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Scattering theory and large time asymptotics of solutions to the Hartree type equations with a long range potential

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Abstract. We study the scattering problem and asymptotics for large time of solutions to the Hartree type equations

$$iu_t = -\frac{1}{2}\Delta u + f(|u|^2)u, \quad (t,x) \in \mathbf{R}\times\mathbf{R}^n, \quad u(0,x) = u_0(x), \quad x \in \mathbf{R}^n, \quad n \geq 1,$$

where the nonlinear interaction term is $f(|u|^2) = V * |u|^2$, $V(x) = \lambda|x|^{-\delta}$, $\lambda \in \mathbf{R}$, $0 < \delta < 1$. We suppose that in the case $n \geq 2$ the initial data $u_0 \in H^{n+2,0} \cap H^{0,n+2}$ and the value $\epsilon = \|u_0\|_{H^{n+2,0}} + \|u_0\|_{H^{0,n+2}}$ is sufficiently small and in one-dimensional case ($n = 1$) we assume that $e^{\beta|x|}u_0 \in L^2$, $\beta > 0$ and the value $\epsilon = \|e^{\beta|x|}u_0\|_{L^2}$ is sufficiently small. Then we prove that there exists a unique final state $\hat{u}_+ \in H^{n+2,0}$ such that the asymptotics

$$u(t,x) = \frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\left(\frac{x}{t}\right) + O(1+t^{1-2\delta})\right) + O(t^{-n/2-\delta})$$

is true as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$ with the following decay estimate $\|u(t)\|_{L^p} \leq C\epsilon t^{\frac{n}{p}-\frac{n}{2}}$, for all $t \geq 1$ and for every $2 \leq p \leq \infty$. Furthermore we show that for $\frac{1}{2} < \delta < 1$ there exists a unique final state $\hat{u}_+ \in H^{n+2,0}$ such that

$$\|u(t) - \exp\left(-\frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\left(\frac{x}{t}\right)\right)U(t)u_+\|_{L^2} = O(t^{1-2\delta})$$

for all $t \geq 1$, and the asymptotic formula

$$u(t,x) = \frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f(|\hat{u}_+|^2)\left(\frac{x}{t}\right)\right) + O(t^{-n/2+1-2\delta}),$$

is valid as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$, where $\hat{\phi}$ denotes the Fourier transform of the function ϕ, $H^{m,s} = \{\phi \in \mathcal{S}'; \|\phi\|_{m,s} = \|(1+x^2)^{s/2}(1-\Delta)^{m/2}\phi\|_{L^2} < \infty\}$, $m, s \in \mathbf{R}$. Analogous results are obtained for the following NLS equation

$$iu_t = -\frac{1}{2}\Delta u + \lambda t^{n-\delta}|u|^2 u$$

with cubic nonlinearity and growing with time coefficient, where $0 < \delta < 1$, $n \geq 1$.

Key words: Scattering theory, Hartree equation, long range potential.

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1. Introduction

This paper is devoted to the study of the asymptotic behavior for large time of small solutions to the Cauchy problem for the Hartree type equation

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta u + f\left(|u|^2\right)u, & (t,x) \in \mathbf{R}\times\mathbf{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where

$$f\left(|u|^2\right) = V * |u|^2 = \int V(x-y)|u|^2(y)dy,$$

$$V(x) = \lambda|x|^{-\delta}, \quad \lambda \in \mathbf{R}, \quad 0 < \delta < n \quad \text{and} \quad n \geq 1.$$

We first survey the previous results without "size restriction" on the data. The Cauchy problem (1.1) was studied in [7], [18] for $0 < \delta < n$ and the global existence, uniqueness and smoothing effect of solutions of (1.1) for $0 < \delta < \min(2, n)$ were shown in [15] by using the space time estimates of the free Schrödinger evolution group and L^2 conservation law. In [14], [17] the time decay of global solutions to (1.1) was obtained when $0 < \delta < n$ and $\lambda > 0$. More precisely, the following time decay estimates

$$\|u(t)\|_{L^{2+2\delta/n}} \leq C(1+|t|)^{-\delta/4} \tag{1.2}$$

for the case $0 < \delta \leq 3/2$, $n \geq 2$ and $\|u(t)\|_{L^p} \leq C(1+|t|)^{\frac{n}{p}-\frac{n}{2}}$, where $2 \leq p < \frac{2n}{2\gamma-n}$ for the case $\frac{3}{2} < \delta < n$, $n \geq 2$ were proved by using the pseudo-conformal conservation law and L^p–L^q time decay estimates of the free Schrödinger evolution group if the initial data $u_0 \in H^{\gamma,0} \cap H^{0,\gamma}$, where $\gamma > n/2$. If the initial data $u_0 \in H^{\gamma,0} \cap H^{0,\gamma}$, where $\gamma > n/2$ are sufficiently small the optimal time decay estimate $\|u(t)\|_{L^p} \leq C(1+|t|)^{\frac{n}{p}-\frac{n}{2}}$ with any $2 \leq p \leq \infty$ was proved for the supercritical values $1 < \delta < n$, for any $\lambda \in \mathbf{R}$. The scattering problem for the Hatree type equation (1.1) with supercritical powers $1 < \delta < n$ was developed in [7], [13], [17]. Equation (1.1) with critical value $\delta = 1$ is known as Hartree equation. For this equation in the three dimensional case the following L^∞ time decay estimate of solutions

$$\|u(t)\|_{L^\infty} \leq C(1+|t|)^{-1/2} \tag{1.3}$$

was obtained in [9] for large initial data $u_0 \in H^{2,0} \cap H^{0,2}$. This result is an improvement of [2]. The estimates (1.2) and (1.3) are not sufficient for the study of the scattering problem. Recently in [10, 12] we obtained the asymptotic behavior for large time of small solutions for (1.1) with critical power $\delta = 1$ and we showed the existence of the modified scattering states. To explain these results more precisely we introduce the following notations and functional spaces. We let $\partial_j = \partial/\partial x_j$, $\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbf{N} \cup \{0\})^n$, $|\alpha| = \sum_{j=1}^n \alpha_j$. And let $\mathcal{F}\phi$ or $\hat{\phi}$ be the Fourier transform of ϕ defined by $\mathcal{F}\phi(\xi) = \frac{1}{(2\pi)^{n/2}}\int e^{-ix\cdot\xi}\phi(x)dx$ and $\mathcal{F}^{-1}\phi(x)$ or $\check{\phi}(x)$ be the inverse Fourier transform of ϕ, i.e. $\mathcal{F}^{-1}\phi(x) = \frac{1}{(2\pi)^{n/2}}\int e^{ix\cdot\xi}\phi(\xi)d\xi$.

We introduce some function spaces. As usually we denote the Lebesgue space as $L^p = L^p(\mathbf{R}^n) = \{\phi \in \mathcal{S}'; \|\phi\|_p < \infty\}$, where $\|\phi\|_p = \left(\int |\phi(x)|^p dx\right)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_\infty = \text{ess. sup}\{|\phi(x)|; x \in \mathbf{R}^n\}$ if $p = \infty$. the weighted Sobolev space is defined by

$$\begin{aligned} H^{m,s} &= H^{m,s}(\mathbf{R}^n) \\ &= \left\{\phi \in \mathcal{S}'; \|\phi\|_{m,s} = \left\|\left(1+x^2\right)^{s/2}(1-\Delta)^{m/2}\phi\right\| < \infty\right\}, \end{aligned}$$

$m, s \in \mathbf{R}$ and the homogeneous Sobolev space is given by

$$\dot{H}^{m,s} = \dot{H}^{m,s}(\mathbf{R}^n) = \left\{\phi \in \mathcal{S}'; \left\||x|^s(-\Delta)^{m/2}\phi\right\|_2 < \infty\right\}$$

with seminorm $\|\phi\|_{\dot{H}^{m,s}} = \left\||x|^s(-\Delta)^{m/2}\phi\right\|_2$. Also we consider the analytic function space

$$\mathcal{H}_\sigma^s = \mathcal{H}_\sigma^s(\mathbf{R}^n) = \left\{\phi \in L^2(\mathbf{R}^n); \left\|(1+p^2)^{s/2}e^{\sigma|p|}\hat{\phi}(p)\right\|_2 < \infty\right\},$$

$s \in \mathbf{R}$, $\sigma > 0$ with a norm $\|\phi\|_{\mathcal{H}_\sigma^s} = \left\|(1+p^2)^{s/2}e^{\sigma|p|}\hat{\phi}(p)\right\|_2$, which can be expressed in the x-representation in terms of the analyticity in the strip $-\sigma \le \operatorname{Im} z_j \le \sigma$, $1 \le j \le n$ via the following norm $\sum_\nu \|\phi(\cdot + i\nu\sigma)\|_{s,0}$, where the sum is over the all possible values of the vector $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, where $\nu_j = 1$ or $\nu_j = -1$, $1 \le j \le n$. Indeed we have the inequality

$$\|\phi\|_{\mathcal{H}_\sigma^s} \le \sum_\nu \|\phi(\cdot + i\nu\sigma)\|_{s,0} \le 2^n\|\phi\|_{\mathcal{H}_\sigma^s}.$$

We let $(\psi, \varphi) = \int \psi(x) \cdot \overline{\varphi}(x)dx$. By $C(I; E)$ we denote the space of continuous functions from an interval I to a Banach space E.

