

On global properties of solutions of the equation

$$y'(t) = ay(t - by(t))$$

Svatoslav STANĚK*

(Received February 22, 1999)

Abstract. Global properties of all maximal solutions of the iterative functional differential equation $x'(t) = a[x(x(t)) - x(t)] + 1$ are considered. Using a correspondence among solutions of the above equation and those of the functional differential equation $y'(t) = ay(t - by(t))$, global properties of all maximal solutions of the last equation are described.

Key words: iterative functional differential equation, existence, maximal solution, global properties, asymptotic formula, Tychonoff-Schauder fixed point theorem.

1. Introduction

Hartung and Turi [3] studied “small” solutions corresponding to small (in sup norm) initial functions of the initial value problem (IVP for short)

$$y'(t) = ay(t - b|y(t)|), \quad t \geq 0, \quad (1)$$

$$y(t) = \Phi(t), \quad t \leq 0, \quad (2)$$

where $a > 0$, $b > 0$ are constants and Φ is Lipschitz-continuous on $(-\infty, 0]$. They proved that any solution $y(t)$ of (1), (2) with $\Phi(0) \neq 0$ is nonvanishing on $[0, \infty)$ and $y(t)$ is identically zero for $t \geq 0$ provided $\Phi(0) = 0$. Moreover, if $y(t)$ is a solution of IVP (1), (2) corresponding to the initial function $\Phi(t)$, then $-y(t)$ is a solution of IVP (1), (2) corresponding to the initial function $-\Phi(t)$ and so, without loss of generality, we can consider the equation

$$y'(t) = ay(t - by(t)), \quad t \geq 0 \quad (3)$$

instead of (1) for $\Phi(0) \geq 0$. Hartung and Turi [3] also showed that for $\Phi(0) > 0$ the following two statements are equivalent:

1991 Mathematics Subject Classification : 34K15, 34K25.

*Supported by grant no. 201/98/0318 of the Grant Agency of Czech Republic

(i) There exists $K > 0$ such that the solution $y(t)$ of IVP (1), (2) satisfies

$$t - b|y(t)| \geq -K \quad \text{for } t \geq 0;$$

(ii) There exist $T \in \mathbb{R}$ and $\alpha \in C^1([0, \infty))$ such that the solution of IVP (1), (2) has the form

$$y(t) = \frac{1}{b} \left(t + T + \alpha(t) \right), \quad t \geq 0,$$

where $\lim_{t \rightarrow \infty} \alpha(t) = 0$ and $\lim_{t \rightarrow \infty} \alpha'(t) = 0$;

and gave necessary and sufficient conditions imposed upon Φ for which statement (i) is satisfied.

In this paper we wish to consider the equation

$$y'(t) = ay(t - by(t)), \quad a \neq 0, b \neq 0 \quad (4)$$

without the initial function Φ . Equation (4) is a differential equation with deviating argument depending on state and may change its sign. We show that (4) is equivalent to the iterated functional differential equation

$$x'(t) = a[x(x(t)) - x(t)] + 1, \quad a \neq 0 \quad (5)$$

and consider properties of all maximal solutions of (5). From these properties we can derive properties of all maximal solutions of (4). We shall show, among others, that the asymptotic behavior of maximal solutions of (4) for $a > 0$ and $t \rightarrow \infty$ are close to that of maximal solutions of IVP (1), (2) (with $a > 0$, $b > 0$ and $\Phi(0) > 0$) for which statement (i) is satisfied.

We recall that the global properties of maximal solutions for the first-order iterative differential equations were considered in [1], [4]–[7].

2. Preliminaries

Definition 1 We say that x is a solution of (5) on an interval J if $x \in C^1(J)$ and (5) is satisfied for $t \in J$.

Definition 2 Let x be a solution of (5) on an interval J and y be a solution of (5) on an interval I . We say that y is a continuation of x if $J \subset I$, $J \neq I$ and $x(t) = y(t)$ for $t \in J$. In addition, if $t \geq s$ (resp. $t \leq s$) for any $t \in I - J$ and $s \in J$, then we say that y is a right (resp. left) continuation of x .

Definition 3 We say that x is a maximal solution of (5) if x has no

continuation.

Remark 1 Similarly we define a solution of (4) on an interval J and a maximal solution of (4).

Lemma 1 *If $y(t)$ is a solution of (4) on an interval J , then $x(t) = t - by(t)$ is a solution of (5) on J and conversely, if $x(t)$ is a solution of (5) on an interval J , then $y(t) = \frac{1}{b}(t - x(t))$ is a solution of (4) on J .*

Proof. Let $y(t)$ be a solution of (4) on an interval J and set $x(t) = t - by(t)$ for $t \in J$. Then $x \in C^1(J)$ and

$$\begin{aligned} x'(t) &= 1 - by'(t) = 1 - aby(t - by(t)) = 1 - aby(x(t)) \\ &= 1 + a[x(x(t)) - x(t)] \end{aligned}$$

for $t \in J$; hence $x(t)$ is a solution of (5) on J .

Let $x(t)$ be a solution of (5) on an interval J and set $y(t) = \frac{1}{b}(t - x(t))$ for $t \in J$. Then $y \in C^1(J)$ and

$$\begin{aligned} y'(t) &= \frac{1}{b}(1 - x'(t)) = -\frac{a}{b}[x(x(t)) - x(t)] \\ &= ay(x(t)) = ay(t - by(t)) \end{aligned}$$

for $t \in J$; hence y is a solution of (4) on J . □

The following lemma is obvious.

Lemma 2 *Let $x(t)$ be a solution of (5) on an interval J . Then*

$$x : J \rightarrow J.$$

Remark 2 If $x(t)$ is a solution of (5) on an interval J , then $x \in C^\infty(J)$.

