \neq p^{r_{i+1}}(H/A)[p]$. By Lemma 3.4, there exists a subgroup L of G containing A such that $L[q] = A[q]$ for every prime q, L/A is a p-group, and L is not eventually p-vertical in G. By Proposition 2.4, L is not p-purifiable in G. On the other hand, since every neat hull of L in G is a neat hull of A in G, L is a kernel of p-purifiability in G. By Lemma 2.13, L is p-purifiable in G. This is a contradiction. Hence there exists a non-negative integer m such that $p^{m+k}(H/A)[p] = p^m(H/A)[p]$ for all $k \geqq 0$. Since $p^m(H/A)$ is a p-group and $p^m(H/A)$ is pure in $p^m(G/A)$, $p^m(H/A)$ is divisible. Hence we have

$$p^m(G/A) = p^m(H/A) \oplus K'/A,$$

where K'/A is a direct summand of $p^m(G/A)$. Suppose that $\dim(p^m(H/A)[p]) = \infty$. Then we can write

$$p^m(H/A) = D[p] \oplus \left(\bigoplus_{i=1}^{\infty} \langle d_i + A \rangle\right),$$

where D is a divisible subgroup of $p^m(H/A)$ and $d_i + A \in p^m(H/A)[p]$. As mentioned above, we can choose a set $\{b_i \in G[p] \mid i \geqq 0\}$ such that

$$b_i \in p^{n_i}G[p] \setminus (A \cap p^{n_i}G)[p] + p^{n_i+1}G[p].$$

Let $M = \langle A, d_i + b_i \mid i \geqq 1 \rangle$ and $x \in M[p]$. Then we have $x = a + \Sigma_i \alpha_i(d_i + b_i)$ for some $a \in A$ and integers α_i. Since $a + \Sigma_i \alpha_i d_i \in H[p] = A[p]$, p divides α_i and so $M[p] = A[p]$. If $h_p^{G/A}(d_i + b_i + A) > n_i$, then $b_i + A = p^{n_i+1}g + A$ for some $g \in G$. Then, since A is p-vertical in G, we have $b_i = a + p^{n_i+1}g \in (A + p^{n_i+1}G) \cap p^{n_i}G[p] = (A \cap p^{n_i}G)[p] + p^{n_i+1}G[p]$. This is a contradiction. Hence $h_p^{G/A}(d_i + b_i + A) = n_i$. Let

$$L' = \langle A, b_{2k-1} + d_{2k} + b_{2k} \mid k = 1, 2, \dots \rangle.$$

By a similar proof of Lemma 3.4, L' is not eventually p-vertical in G, L'/A is a p-group, and $L'[q] = A[q]$ for every prime q. Similarly, this is a contradiction. Hence $\dim(p^m(H/A)[p]) < \infty$.

Note that $p^m H$ is a p-pure hull of $A \cap p^m G$ in $p^m G$ and $A \cap p^m G$ is p-vertical in $p^m G$. By Lemma 2.6, we have

$$\frac{p^m G}{A \cap p^m G}[p] = \frac{p^m H}{A \cap p^m G}[p] \oplus \frac{p^m G[p] + (A \cap p^m G)}{A \cap p^m G}.$$

Since $\frac{p^m H}{A \cap p^m G} \cong \frac{p^m H + A}{A} = p^m(H/A)$ is a divisible p-group, there exists a subgroup K of G such that

$$\frac{p^m G}{A \cap p^m G} = \frac{p^m H}{A \cap p^m G} \oplus \frac{K}{A \cap p^m G},$$

where $\frac{K}{A \cap p^m G}[p] = \frac{p^m G[p] + (A \cap p^m G)}{A \cap p^m G}$. Next, we prove that $p^\omega(\frac{K}{A \cap p^m G})[p] = 0$.