The free Schrödinger evolution group $U(t) = e^{it\Delta/2}$ gives us the solution of the linear Cauchy problem (1.1) (with $f = 0$). It can be represented explicitly in the following manner

$$U(t)\phi = \frac{1}{(2\pi it)^{n/2}} \int e^{i(x-y)^2/2t}\phi(y)dy = \mathcal{F}^{-1}e^{-it\xi^2/2}\mathcal{F}\phi.$$

Note that $U(t) = M(t)D(t)\mathcal{F}M(t)$, where $M = M(t) = \exp\left(\frac{ix^2}{2t}\right)$ and $D(t)$ is the dilation operator defined by $(D(t)\psi)(x) = \frac{1}{(it)^{n/2}}\psi\left(\frac{x}{t}\right)$. Then since $D(t)^{-1} = i^n D\left(\frac{1}{t}\right)$ we have

$$U(-t) = \overline{M}\mathcal{F}^{-1}D(t)^{-1}\overline{M} = \overline{M}i^n\mathcal{F}^{-1}D\left(\frac{1}{t}\right)\overline{M},$$

where $\overline{M} = M(-t) = \exp\left(-\frac{ix^2}{2t}\right)$.

Different positive constants might be denoted by the same letter C. In what follows we consider the positive time t only since for the negative one the results are analogous.

In paper [10] we proved that for any initial data $u_0 \in H^{\gamma,0} \cap H^{0,\gamma}$, where $\gamma > \frac{n}{2}$, $n \geq 1$, such that the norm $\|u_0\|_{\gamma,0} + \|u_0\|_{0,\gamma} = \epsilon$ is sufficiently small there exists a unique global solution $u \in C\left(\mathbf{R}; H^{\gamma,0} \cap H^{0,\gamma}\right)$ of the Cauchy problem (1.1) with $\delta = 1$ such that $\|u(t)\|_\infty \leq C\epsilon(1+|t|)^{-n/2}$. Moreover we proved that there exist unique functions $\Phi \in L^\infty$ and $\hat{u}_+ \in L^\infty \cap L^2$ such that

$$\left\| \mathcal{F}(U(-t)u)(t) \exp\left(i\int_1^t f\left(|\hat{u}(\tau)|^2\right)\frac{d\tau}{\tau}\right) - \hat{u}_+ e^{i\Phi}\right\|_k \leq C\epsilon t^{-\xi} \quad \text{for all } t \geq 1,$$

where $k = 2$ or ∞, and

$$\left\| \int_1^t f\left(|\hat{u}(\tau)|^2\right)\frac{d\tau}{\tau} - f\left(|\hat{u}_+|^2\right)\log t - \Phi \right\|_\infty \leq C\epsilon t^{-\xi\eta} \quad \text{for all } t \geq 1,$$

where $0 < \eta < 2/n$, $\gamma > \frac{n}{2}$ and $0 < \xi < \min\left(1, \frac{1}{2}\left(\gamma - \frac{n}{2}\right)\right)$. Furthermore we have the asymptotic formula for large time t uniformly with respect to $x \in \mathbf{R}^n$

$$u(t,x) = \frac{1}{(it)^{n/2}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(i\frac{x^2}{2t} - if\left(|\hat{u}_+|^2\right)\left(\frac{x}{t}\right)\log t\right) + O(\epsilon t^{-\frac{n}{2}-\xi\eta})$$

and the estimate $\|\mathcal{F}(U(-t)u)(t) - \hat{u}_+\exp(-if\,(|\hat{u}_+|)^2\log t)\|_k \le C\epsilon t^{-\xi\eta}$, where $k = 2$ or ∞. From the above asymptotics we got the estimate

$$\left\|u(t) - \exp\left(-if\left(|\hat{u}_+|^2\right)\left(\frac{\cdot}{t}\right)\log t\right)U(t)u_+\right\|_\infty \le C\epsilon t^{-\frac{n}{2}-\xi\eta}.$$

In paper [12] we improved the regularity condition on the initial data $u_0 \in H^{\gamma,0} \cap H^{0,\gamma}$ as follows. If the norm of the initial data $\|u_0\|_{\gamma,0} + \|u_0\|_{0,\gamma} = \epsilon$ is sufficiently small, where $\frac{1}{2} < \gamma < \frac{n}{2}$, $n \ge 2$ then there exists a unique global solution $u \in C\left(\mathbf{R}; H^{\gamma,0} \cap H^{0,\gamma}\right)$ of the Hartree equation (1.1) with $\delta = 1$ such that $\|u(t)\|_p \le C\epsilon(1+|t|)^{\frac{n}{p}-\frac{n}{2}}$, where $\frac{2n}{n-1} \le p < \frac{2n}{n-2\gamma}$. Moreover we showed that for any small initial data $u_0 \in H^{\gamma,0} \cap H^{0,\gamma}$ there exists a unique function $u_+ \in H^{\sigma,0} \cap H^{0,\sigma}$ such that $\left\|u(t) - \exp\left(-if\left(|\hat{u}_+|^2\right)\left(\frac{\cdot}{t}\right)\log t\right)U(t)u_+\right\|_2 \le C\epsilon t^{-\mu}$, where $\frac{1}{2} < \sigma < \gamma < \frac{n}{2}$, $\mu = \min\left(1, \frac{\gamma}{2}\right)$.

We also note that the Hartree type equation (1.1) with $2/3 < \delta \le 1$ if $n \ge 4$ and $\left(\sqrt{17}-1\right)/4 < \delta \le 1$ if $n = 3$ was treated in [19] and the existence of weak modified scattering states was shown without the uniqueness. This result does not say that the modified scattering state is not equal to zero identically and therefore is not sufficient for the scattering theory.

Our purpose in the present paper is to study the scattering problem for the Hartree type equation (1.1) in a more difficult subcritical case $0 < \delta < 1$. We propose here a new method which differs from the previous approach of [10, 12] in the point that we introduce another phase function. This method also gives us a simple proof of existence of modified scattering states for the Hartree equation (1.1) with $\delta = 1$ and can be easily applied to the nonlinear Schrödinger equations with critical and supercritical power nonlinearities.

We now state our results in this paper.

Theorem 1.1 *Let $0 < \delta < 1$. Let in the case $n \ge 2$ the initial data $u_0 \in H^{l,0}(\mathbf{R}^n) \cap H^{0,l}(\mathbf{R}^n)$, $l = n+2$ and the value $\epsilon = \|u_0\|_{H^{l,0}} + \|u_0\|_{H^{0,l}}$ be sufficiently small. In the one-dimensional case $n = 1$ we assume that $e^{\beta|x|}u_0 \in L^2(\mathbf{R})$, $\beta > 0$ and the value $\epsilon = \|e^{\beta|x|}u_0\|_2$ is sufficiently small. Then there exists a unique global solution of the Hartree type equation* (1.1) *such that* $u \in C\left(\mathbf{R}; H^{l,0}(\mathbf{R}^n) \cap H^{0,l}(\mathbf{R}^n)\right)$ *for* $n \geq 1$. *Moreover the following decay estimate*

$$\|u(t)\|_p \leq C\epsilon t^{\frac{n}{p}-\frac{n}{2}}$$

is valid for all $t \geq 1$, *where* $2 \leq p \leq \infty$.

Remark 1.1 The decay rate in Theorem 1.1 is the same as that of the solutions to the linear Schrödinger equation. While the global existence of solutions was known (as mentioned before), however the optimal decay estimate of Theorem 1.1 is a new result.

Theorem 1.2 *Let* u *be the solution of* (1.1) *obtained in Theorem* 1.1. *Then for any* u_0 *satisfying the condition of Theorem* 1.1, *there exists a unique final state* $\hat{u}_+ \in H^{l,0}$ *such that the following asymptotics for* $t \to \infty$

$$\begin{aligned} u(t,x) &= \frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f\left(|\hat{u}_+|^2\right)\left(\frac{x}{t}\right) + O(1+t^{1-2\delta})\right) \\ &\quad + O(t^{-\frac{n}{2}-\delta}) \end{aligned}$$

is valid uniformly with respect to $x \in \mathbf{R}^n$.

For the values $\delta \in \left(\frac{1}{2}, 1\right)$ we obtain the existence of the modified scattering states.

Theorem 1.3 *Let* u *be the solution of* (1.1) *obtained in Theorem* 1.1 *and* $\frac{1}{2} < \delta < 1$. *Then there exists a unique final state* $\hat{u}_+ \in H^{l,0}$ *such that the following asymptotics*

$$\begin{aligned} u(t,x) &= \frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f\left(|\hat{u}_+|^2\right)\left(\frac{x}{t}\right)\right) \\ &\quad + O(t^{-\frac{n}{2}+1-2\delta}) \end{aligned}$$

is valid for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}^n$ *and the estimate*

$$\left\| u(t) - \exp\left(-\frac{it^{1-\delta}}{1-\delta}f\left(|\hat{u}_+|^2\right)\left(\frac{x}{t}\right)\right)U(t)u_+\right\|_2 \leq Ct^{1-2\delta}$$

is true for all $t \geq 1$.

Remark 1.2 Note that in the region $0 < \delta \leq \frac{1}{2}$ the value of the phase in the large time asymptotic formula of the solution is determined in Theorem 1.2 with accuracy of the summand growing as $t^{1-2\delta}$ and for the region

$\delta \in \left(\frac{1}{2}, 1\right)$ the phase is evaluated up to a constant summand. In the case $\frac{1}{2} < \delta < 1$ Theorem 1.3 makes the value of the phase in the asymptotic formula precisely.

For the convenience of the reader we explain our strategy of the proof of Theorems 1.1–1.3 in the case of $n \geq 2$. We can easily prove (see paper [16]) the local in time existence of solutions to the Cauchy problem (1.1) by virtue of the contraction mapping principle in the closed ball $\mathcal{X}_{T,2\epsilon}$ (with a center at the origin and a radius 2ϵ) of the following functional space $\mathcal{X}_T = \{\varphi \in C\left([0,T]; L^2\right); \|\varphi\|_{\mathcal{X}_T} < \infty\}$, where $\|u\|_{\mathcal{X}_T} = \sup_{0 \leq t \leq T} \left(\|u(t)\|_{m,0} + \|U(-t)u(t)\|_{0,j}\right)$ with any $m, j \in \mathbf{N}$:

Proposition 1.1 *Let the initial data $u_0 \in H^{m,0} \cap H^{0,j}$, $m, j \in \mathbf{N}$ and let the norm $\epsilon = \|u_0\|_{m,0} + \|u_0\|_{0,j}$ be sufficiently small. Then there exists a time $T > 1$ and a unique solution of the Cauchy problem* (1.1) *with* $0 < \delta < 1$ *such that $u(t) \in C\left([0,T]; H^{m,0}\right)$, $U(-t)\ u(t) \in C\left([0,T]; H^{0,j}\right)$ and $\|u\|_{\mathcal{X}_T} \leq 2\epsilon$.*

Via Proposition 1.1 we can define a new function $v = \mathcal{F} e^{\frac{ix^2}{2t}} U(-t)u(t)$, which satisfies the following equation

$$iv_t + \frac{1}{2t^2}\Delta v = t^{-\delta} f\left(|v|^2\right) v. \tag{1.4}$$

To treat the nonlinear term we introduce a phase function g such that

$$\begin{cases} g_t - \dfrac{\mu}{2t^2}\Delta g = t^{-\delta} f\left(|v|^2\right) + \dfrac{1}{2t^2}(\nabla g)^2, & t > 1, \\ g(1) = 0, \end{cases} \tag{1.5}$$

where $\mu \geq 0$ is an arbitrary constant. The function g is real valued and well defined by v since the equation (1.5) is a nonlinear parabolic equation (when $\mu > 0$). We put $w = ve^{ig}$, then w is also well defined for $1 \leq t \leq T$. Furthermore if we multiply (1.4) by e^{ig} and use (1.5) we easily see that w satisfies the Cauchy problem

$$\begin{cases} w_t = \dfrac{1}{t^2}\nabla w \nabla g + \dfrac{i}{2t^2}\Delta w + \dfrac{1+i\mu}{2t^2} w\Delta g, & t > 1, \\ w(1) = v(1) = \mathcal{F} e^{\frac{ix^2}{2}} U\left(-1\right) u(1). \end{cases} \tag{1.6}$$

To obtain the desired result we need to show the global in time existence of solutions to the system of equations (1.5)–(1.6) under the condition that

$\|w(1)\|_{l,0}+\|w(1)\|_{0,l} = \|U(-1)\,u(1)\|_{0,l}+\|u(1)\|_{l,0} \leq 2\epsilon$. We use the identity $f\left(|v|^2\right) = f\left(|w|^2\right) = C(-\Delta)^{-(n-\delta)/2}|w|^2$, (see [22]), whence we see that the nonlinearity $f\left(|v|^2\right)$ possesses a regularizing property (which is sufficient for our purposes in the case $n \geq 2$). For the one-dimensional case ($n = 1$) we have analogous results, however we need to assume stronger conditions on the regularity properties of the initial data $w(1)$ since the system (1.5)–(1.6) has the derivative loss. For details, see Section 3.

In the case of the Hartree equation (1.1) (with $\delta = 1$) in paper [10] we also have introduced a phase function $\tilde{g}$ in order to eliminate a divergent term in the equation. The phase function $\tilde{g}$ previously used in [10] is determined by the equation $\tilde{g}_t = it^{-1}f\left(|\tilde{v}|^2\right)$, where $\tilde{v} = \mathcal{F}U(-t)u(t)$. Thus our phase function g here is slightly different from that used in [10] and only the leading terms of the large time asymptotics of these phase functions coincide.

The situation for the Hartree type equation (1.1) is similar to that for the nonlinear Schrödinger equation with the power nonlinearity $f\left(|u|^2\right) = \lambda|u|^{\rho-1}$. Roughly speaking, a potential $V = \lambda|x|^{-\delta}$ in (1.1) corresponds to the power $\rho = 1 + 2\delta/n$, so that the NLS equation with the critical power $\rho = 1 + 2/n$ is the analogue of the Hartree equation (1.1) with $\delta = 1$. Thus from the point of view of large time behavior of solutions the Hartree type equation (1.1) corresponds to the following nonlinear Schrödinger equation

$$i\partial_t u = -\frac{1}{2}\Delta u + \lambda|u|^{2\delta/n}u. \tag{1.7}$$

The Cauchy problem for the nonlinear Schrödinger equation (1.7) was studied in [1], [6] and the scattering theory for the supercritical power $\delta > 1$ was developed in [5], [16], [23]. For the critical case $\delta = 1$ it was shown in [20] for $n = 1$ and in [10] for $n = 2, 3$ that the modified scattering states exist. The existence of the modified wave operator was proved in [21] for $n = 1$ and in [4] for $n = 2, 3$. In the present paper we restrict our attention to the more difficult subcritical case $0 < \delta < 1$ in the equation (1.1) which corresponds to (1.7) with $0 < \delta < 1$. We hope that our method can be applied also to the case of the equation (1.7). (However in the case of the NLS equation (1.7) we have a difficulty that when estimating the derivatives and the analytical properties of the solution, the nonlinearity $|u|^{2\delta/n}$ gives us a singularity at the origin and so we need some estimates of the solution u from the below in order to use our approach).

On the other hand similar results can be easily obtained for the following NLS equation with cubic nonlinearity with growing in time coefficient

$$i\partial_t u = -\frac{1}{2}\Delta u + \lambda t^{n-\delta}|u|^2 u, \tag{1.8}$$

where $0 < \delta < 1$, $n \geq 1$. We have the following Theorems which are analogous to Theorems 1.1–1.3 concerning the one dimensional case.

Theorem 1.1′ *Let $0 < \delta < 1$, $n \geq 1$ and the initial data $e^{\beta|x|}u_0(x) \in L^2(\mathbf{R}^n)$, $\beta > 0$ be such that the norm $\epsilon = \left\|e^{\beta|x|}u_0\right\|_2$ be sufficiently small. Then there exists a unique global solution of the Cauchy problem for equation (1.8) such that $u \in C\left(\mathbf{R}; H^{l,0}(\mathbf{R}^n)\right)$ $l = n+2$. Moreover the following decay estimate $\|u(t)\|_p \leq C\epsilon t^{\frac{n}{p}-\frac{n}{2}}$ is valid for all $t \geq 1$, where $2 \leq p \leq \infty$.*

Theorem 1.2′ *Let u be the solution of the Cauchy problem for equation (1.8) obtained in Theorem 1.1′. Then for any initial data u_0 satisfying the conditions of Theorem 1.1′, there exists a unique final state $\hat{u}_+ \in H^{l,0}$ such that the following asymptotics for $t \to \infty$*

$$\begin{aligned} u(t,x) &= \frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f\left(|\hat{u}_+|^2\right)\left(\frac{x}{t}\right) + O(1+t^{1-2\delta})\right) \\ &\quad + O(t^{-\frac{n}{2}-\delta}) \end{aligned}$$

is valid uniformly with respect to $x \in \mathbf{R}^n$.

Finally if $\delta \in \left(\frac{1}{2}, 1\right)$ we are able to obtain the existence of the modified scattering states.