Suppose that $p^\omega(\frac{K}{A \cap p^m G})[p] \neq 0$. By Lemma 3.5(1), there exists a subgroup $\frac{L''}{A \cap p^m G}$ of $(\frac{p^m G}{A \cap p^m G})_p$ such that $(\frac{L''}{A \cap p^m G})[p] = (\frac{p^m H}{A \cap p^m G})[p]$ and $\frac{L''}{A \cap p^m G}$ is not p-purifiable in $\frac{p^m G}{A \cap p^m G}$. Since $\frac{L''}{A \cap p^m G}$ is a p-group, $L''[q] = (A \cap p^m G)[q]$ for every prime $q \neq p$. Let $x \in L''[p]$. Since $x + A \in (\frac{p^m H}{A \cap p^m G})[p] \cap \frac{p^m G[p] + (A \cap p^m G)}{A \cap p^m G} = 0$, we have $L''[p] = (A \cap p^m G)[p]$. Hence $L''[q] = (A \cap p^m G)[q]$ for every prime q. Since a neat hull of L'' implies one of $A \cap p^m G$, L'' is a kernel of p-purifiability in $p^m G$ by Proposition 3.2(1). Therefore L'' is p-purifiable in $p^m G$ by Lemma 2.13. Let M' be a p-pure hull of L'' in $p^m G$. By Proposition 1.8, $A \cap p^m G$ is p-purifiable in M'. Let N be a p-pure hull of $A \cap p^m G$ in M'. Then

$$p^k M[p] \subseteq L''[p] = (A \cap p^m G)[p] \subseteq N$$

for some integer $k \geqq 0$. Since $\frac{M'}{A \cap p^m G}$ is a p-group, we have $p^k M \subseteq N$ by Lemma 3.3. Then $(\frac{L''}{A \cap p^m G})[p] \subseteq p^\omega(\frac{p^m G}{A \cap p^m G})[p] \cap (\frac{M'}{A \cap p^m G})[p] = p^\omega(\frac{M'}{A \cap p^m G})[p] = \cap_n(\frac{p^n M' + (A \cap p^m G)}{A \cap p^m G})[p] \subseteq (\frac{N}{A \cap p^m G})[p]$ and $(\frac{p^m G}{A \cap p^m G})[p] = (\frac{p^m H}{A \cap p^m G})[p] \oplus (\frac{K}{A \cap p^m G})[p] = (\frac{L''}{A \cap p^m G})[p] \oplus (\frac{K}{A \cap p^m G})[p] = (\frac{N}{A \cap p^m G})[p] \oplus (\frac{K}{A \cap p^m G})[p]$. Since $\dim((H/A)[p]) < \infty$, we have $(\frac{L''}{A \cap p^m G})[p] = (\frac{N}{A \cap p^m G})[p]$. Hence $p^k(\frac{M'}{A \cap p^m G})[p] \subseteq (\frac{N}{A \cap p^m G})[p] = (\frac{L''}{A \cap p^m G})[p]$ and so $\frac{L''}{A \cap p^m G}$ is p-purifiable in $\frac{p^m G}{A \cap p^m G}$ by Proposition 1.8. This is a contradiction. Hence $p^\omega(\frac{K}{A \cap p^m G})[p] = 0$.

($\Leftarrow$) By Proposition 2.8, Lemma 2.9, and Proposition 3.2(2), we may assume that

$$(G/A) = (H/A) \oplus (K/A),$$

where H/A is a divisible subgroup of $(G/A)_p$ with $\dim((H/A)[p]) < \infty$ and K/A is a subgroup of G/A such that $p^{\omega}(K/A)[p] = 0$ and $(K/A)[p] = \frac{G[p]+A}{A}$.

Let N' be a neat hull of A in G. Then

$$(G/A)[p] = (N'/A)[p] \oplus \frac{G[p]+A}{A}.$$

Since N'/A is torsion and $\dim((N'/A)[p]) = \dim((H/A)[p]) < \infty$, by Lemma 3.5(2), there exists a p-pure subgroup M''/A of $T(G/A)$ containing $\frac{N'+H}{A}$ such that $p^k(M''/A)_p \subseteq H/A$ for some integer $k \geqq 0$.