Theorem 1.3′ *Let u be the solution of the Cauchy problem for equation (1.8) obtained in Theorem 1.1′ and $\frac{1}{2} < \delta < 1$. Then there exists a unique final state $\hat{u}_+ \in H^{l,0}$ such that the following asymptotics*

$$\begin{aligned} u(t,x) &= \frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta}f\left(|\hat{u}_+|^2\right)\left(\frac{x}{t}\right)\right) \\ &\quad + O(t^{-\frac{n}{2}+1-2\delta}) \end{aligned}$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$ and the estimate

$$\left\|u(t) - \exp\left(-\frac{it^{1-\delta}}{1-\delta}f\left(|\hat{u}_+|^2\right)\left(\frac{x}{t}\right)\right)U(t)u_+\right\|_2 \leq Ct^{1-2\delta}$$

is true for all $t \geq 1$.

We believe that our method is interesting from the mathematical point of view since it suggests a possibility to study the large time asymptotic behavior of small solutions to many important subcritical nonlinear evolution equations, such as the Korteweg-de Vries and Benjamin-Ono equations. From the physical point of view the Hartree type equation (1.1) helps us to explain the scattering theory with long range potentials. We have a conjecture that the Korteweg-de Vries and Benjamin-Ono equations correspond to the subcritical value $\delta = \frac{1}{2}$ in the Hartree type equation (1.1). Note that in paper [11] we constructed the modified scattering states for the modified Benjamin-Ono equation which are similar to the case of the Hartree equation (1.1) with the critical value $\delta = 1$. We also have a hypothesis that if the initial data have a zero mean value $\int u_0(x)dx = 0$ then the solutions of the modified Korteweg-de Vries equation have the same scattering properties.

We organize our paper as follows. In Section 2 we prepare some preliminary estimates. Lemma 2.1 is the usual Sobolev inequality. We need Lemma 2.2 to treat the nonlinear term. Section 3 is devoted to the proof of Theorems 1.1–1.3. In the case $n \geq 2$ using the local existence Theorem 3.1 we prove a-priory estimates of the solutions to the system (1.5)–(1.6) in Lemma 3.1. For the one-dimensional case we apply the local existence Theorem 3.2 in the analytic functional space to get the estimates of the solutions to the system (1.5)–(1.6) in Lemma 3.2. The rest of Section 3 is devoted to the proof of Theorems 1.1–1.3. The proof of Theorems 1.1′–1.3′ is the same as that of Theorems 1.1–1.3 in one-dimensional case by use of the analytic functional spaces, so we omit it. (However we give some necessary alterations concerning NLS equation while proving Theorems 1.1–1.3.)

2. Preliminaries

We first state the well-known Sobolev embedding inequality (for the proof, see, e.g., [3]).

Lemma 2.1 *Let q, r be any numbers satisfying $1 \leq q, r \leq \infty$, and let j, m be any real numbers satisfying $0 \leq j < m$. Then the following inequality is valid*

$$\left\|(-\Delta)^{j/2}u\right\|_p \leq C\left\|(-\Delta)^{m/2}u\right\|_r^a \|u\|_q^{1-a}$$

if the right-hand side is bounded, where C is a constant depending only on m, n, j, q, r, a, here $\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}$ and a is any real number *from the interval* $\frac{j}{m} \le a \le 1$, *with the following exception: if* $m - j - \frac{n}{r}$ *is nonnegative and integer, then* $a = \frac{j}{m}$.

The following lemma is used for obtaining estimates of the nonlinear term.

Lemma 2.2 *We have the following estimates*

$$\|\phi\psi\|_{l,0} \le C\|\phi\|_{l,0}\left(\|\psi\|_\infty + \|\psi\|_{\dot{H}^{l,0}}\right),$$

$$\sum_{j=1}^{n}\left|\mathrm{Re}\left(\partial_j^l\phi, \partial_j^l\left(\nabla\psi\cdot\nabla\phi\right)\right)\right| \le C\|\phi\|_{l,0}^2\left(\|\psi\|_\infty + \|\psi\|_{\dot{H}^{k,0}}\right)$$

and

$$\left|\left(\partial_j^k\psi, \partial_j^k\left(\nabla\psi\right)^2\right)\right| \le C\left(\|\psi\|_\infty + \|\psi\|_{\dot{H}^{k,0}}\right)^2\|\partial_j^k\psi\|_2, \quad j = 1, \ldots, n$$

if the right-hand sides are bounded, where ψ *is a real valued function,* ϕ *is a complex valued function,* $l = n + 2$, $k = l + 2$, $n \ge 2$.

Proof. By the Leibnitz rule we have

$$\begin{aligned}\sum_{|\alpha|=l}\|\partial^\alpha(\psi\phi)\|_2 \le & \sum_{|\alpha|=l}\|\phi\partial^\alpha\psi\|_2 + \sum_{|\alpha|=l}\|\psi\partial^\alpha\phi\|_2 \\ & + C\sum_{m=1}^{l-1}\sum_{|\alpha|=l-m}\sum_{|\beta|=m}\left\|\partial^\alpha\psi\partial^\beta\phi\right\|_2 \\ \le & \|\phi\|_\infty\sum_{|\alpha|=l}\|\partial^\alpha\psi\|_2 + \|\psi\|_\infty\sum_{|\alpha|=l}\|\partial^\alpha\phi\|_2 \\ & + C\sum_{|\alpha|=l}\|\partial^\alpha\psi\|_2^a\|\psi\|_\infty^{1-a}\sum_{|\alpha|=l}\|\partial^\alpha\phi\|_2^b\|\phi\|_2^{1-b},\end{aligned}$$

where we have applied the Hölder's inequality with $\frac{1}{p}+\frac{1}{q} = \frac{1}{2}$ and Lemma 2.1 with $\frac{1}{p} = \frac{l-m}{n} + a\left(\frac{1}{2} - \frac{l}{n}\right) \in \left[0, \frac{1}{2}\right]$ and $\frac{1}{q} = \frac{m}{n} + b\left(\frac{1}{2} - \frac{l}{n}\right) + \frac{1-b}{2} \in \left[0, \frac{1}{2}\right]$, $b = 1 - a + \frac{an}{2l}$, here $a = a(m)$ is such that $0 < a < 1$ for $1 \le m \le l - 1$. (It is easy to see that such values of a exist, they are from a nonempty interval $\left(\max\left(0, \frac{2l-2m-n}{2l-n}\right), \min\left(1, \frac{2l-2m}{2l-n}\right)\right)$.) Therefore we get

$$\sum_{|\alpha|=l}\|\partial^\alpha(\psi\phi)\|_2 \le C\|\phi\|_{l,0}\left(\|\psi\|_\infty + \sum_{|\alpha|=l}\|\partial^\alpha\psi\|_2\right).$$

This implies the first estimate of the lemma. Since ψ is a real valued function we have by integrating by parts and Hölder's inequality

$$\left|\mathrm{Re}\left(\partial_j^l\phi, \nabla\psi\cdot\nabla\partial_j^l\phi\right)\right| \le C\|\Delta\psi\|_\infty\|\partial_j^l\phi\|_2^2.$$

Taking the inequality into account and using the Leibnitz we obtain

$$\begin{aligned}&\left|\mathrm{Re}\left(\partial_j^l\phi, \partial_j^l\left(\nabla\psi\cdot\nabla\phi\right)\right)\right| \\ &\le C\left\|\partial_j^l\phi\right\|_2\sum_{m=0}^{l-1}\sum_{|\alpha|=l-m+1}\sum_{|\beta|=m+1}\left\|\partial^\alpha\psi\partial^\beta\phi\right\|_2\end{aligned}$$

and as above taking $\frac{1}{p}=\frac{k-m}{n}+a\left(\frac{1}{2}-\frac{k}{n}\right)\in\left[0,\frac{1}{2}\right]$ and $\frac{1}{q}=\frac{m}{n}+b\left(\frac{1}{2}-\frac{l}{n}\right)+\frac{1-b}{2}\in\left[0,\frac{1}{2}\right]$, where $b=\frac{k}{l}+\frac{an}{2l}-\frac{ak}{l}$, $a=a(m)$ is such that $0<a<1$ and $0<b<1$ for $1\le m\le l-1$ (We can take such values of a from the interval $\left(\max\left(0,\frac{2k-2l}{2k-n},\frac{2k-2m-n}{2k-n}\right),\min\left(1,\frac{2k-2m}{2k-n}\right)\right)$) we get

$$\begin{aligned}&\sum_{m=0}^{l-1}\sum_{|\alpha|=l-m+1}\sum_{|\beta|=m+1}\left\|\partial^\alpha\psi\partial^\beta\phi\right\|_2 \\ &\le\|\Delta\psi\|_\infty\sum_{|\alpha|=l}\|\partial^\alpha\phi\|_2+C\sum_{m=1}^{l-1}\sum_{|\alpha|=k-m}\sum_{|\beta|=m}\left\|\partial^\alpha\psi\partial^\beta\phi\right\|_2 \\ &\le\|\Delta\psi\|_\infty\sum_{|\alpha|=l}\|\partial^\alpha\phi\|_2+C\|\phi\|_{l,0}\left(\|\psi\|_\infty+\sum_{|\alpha|=k}\|\partial^\alpha\psi\|_2\right) \\ &\le C\|\phi\|_{l,0}\left(\|\psi\|_\infty+\|\psi\|_{\dot{H}^{k,0}}\right),\end{aligned}$$

whence the second estimate of the lemma follows. And finally by virtue of the Hölder inequality with $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and Lemma 2.1 with $\frac{1}{p}=\frac{k+2-m}{n}+a\left(\frac{1}{2}-\frac{k}{n}\right)\in\left[0,\frac{1}{2}\right]$ and $\frac{1}{q}=\frac{m}{n}+b\left(\frac{1}{2}-\frac{k}{n}\right)\in\left[0,\frac{1}{2}\right]$, where $a=a(m)$ and $b=1-a+\frac{4}{2k-n}$ satisfy the inequalities $0<a<1$, $0<b<1$ for all $3\le m\le k-1$ (we can choose a from the nonempty interval $\left(\max\left(0,\frac{2k-2m+4-n}{2k-n},\frac{4}{2k-n}\right),\min\left(1,\frac{2k-2m+4}{2k-n},\frac{2k-n+4}{2k-n}\right)\right)$) we get

$$
\begin{aligned}
&\left|\left(\partial_j^k\psi, \partial_j^k(\nabla\psi)^2\right)\right| \\
&\leq C\sum_{m=0}^{k}\left|\left(\partial_j^k\psi, \nabla\partial_j^{k-m}\psi\cdot\partial_j^m\nabla\psi\right)\right| \\
&\leq C\|\partial_j^k\psi\|_2^2\|\Delta\psi\|_\infty + C\|\partial_j^k\psi\|_2\sum_{m=2}^{k}\sum_{|\alpha|=k+2-m}\sum_{|\beta|=m}\left\|\partial^\alpha\psi\cdot\partial^\beta\psi\right\|_2 \\
&\leq C\|\partial_j^k\psi\|_2\sum_{|\alpha|=k}\|\partial^\alpha\psi\|_2\|\Delta\psi\|_\infty \\
&\quad + C\|\partial_j^k\psi\|_2\sum_{m=3}^{k-1}\sum_{|\alpha|=k+2-m}\sum_{|\beta|=m}\left\|\partial^\alpha\psi\cdot\partial^\beta\psi\right\|_2 \\
&\leq C\|\partial_j^k\psi\|_2\sum_{|\alpha|=k}\|\partial^\alpha\psi\|_2\left(\|\psi\|_\infty + \sum_{|\alpha|=k}\|\partial^\alpha\psi\|_2\right) \\
&\quad + C\|\partial_j^k\psi\|_2\sum_{|\alpha|=k}\|\partial^\alpha\psi\|_2^a\|\psi\|_\infty^{1-a}\sum_{|\alpha|=k}\|\partial^\alpha\psi\|_2^b\|\psi\|_\infty^{1-b} \\
&\leq C\left(\|\psi\|_\infty + \sum_{|\alpha|=k}\|\partial^\alpha\psi\|_2\right)^2\|\partial_j^k\psi\|_2.
\end{aligned}
$$

Lemma 2.2 is proved. □

3. Proof of Theorems

We have

$$
\begin{aligned}
&D(t)^{-1}\overline{M}\left(f\left(|u|^2\right)u\right) \\
&\quad = D(t)^{-1}\overline{M}\int\frac{|u(t,\xi)|^2}{|x-\xi|^\delta}d\xi u(t,x) \\
&\quad = D(t)^{-1}\int\frac{|\overline{M}u(t,\xi)|^2}{|x-\xi|^\delta}d\xi\overline{M}u(t,x) \\
&\quad = i^{n/2}t^{n/2}\int\frac{|\overline{M}u(t,\xi)|^2}{|tx-\xi|^\delta}d\xi e^{-ix^2t/2}u(t,tx) \\
&\quad = t^{-\delta}f\left(|D(t)^{-1}\overline{M}u|^2\right)D(t)^{-1}\overline{M}u = t^{-\delta}f\left(|v|^2\right)v,
\end{aligned}
$$

where

$$v(t) = D(t)^{-1}\overline{M}u = \mathcal{F}M(t)U(-t)u(t).$$

Using the identity $U(-t) = \overline{M}\mathcal{F}^{-1}D(t)^{-1}\overline{M}$ for the free Schrödinger evolution group we translate the Hartree equation (1.1)

$$iu_t = -\frac{1}{2}\Delta u + f\left(|u|^2\right)u$$

to the following equation

$$\begin{aligned} i(U(-t)u)_t &= U(-t)f\left(|u|^2\right)u = \overline{M}\mathcal{F}^{-1}D(t)^{-1}\overline{M}f\left(|u|^2\right)u \\ &= \overline{M}\mathcal{F}^{-1}t^{-\delta}f\left(|v|^2\right)v, \end{aligned}$$

whence we get

$$i\left(MU(-t)u\right)_t = \frac{x^2}{2t^2}MU(-t)u + t^{-\delta}\mathcal{F}^{-1}f\left(|v|^2\right)v$$

and finally as in paper [13] we obtain

$$iv_t = t^{-\delta}f\left(|v|^2\right)v + \mathcal{F}\left(\frac{x^2}{2t^2}MU(-t)u\right) = t^{-\delta}f\left(|v|^2\right)v - \frac{1}{2t^2}\Delta v. \tag{3.1}$$

Note that for the case of the NLS equation (1.8) we get easily $D(t)^{-1}\overline{M}|u|^2u = t^{-n}|v|^2v$ and therefore from the NLS equation (1.8) we obtain the equation $iv_t = t^{-\delta}|v|^2v - \frac{1}{2t^2}\Delta v$, which is similar to (3.1). Then changing the dependent variable $v = we^{-ig}$ we get

$$\begin{aligned} iw_t &= w\left(t^{-\delta}f\left(|we^{-ig}|^2\right) - g_t\right) - \frac{1}{2t^2}e^{ig}\Delta\left(e^{-ig}w\right) \\ &= w\left(t^{-\delta}f\left(|we^{-ig}|^2\right) - g_t\right) \\ &\quad - \frac{1}{2t^2}(\Delta w - 2i\nabla w\nabla g - w(\nabla g)^2 - iw\Delta g). \end{aligned}$$

We now choose the function g in such a way that it satisfies the equation

$$g_t = t^{-\delta}f\left(|we^{-ig}|^2\right) + \frac{1}{2t^2}(\nabla g)^2 + \frac{\mu}{2t^2}\Delta g$$

with the initial condition $g(1,x) = 0$, $\mu \geq 0$ is arbitrary (below we choose

$\mu = 1$ in Lemma 3.1 and $\mu = 0$ in Lemma 3.2). Then we obtain

$$iw_t = \frac{1}{2t^2}\left(-\Delta w + 2i\nabla w \nabla g + (i-\mu)\, w\Delta g\right)$$

and $w(1) = v(1)$. Note that the function g remains to be real for all $t > 1$ and so $|v| = |w|$. Thus from (3.1) we get the following system

$$\begin{cases} w_t = \dfrac{1}{t^2}\nabla w \nabla g + \dfrac{i}{2t^2}\Delta w + \dfrac{1+i\mu}{2t^2} w\Delta g, \\ g_t = t^{-\delta} f\left(|w|^2\right) + \dfrac{1}{2t^2}(\nabla g)^2 + \dfrac{\mu}{2t^2}\Delta g, \\ g(1) = 0,\ w(1) = v(1) = \mathcal{F} e^{\frac{ix^2}{2}} U(-1) u(1). \end{cases} \tag{3.2}$$

Note that $f\left(|w|^2\right) = C(-\Delta)^{-(n-\delta)/2}|w|^2$ (see [22]).

By virtue of Proposition 1.1 we may assume that

$$\|w\,(1)\,\|_{l,0} + \|w\,(1)\,\|_{0,l} = \|u\,(1)\,\|_{l,0} + \|U\,(-1)\,u\,(1)\,\|_{0,l} \le 2\epsilon.$$

In order to obtain the desired result we need to prove the global existence in time of solutions to (3.2). To clarify the idea of the proof of the Theorems we use the following local existence theorem which can be shown by the contraction mapping principle.

Theorem 3.1 *Suppose that $w(1) \in H^{l,0} \cap H^{0,l}$. Then there exists a time $T > 1$ and a unique solution to the Cauchy problem for the system of equations (3.2) with $\mu = 1$ such that $w \in C\left([1,T], H^{l,0} \cap H^{0,l}\right)$, $g \in C([1,T], \dot{H}^{k,0} \cap L^\infty)$, where $l = n+2$, $k = l+2$, $n \ge 2$.*

We now prove

Lemma 3.1 *Suppose that the initial data $v(1)$ are such that the value*

$$\epsilon = \|v(1)\|_{l,0} + \|v(1)\|_{0,l}$$

is sufficiently small. Then the following estimates are valid

$$\|w\|_{l,0} + \|w\|_{0,l} + t^{\delta-1}\left(\|g\|_\infty + \|g\|_{\dot{H}^{l,0}}\right) + t^{\frac{\delta}{2}-1}\|g\|_{\dot{H}^{k,0}} < 5\epsilon, \tag{3.3}$$

for any $t \ge 1$, where $l = n+2$, $k = l+2$, $n \ge 2$.