Now we prove that M'' is p-pure in G. Let $pg \in M''$ with $g \in G$. Since M''/A is p-pure in G/A, there exists $x \in M''$ such that

$$g - x + A \in (G/A)[p] = (H/A)[p] \oplus \frac{G[p]+A}{A}.$$

Since $g - x = h + g_0 + a$ for some $h \in H$, $g_0 \in G[p]$, and $a \in A$, we have $g - g_0 = h + x + a \in M''$. Hence $pg = p(h + x + a) \in pM''$. Suppose by induction that $M'' \cap p^nG = p^nM''$. Let $p^{n+1}g \in M''$ with $g \in G$. Since M''/A is p-pure in G/A, there exists $x \in M''$ such that $p^{n+1}g - p^{n+1}x \in A$. By Lemma 2.6, we have

$$\begin{aligned} p^n(G/A)[p] &= p^n(H/A)[p] \oplus \frac{p^nG[p]+A}{A} \\ &= (H/A)[p] \oplus \frac{p^nG[p]+A}{A}. \end{aligned}$$

Then there exist $h \in H$, $p^ng_0 \in G[p]$, and $a \in A$ such that $p^ng - p^nx = h + p^ng_0 + a$. Since $p^n(g - g_0) = p^nx + h + a \in M'' \cap p^nG = p^nM''$, we have $p^{n+1}g = p^{n+1}(g - g_0) = p^{n+1}x'$ for some $x' \in M''$. Hence M'' is p-pure in G. Since

$$\frac{p^kM''[p]+A}{A} \subseteq p^k(M''/A)[p] \subseteq H/A,$$

we have

$$\frac{p^kM''[p]+A}{A} \subseteq H/A \cap \frac{p^kG[p]+A}{A} = 0.$$

Hence $p^k M''[p] \subseteq A \subseteq N'$. Since M''/A is torsion, M''/N is torsion. By Proposition 1.8, N' is p-purifiable in G. □

By Proposition 3.1 and the Main Theorem, we have:

Corollary 3.7 *Let G be an arbitrary abelian group and A a subgroup of G. Suppose that A is p-vertical in G for every prime p. Then A is a kernel of purifiability in G if and only if, for every prime p, one of the three conditions in the Main Theorem* 3.7 *holds.* □

Comparing Corollary 3.7 with Megibben's result Theorem 1.11, we can see easily that the condition for a subgroup to be a kernel of purifiability is weaker than the condition for it to be a kernel of purity.

We occasionally use the expression "a maximal p-vertical subgroup M of a p-vertical subgroup A in an arbitrary abelian group G" meaning implicitely that the subgroup M is maximal among the p-vertical subgroups of G containing A having properties that $M[p] = A[p]$ and M/A is a p-group. The existence of maximal p-vertical subgroups for every p-vertical subgroup of G is guaranteed by [10, Proposition 3.1].

Definition 3.8 Let G be an arbitrary abelian group and A a subgroup of G. Suppose that A is eventually p-vertical in G such that there exists a non-negative integer m such that $V_{p,n}(G, A) = 0$ for all $n \geqq m$. A is said to be strongly p-purifiable in G if all maximal p-vertical extensions of $A \cap p^m G$ in $p^m G$ are p-pure in $p^m G$.

Lemma 3.9 *Let G be an arbitrary abelian group and A a p-vertical subgroup of G. If A is a kernel of p-purifiability in G, then every maximal p-vertical subgroup of A is a p-pure hull of A in G.*

Proof. Let M be a maximal p-vertical subgroup of G containing A. Since M/A is a p-group, we have $M[q] = A[q]$ for every prime $q \neq p$. Let L be a neat hull of M in G. Then L becomes a neat hull of A in G. By hypothesis, there exists a p-pure hull H of L in G. Since H/L is a p-group and L/M is torsion, H/M is torsion. Moreover, we have $p^m H[p] \subseteq L[p] = M[p]$ for some integer $m \geqq 0$. By Proposition 1.8, M is p-purifiable in G. Since M is maximal p-vertical in G, M is p-pure in G. By Proposition 1.5, M is a p-pure hull of A in G. □

From Proposition 3.2 and Lemma 3.9 combined, it immediately follows:

Theorem 3.10 *Let G be an arbitrary abelian group and A a p-vertical subgroup of G. If A is a kernel of p-purifiability in G, then A is strongly p-purifiable in G.* □

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