Proof. By the contrary we suppose that estimate (3.3) is violated for some time. By Theorem 3.1 and continuity of the left hand side of (3.3) we can find a maximal time $T>1$ such that nonstrict inequality (3.3) is valid for all $t\in[1,T]$. We estimate the following norms $J=\|w\|_{l,0}$ and

$$I=t^{\delta-1}\Big(\|g\|_\infty+\sum_{|\alpha|=l}\|\partial^\alpha g\|_2\Big)+t^{\frac{\delta}{2}-1}\sum_{|\alpha|=k}\|\partial^\alpha g\|_2$$

of the functions w and g on $[1,T]$. Differentiating (3.2) (with $\mu=1$) with respect to x_j and integrating by parts we get

$$\begin{aligned}\frac{d}{dt}\left(\partial_j^l w,\partial_j^l w\right) &= \mathrm{Re}\,\frac{2}{t^2}\left(\partial_j^l w,\partial_j^l\left(\nabla g\cdot\nabla w\right)\right)\\ &\quad+\mathrm{Re}\,\frac{1+i}{t^2}\left(\partial_j^l w,\partial_j^l\left(w\Delta g\right)\right),\end{aligned}$$

whence by the first two estimates of Lemma 2.2 we obtain

$$\frac{d}{dt}J^2\le Ct^{-1-\delta/2}IJ^2\le C\epsilon^3t^{-1-\delta/2}$$

and integration with respect to t over $[1,t]$, $1\le t\le T$ gives $J\le 2\epsilon$. Analogously by virtue of the third estimate of Lemma 2.2 we find

$$\begin{aligned}\frac{d}{dt}\left\|\partial_j^k g\right\|_2^2 &\le 2t^{-\delta}\left|\left(\partial_j^k g,\partial_j^k f\left(|w|^2\right)\right)\right|\\ &\quad+\frac{1}{t^2}\left|\left(\partial_j^k g,\partial_j^k(\nabla g)^2\right)\right|-\frac{1}{t^2}\left\|\nabla\partial_j^k g\right\|_2^2\\ &\le Ct^{-\delta}\left\|(-\Delta)^{\delta/2}r_j\right\|_2\left\|\partial_j^k(-\Delta)^{-\delta/2}f\left(|w|^2\right)\right\|_2\\ &\quad+Ct^{-\delta}\|r_j\|_2I^2-\frac{1}{t^2}\|\nabla r_j\|_2^2,\end{aligned}$$

where $r_j=\partial_j^k g$ and $k=n+4$. From Lemma 2.1 we have the estimate $\left\|(-\Delta)^{\delta/2}r_j\right\|_2\le C\|r_j\|_2^{1-\delta}\|\nabla r_j\|_2^\delta$ since $\delta\in(0,1)$. Then using the Young's inequality $ab\le\frac{a^p}{p}+\frac{b^q}{q}$, where we take $a=C\|r_j\|_2^{1-\delta}\|\partial_j^k(-\Delta)^{-\delta/2}f\|_2$ and $b=t^{-\delta}\|\nabla r_j\|_2^\delta$, $p=\frac{2}{2-\delta}$, $q=\frac{2}{\delta}$, so that $\frac{1}{p}+\frac{1}{q}=1$, we get

$$\begin{aligned}&Ct^{-\delta}\left\|(-\Delta)^{\delta/2}r_j\right\|_2\left\|\partial_j^k(-\Delta)^{-\delta/2}f\left(|w|^2\right)\right\|_2\\ &\le C\frac{2-\delta}{2}\left(\|r_j\|_2^{1-\delta}\left\|\partial_j^k(-\Delta)^{-\delta/2}f\right\|_2\right)^{\frac{2}{2-\delta}}+\frac{\delta}{2}\frac{1}{t^2}\|\nabla r_j\|_2^2.\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dt}\|r_j\|_2^2 &\le C\left(\|r_j\|_2^{1-\delta}\|\partial_j^k(-\Delta)^{-\delta/2}f\|_2\right)^{\frac{2}{2-\delta}} + Ct^{-\delta}\|r_j\|_2 I^2\\ &\le CJ^{\frac{4}{2-\delta}}\|r_j\|_2^{\frac{2-2\delta}{2-\delta}} + Ct^{-\delta}\|r_j\|_2 I^2 \le C\epsilon^4 t^{1-\delta}\end{aligned}$$

since for all $n \ge 2$ we have

$$\begin{aligned}\left\|\partial_j^k(-\Delta)^{-\delta/2}f\left(|w|^2\right)\right\|_2 &\le C\left\|\partial_j^k(-\Delta)^{-n/2}|w|^2\right\|_2\\ &\le C\left\|\Delta^2|w|^2\right\|_2 \le C\|w\|_{l,0}^2.\end{aligned}$$

Integration with respect to t yields $\|r_j\|^2 \le C\epsilon^4 t^{2-\delta}$. Thus $\|g\|_{\dot H^{k,0}} \le C\epsilon^2 t^{1-\delta/2} < \epsilon t^{1-\delta/2}$. For the norm $\dot H^{l,0}$ directly integrating equation (3.2) we get

$$\begin{aligned}\|g\|_{\dot H^{l,0}} &= \left\|\int_1^t g_t dt\right\|_{\dot H^{l,0}} \le \int_1^t \tau^{-\delta}\left\|f\left(|w|^2\right)\right\|_{\dot H^{l,0}} d\tau\\ &\quad + \frac{1}{2}\int_1^t \left(\|(\nabla g)^2\|_{\dot H^{l,0}} + \|\Delta g\|_{\dot H^{l,0}}\right)\frac{d\tau}{\tau^2}\\ &\le C\int_1^t \left(\tau^{-\delta}J^2 + \tau^{-2}\|g\|_{\dot H^{k,0}}\right)d\tau\\ &\le C\epsilon^2 t^{1-\delta} < \epsilon t^{1-\delta}.\end{aligned}$$

In the same way we estimate the L^∞ norm to get

$$\begin{aligned}\|g\|_\infty &= \left\|\int_1^t g_t dt\right\|_\infty \le \int_1^t \tau^{-\delta}\left\|f\left(|w|^2\right)\right\|_\infty d\tau\\ &\quad + \frac{1}{2}\int_1^t \left(\left\|(\nabla g)^2\right\|_\infty + \|\Delta g\|_\infty\right)\frac{d\tau}{\tau^2}\\ &\le C\left(\epsilon^2 + \epsilon^{1+\frac{4}{n+7}}\right)\int_1^t \tau^{-\delta}d\tau < \epsilon t^{1-\delta}\end{aligned}$$

since by Lemma 2.1 with $j = 2$, $m = k - \frac{1}{2} = n + \frac{7}{2}$, $r = 2$, $q = \infty$, $p = \infty$, $a = \frac{4}{n+7}$ we get

$$\|\Delta g\|_\infty \le C\|g\|_{\dot H^{k,0}}^{\frac{4}{n+7}}\|g\|_\infty^{\frac{n+3}{n+7}} \le C\epsilon^{1+\frac{4}{n+7}}t^{1-\delta/2}.$$

Therefore we see that

$$I = t^{\delta-1}\Big(\|g\|_\infty + \sum_{|\alpha|=l} \|\partial^\alpha g\|_2 \Big) + t^{\frac{\delta}{2}-1} \sum_{|\alpha|=k} \|\partial^\alpha g\|_2 < 3\epsilon.$$

Similarly, multiplying (3.2) by $x_j^{2l}\bar{w}$ and integrating by parts, we easily obtain the estimate $\|w\|_{0,l} \le \epsilon$. Hence we get (3.3) for all $t \in [1,T]$. The contradiction obtained yields the result of Lemma 3.1. □

Let us consider the one-dimensional case $n=1$. Note that from our supposition on the initial data $\|e^{\beta|x|}u_0(x)\|_2 \le \epsilon$ we have $v(1) = \mathcal{F}e^{ix^2/2}u_0(x) \in \mathcal{H}^3_{\beta'}$ and $\|v(1)\|_{\mathcal{H}^3_{\beta'}} \le 2\epsilon$, where $0<\beta'<\beta$. Below we will omit the prime. As in the case $n \ge 2$ we assume that the following local existence theorem holds. Denote $\sigma = \sigma(t) = \beta t^{-\gamma}$, where $\beta > 0$, $0 < \gamma \le \frac{\delta}{2}$.

Theorem 3.2 *Suppose that $v(1) \in \mathcal{H}^3_\beta$. Then there exists a time $T>1$ and a unique solution to the Cauchy problem for the system of equations (3.2) with $\mu = 0$ such that $w \in C\left([1,T],\mathcal{H}^3_\sigma\right)$, $g \in C\left([1,T],L^\infty\right)$, $g_x \in C\left([1,T],\mathcal{H}^3_\sigma\right)$.*

Now let us prove the following estimates.

Lemma 3.2 *Suppose that the initial data $v(1)$ are such that the value $\epsilon = \|v(1)\|_{\mathcal{H}^3_\beta}$ is sufficiently small. Then the following estimate is valid*

$$\|w\|_{\mathcal{H}^3_\sigma} + t^{\delta-1}\left(\|g\|_\infty + \|g\|_{\dot{H}^{3,0}}\right) + t^{\delta-1-\gamma}\|g_x\|_{\mathcal{H}^3_\sigma} < 5\epsilon \tag{3.4}$$

for any $t \ge 1$.

Proof. Denote $h = t^{\delta-1-\gamma}g_x$, where $\gamma \in \left(0,\frac{\delta}{2}\right]$. Then from the system (3.2) (with $\mu = 0$) we get

$$\begin{cases} w_t = \dfrac{1}{2t^2}\left(2t^{1+\gamma-\delta}hw_x + iw_{xx} + t^{1+\gamma-\delta}wh_x\right), \\ h_t = t^{-1-\gamma}\partial_x f\left(|w|^2\right) + t^{\gamma-1-\delta}hh_x - \dfrac{1+\gamma-\delta}{t}h, \\ h(1) = 0, \quad w(1) = v(1). \end{cases} \tag{3.5}$$

We consider a symbol $\hat{E}(p) = (1+|p|)^3 e^{\sigma|p|}$, where $\sigma = \sigma(t) = \beta t^{-\gamma}$ decays with time. Note that $\hat{E}_t(p) = \sigma'(t)|p|\hat{E}(p)$. Taking Fourier transform of (3.5), multiplying the result by $\hat{E}^2(p)\overline{\hat{w}(t,p)}$, integrating with respect to

$p \in \mathbf{R}$ and taking the real part of the result we obtain

$$\begin{cases} \dfrac{d}{dt}\|w\|^2_{\mathcal{H}^3_\sigma} - 2\sigma' \left\| \sqrt{|p|}\hat{E}\hat{w} \right\|_2^2 = 2\,\mathrm{Re}(\hat{E}\hat{w}, \hat{E}\hat{G}_1), \\ \dfrac{d}{dt}\|h\|^2_{\mathcal{H}^3_\sigma} - 2\sigma' \left\| \sqrt{|p|}\hat{E}\hat{h} \right\|_2^2 = 2\,\mathrm{Re}(\hat{E}\hat{h}, \hat{E}\hat{G}_2) - 2\dfrac{1+\gamma-\delta}{t}\|h\|^2_{\mathcal{H}^3_\sigma}, \end{cases} \tag{3.6}$$

where $G_1 = t^{\gamma-1-\delta}\left(hw_x + \frac{1}{2}wh_x\right)$, $G_2 = t^{-1-\gamma}\partial_x f\left(|w|^2\right) + t^{\gamma-1-\delta}hh_x$. By the Schwarz and Sobolev inequalities we get

$$\begin{aligned} &\left|\mathrm{Re}\,(\hat{E}\hat{w}, \hat{E}\hat{G}_1)\right| \\ &\le \|w\|_{\mathcal{H}^{7/2}_\sigma}\|G_1\|_{\mathcal{H}^{5/2}_\sigma} \\ &\le Ct^{\gamma-1-\delta}\|w\|_{\mathcal{H}^{7/2}_\sigma}\Big(\|h(t,\cdot+i\sigma)w_x(t,\cdot+i\sigma)\|_{5/2,0} \\ &\quad + \|h(t,\cdot-i\sigma)w_x(t,\cdot-i\sigma)\|_{5/2,0} + \|h_x(t,\cdot+i\sigma)w(t,\cdot+i\sigma)\|_{5/2,0} \\ &\quad\quad + \|h_x(t,\cdot-i\sigma)w(t,\cdot-i\sigma)\|_{5/2,0}\Big) \\ &\le Ct^{\gamma-1-\delta}\|w\|_{\mathcal{H}^{7/2}_\sigma}(\|h\|_{\mathcal{H}^3_\sigma}\|w\|_{\mathcal{H}^{7/2}_\sigma} + \|h\|_{\mathcal{H}^{7/2}_\sigma}\|w\|_{\mathcal{H}^3_\sigma}), \end{aligned}$$

where we have used the fact that

$$\begin{aligned} &\|h(t,\cdot+i\sigma)w_x(t,\cdot+i\sigma)\|_{5/2,0} \\ &\le C\Big(\|h(t,\cdot+i\sigma)\|_{1,0}\|w(t,\cdot+i\sigma)\|_{7/2,0} \\ &\quad + \|h(t,\cdot+i\sigma)\|_{5/2,0}\|w(t,\cdot+i\sigma)\|_{2,0}\Big) \\ &\le C\left(\|h\|_{\mathcal{H}^1_\sigma}\|w\|_{\mathcal{H}^{7/2}_\sigma} + \|h\|_{\mathcal{H}^{5/2}_\sigma}\|w\|_{\mathcal{H}^2_\sigma}\right), \end{aligned}$$

and since $|w|^2 = w\overline{w}$ has an analytic continuation as $w(t,z)\overline{w(t,\bar{z})}$ for the complex values of the independent variable z in a strip $-\sigma \le \mathrm{Im}\,z \le \sigma$ we obtain

$$\begin{aligned} &\left|\mathrm{Re}\,(\hat{E}\hat{h}, \hat{E}\hat{G}_2)\right| \\ &\le \|h\|_{\mathcal{H}^{7/2}_\sigma}\|G_2\|_{\mathcal{H}^{5/2}_\sigma} \\ &\le C\|h\|_{\mathcal{H}^{7/2}_\sigma}\Big(t^{-1-\gamma}\Big(\|w(t,\cdot+i\sigma)\overline{w(t,\cdot-i\sigma)}\|_{\delta+5/2,0} \\ &\quad + \|w(t,\cdot-i\sigma)\overline{w(t,\cdot+i\sigma)}\|_{\delta+5/2,0}\Big) \\ &\quad + t^{\gamma-1-\delta}\left(\|h^2(t,\cdot+i\sigma)\|_{7/2,0} + \|h^2(t,\cdot-i\sigma)\|_{7/2,0}\right)\Big) \end{aligned}$$

$$\leq C\|h\|_{\mathcal{H}_\sigma^{7/2}}\left(t^{-1-\gamma}\|w\|_{\mathcal{H}_\sigma^{7/2}}\|w\|_{\mathcal{H}_\sigma^3}+t^{\gamma-1-\delta}\|h\|_{\mathcal{H}_\sigma^{7/2}}\|h\|_{\mathcal{H}_\sigma^3}\right).$$

Now as in the proof of Lemma 3.1 we argue by contradiction. We use the nonstrict estimate (3.4) on some maximal interval $[1,T]$. Via system (3.6) we get for the norm $J^2=\|w\|^2_{\mathcal{H}_\sigma^3}+\|h\|^2_{\mathcal{H}_\sigma^3}$

$$\begin{aligned}\frac{d}{dt}J^2 &\leq Ct^{-1-\gamma}J^2-2\beta\gamma t^{-1-\gamma}\left(\|w\|^2_{\mathcal{H}_\sigma^{7/2}}+\|h\|^2_{\mathcal{H}_\sigma^{7/2}}\right)\\ &\quad+C(t^{-1-\gamma}+t^{\gamma-1-\delta})J\left(\|w\|^2_{\mathcal{H}_\sigma^{7/2}}+\|h\|^2_{\mathcal{H}_\sigma^{7/2}}\right)\\ &\leq C\epsilon t^{-1-\gamma}J^2,\end{aligned}$$

whence by the Gronwall's inequality we obtain $J(t)\leq J(1)e^{C\epsilon}<2\epsilon$.

In the same way as in Lemma 3.1 we estimate the value $t^{\delta-1}\left(\|g\|_\infty+\|g\|_{\dot{H}^{3,0}}\right)$. Lemma 3.2 is proved. □

We are now in a position to prove Theorems 1.1–1.3.

Proof of Theorem 1.1. From Lemma 3.1 for $n\geq 2$ and from Lemma 3.2 for $n=1$ we find that there exists a unique global solution u of (1.1) such that $u\in C\left(\mathbf{R}^+;H^{n+2,0}\cap H^{0,n+2}\right)$. Using the identity $\mathcal{F}MU(-t)u(t)=w(t)\exp\left(-ig(t)\right)$ and our trivial representation of the solution

$$\begin{aligned}u(t) &= M(t)D(t)w(t)\exp(-ig)\\ &= \frac{1}{(it)^{n/2}}M(t)w\left(t,\frac{x}{t}\right)\exp\left(-ig\left(t,\frac{x}{t}\right)\right)\end{aligned}$$

we easily get

$$\begin{aligned}\|u(t)\|_p &\leq Ct^{-n/2}\left\|w\left(t,\frac{\cdot}{t}\right)\right\|_p\leq Ct^{-n/2}\left(\int\left|w\left(t,\frac{x}{t}\right)\right|^p dx\right)^{1/p}\\ &= Ct^{n/p-n/2}\left(\int|w(t,y)|^p dy\right)^{1/p}=Ct^{n/p-n/2}\|w\|_p\\ &\leq Ct^{n/p-n/2}\|w\|_{n/2-n/p,0}\leq C\epsilon t^{n/p-n/2}\end{aligned}$$

for all $p\geq 2$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We have via Lemma 2.2, Lemma 3.1 and Lemma 3.2

$$\|w(t)-w(s)\|_{n,0}\leq\int_s^t\|w_\tau(\tau)\|_{n,0}d\tau$$

$$\leq C\int_s^t \left(\|\nabla g\nabla w\|_{n,0} + \|\Delta w\|_{n,0} + \|w\Delta g\|_{n,0}\right)\frac{d\tau}{\tau^2}$$
$$\leq C\epsilon \int_s^t \frac{d\tau}{\tau^{1+\delta}} \leq C\epsilon s^{-\delta} \tag{3.7}$$

for all $1 < s < t$. Therefore there exists a unique limit $W_+ \in H^{n,0}$ such that $\lim_{t\to\infty} w(t) = W_+$ in $H^{n,0}$ and thus we get

$$\begin{aligned} u(t,x) &= \frac{1}{(it)^{\frac{n}{2}}} M(t) w\left(t, \frac{x}{t}\right) e^{-ig\left(t,\frac{x}{t}\right)} \\ &= \frac{1}{(it)^{\frac{n}{2}}} M(t) W_+\left(\frac{x}{t}\right) e^{-ig\left(t,\frac{x}{t}\right)} + O(\epsilon t^{-\frac{n}{2}-\delta}) \end{aligned}$$

uniformly with respect to $x \in \mathbf{R}^n$ since for all $2 \leq p \leq \infty$ we have the estimate

$$\begin{aligned} &\left\| u(t) - \frac{1}{(it)^{n/2}} M(t) W_+\left(\frac{\cdot}{t}\right) e^{-ig\left(t,\frac{\cdot}{t}\right)} \right\|_p \\ &\leq Ct^{-n/2} \left\| w\left(t, \frac{\cdot}{t}\right) - W_+\left(\frac{\cdot}{t}\right) \right\|_p \\ &\leq Ct^{n/p-n/2} \|w(t) - W_+\|_p \leq Ct^{n/p-n/2} \|w(t) - W_+\|_{n/2-n/p,0} \\ &\leq C\epsilon t^{n/p-n/2-\delta}. \end{aligned}$$

Note that analogously to (3.7) we see that $\|w(t) - w(s)\|_{l,0} \leq C\epsilon s^{-\delta/2}$ therefore $W_+ \in H^{l,0}$. For the phase g we write the identity

$$\begin{aligned} g(t) &= \int_1^t f\left(|w|^2\right) \frac{d\tau}{\tau^\delta} + \int_1^t \left((\nabla g)^2 + \mu\Delta g\right) \frac{d\tau}{2\tau^2} \\ &= f\left(|W_+|^2\right) \frac{t^{1-\delta}}{1-\delta} + \Phi(t), \end{aligned}$$

where

$$\begin{aligned} \Phi(t) =& -\frac{1}{1-\delta} f\left(|W_+|^2\right) + \Psi(t) + \left(f\left(|w(t)|^2\right) - f\left(|W_+|^2\right)\right) \frac{(t^{1-\delta} - 1)}{1-\delta} \\ &+ \int_1^t \left((\nabla g)^2 + \mu\Delta g\right) \frac{d\tau}{2\tau^2}, \end{aligned}$$

$$\Psi(t) = \int_1^t \left(f\left(|w(\tau)|^2\right) - f\left(|w(t)|^2\right)\right) \frac{d\tau}{\tau^\delta}.$$

Since

$$\|f\left(|w(t)|^2\right) - f\left(|w(\tau)|^2\right)\|_\infty \le C\epsilon\|w(t) - w(\tau)\|_{n,0} \le C\epsilon^2\tau^{-\delta}$$

we get

$$g = \frac{t^{1-\delta}}{1-\delta} f\left(|W_+|^2\right) + O(1 + t^{1-2\delta})$$

uniformly in $x \in \mathbf{R}^n$. From these estimates the result of Theorem 1.2 follows with $\hat{u}_+ = W_+$. □

Proof of Theorem 1.3. We have

$$\begin{aligned}
\Phi(t) - \Phi(s) = & \int_s^t \left(f\left(|w(\tau)|^2\right) - f\left(|w(t)|^2\right)\right)\frac{d\tau}{\tau^\delta} \\
& - \left(f\left(|w(t)|^2\right) - f\left(|w(s)|^2\right)\right)\frac{s^{1-\delta}-1}{1-\delta} \\
& + \left(f\left(|w(t)|^2\right) - f\left(|W_+|^2\right)\right)\frac{t^{1-\delta}-1}{1-\delta} \\
& - \left(f\left(|w(s)|^2\right) - f\left(|W_+|^2\right)\right)\frac{s^{1-\delta}-1}{1-\delta} \\
& + \int_s^t \left((\nabla g(\tau))^2 + \mu\Delta g(\tau)\right)\frac{d\tau}{2\tau^2},
\end{aligned} \tag{3.8}$$

where $1 < s < t$. We apply Lemma 3.1–3.2 and (3.7) to (3.8) to get

$$\|\Phi(t) - \Phi(s)\|_{\dot{H}^{l,0}} + \|\Phi(t) - \Phi(s)\|_\infty \le C\epsilon s^{1-2\delta}$$

for $1 < s < t$. This implies that there exists a unique limit $\Phi_+ = \lim_{t\to\infty}\Phi(t) \in \dot{H}^{l,0}(\mathbf{R}^n) \cap L^\infty$ such that

$$\|\Phi(t) - \Phi_+\|_{\dot{H}^{l,0}} + \|\Phi(t) - \Phi_+\|_\infty \le C\epsilon t^{1-2\delta} \tag{3.9}$$

since we now consider the case $\frac{1}{2} < \delta < 1$. By virtue of (3.9) we have

$$\left\|g(t) - \frac{t^{1-\delta}}{1-\delta} f\left(|W_+|^2\right) - \Phi_+\right\|_\infty \le C\epsilon t^{1-2\delta}. \tag{3.10}$$

We now put $\hat{u}_+ = W_+ \exp(-i\Phi_+)$. Therefore we obtain the asymptotics

$$\begin{aligned}
u(t,x) = & \frac{1}{(it)^{\frac{n}{2}}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - \frac{it^{1-\delta}}{1-\delta} f\left(|\hat{u}_+|^2\right)\left(\frac{x}{t}\right)\right) \\
& + O(t^{-n/2+1-2\delta})
\end{aligned}$$

for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^n$. Via (3.10), Lemma 3.1 for $n \geq 2$ and Lemma 3.2 for the one-dimensional case we have

$$\begin{aligned}
&\left\| \mathcal{F}MU(-t)u(t) - \hat{u}_+ \exp\left(-i \frac{t^{1-\delta}}{1-\delta} f\left(|\hat{u}_+|^2 \right) \right) \right\|_2 \\
&= \left\| w(t) \exp\left(-ig(t) \right) - W_+ \exp\left(-i \frac{t^{1-\delta}}{1-\delta} f\left(|\hat{W}_+|^2 \right) - i\Phi_+ \right) \right\|_2 \\
&\leq \| w(t) - W_+ \| + \left\| W_+ \| \| g(t) - f\left(|W_+|^2 \right) \frac{t^{1-\delta}}{1-\delta} - \Phi_+ \right\|_\infty \\
&\leq C\epsilon t^{1-2\delta},
\end{aligned}$$

whence we get

$$\begin{aligned}
&\left\| u(t) - \exp\left(-i \frac{t^{1-\delta}}{1-\delta} f\left(|\hat{u}_+|^2 \right) \left(\frac{x}{t} \right) \right) U(t) u_+ \right\|_2 \\
&= \left\| u(t) - M(t)D(t) \exp\left(-i \frac{t^{1-\delta}}{1-\delta} f\left(|\hat{u}_+|^2 \right) \right) \mathcal{F}M(t) u_+ \right\|_2 \\
&\leq \left\| M(t)D(t) \left(\mathcal{F}M(t)U(-t)u(t) - \hat{u}_+ \exp\left(-i \frac{t^{1-\delta}}{1-\delta} f\left(|\hat{u}_+|^2 \right) \right) \right) \right\|_2 \\
&\quad + \left\| M(t)D(t) \exp\left(-i \frac{t^{1-\delta}}{1-\delta} f\left(|\hat{u}_+|^2 \right) \right) \mathcal{F}\left(M(t) - 1 \right) u_+ \right\|_2 \\
&\leq Ct^{1-2\delta} + \| \mathcal{F}\left(M(t) - 1 \right) u_+ \|_2 \leq Ct^{1-2\delta} + Ct^{-1} \left\| x^2 u_+ \right\|_2 \\
&\leq Ct^{1-2\delta}
\end{aligned}$$

since $\left\| x^2 u_+ \right\|_2 = \| \Delta \hat{u}_+ \|_2 = \left\| \Delta \left(W_+ e^{i\Phi_+} \right) \right\|_2 \leq C\epsilon$. This completes the proof of Theorem 1.3. □

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