Lemma 3 *A function $x(t)$ is a solution of (5) on an interval J if and only if the function $z(t) = -x(-t)$, $t \in I = \{t : -t \in J\}$ is a solution of the equation*

$$z'(t) = -a[z(z(t)) - z(t)] + 1 \tag{6}$$

on I .

Proof. Let $x(t)$ be a solution of (5) on an interval J and set $z(t) = -x(-t)$ for $t \in I$. Then

$$z'(t) = x'(-t) = a[x(x(-t)) - x(-t)] + 1 = -a[z(z(t)) - z(t)] + 1$$

and so $z(t)$ is a solution of (6) on I .

Let $z(t)$ be a solution of (6) on an interval I and set $x(t) = -z(-t)$ for $t \in J = \{t : -t \in I\}$. Then we can verify that $x(t)$ is a solution of (5) on the interval J . \square

Remark 3 By Lemma 3, it is sufficient to consider only solutions of (5) with $a > 0$.

Lemma 4 *Let $x(t)$ be a maximal solution of (5) with $a > 0$ on an interval J and $x(t_0) = t_0$ for a $t_0 \in J$. Then*

$$J = \mathbb{R} \quad \text{and} \quad x(t) \equiv t.$$

Proof. We see that the function $z(t) = t$ for $t \in \mathbb{R}$ is a maximal solution of (5). To prove our lemma it is sufficient to show that $x = z$. Set $w(t) = x(t) - z(t)$ for $t \in J$. Then $w(t_0) = 0$. Assume $w(t) \not\equiv 0$ on J . Then there exists a $t_1 \in J$ such that $w(t_1) = 0$ and let, for example, $(t_1, \infty) \cap J \neq \emptyset$ and

$$p(t) = \max\{|w(s)| : t_1 \leq s \leq t\} > 0$$

for $t \in J$, $t > t_1$ (analogously for $(-\infty, t_1) \cap J \neq \emptyset$ and $\max\{|w(s)| : t \leq s \leq t_1\} > 0$ for $t \in J$, $t < t_1$). Let $[c, d] \subset J$ be a compact interval such that $t_1 \in (c, d)$ provided t_1 is an inner point of J , otherwise $c = t_1$, and set $M = \max\{|x'(t)| : c \leq t \leq d\}$. Since $x(t_1) = t_1$, there exists a positive number ε , $\varepsilon \leq d - t_1$, such that $x(t) \in [c, d]$ for $t \in [t_1, t_1 + \varepsilon]$ (see Lemma 2). By the Taylor formula,

$$w'(t) = a[x(x(t)) - x(t)] = ax'(\xi)(x(t) - t) = ax'(\xi)w(t) \quad (7)$$

for $t \in [t_1, t_1 + \varepsilon]$, where $\xi (= \xi(t))$ lies between $x(t)$ and t , and so $\xi \in [c, d]$. Then (cf. (7)) $|w'(t)| \leq aM|w(t)|$ and

$$|w(t)| \leq \int_{t_1}^t |w'(s)| ds \leq aM \int_{t_1}^t |w(s)| ds \leq aMp(t)(t - t_1),$$

$$t \in [t_1, t_1 + \varepsilon].$$

Hence $p(t) \leq aMp(t)(t - t_1)$ on $[t_1, t_1 + \varepsilon]$ and since $p(t) > 0$ for $t \in (t_1, t_1 + \varepsilon]$,

$$1 \leq aM(t - t_1)$$

for $(t_1, t_1 + \varepsilon]$, a contradiction. We have proved that $x(t) = z(t) (= t)$ for

$t \in J$ and since $x(t)$ is a maximal solution, $J = \mathbb{R}$ and $x(t) \equiv t$. \square

Lemma 5 *Let $x(t)$ be a maximal solution of (5) with $a > 0$ on an interval J . Then $x'(t_0) = 1$ for some $t_0 \in J$ if and only if $J = \mathbb{R}$ and $x(t) \equiv t$.*

Proof. We see that $x'(t_0) = 1$ for some $t_0 \in J$ if and only if $x(T) = T$ with $T = x(t_0)$. By Lemma 4, $x(T) = T$ for some $T \in J$ if and only if $J = \mathbb{R}$ and $x(t) \equiv t$. \square

Denote by \mathcal{A}^+ the set of all maximal solutions x of (5) with $a > 0$ such that $x' < 1$, and by \mathcal{B}^+ the set of all maximal solutions x of (5) with $a > 0$ such that $x' > 1$.

Lemma 6 *Let $x(t)$ be a maximal solution of (5) with $a > 0$ on J and let $x(t) \not\equiv t$. Then either $x \in \mathcal{A}^+$ or $x \in \mathcal{B}^+$.*

Proof. By Lemma 5, $x'(t) \neq 1$ for $t \in J$. Then either $x'(t) < 1$ or $x'(t) > 1$ on J , and consequently either $x \in \mathcal{A}^+$ or $x \in \mathcal{B}^+$. \square

Lemma 7 *Let $x(t)$ be a maximal solution of (5) with $a > 0$ on J , $x(t) \not\equiv t$. Then $x \in \mathcal{A}^+$ if and only if $x(t) < t$ for $t \in J$ and $x \in \mathcal{B}^+$ if and only if $x(t) > t$ for $t \in J$.*

Proof. Since $x \in \mathcal{A}^+$ if and only if $x(x(t)) - x(t) < 0$ for $t \in J$ and $x(t) \neq t$ on J by Lemma 4, we see that $x \in \mathcal{A}^+$ if and only if $x(t) < t$ for $t \in J$. Analogously for $x \in \mathcal{B}^+$. \square

3. Set \mathcal{A}^+

Lemma 8 *Let $x \in \mathcal{A}^+$ be defined on J . Then $x'(t) \neq 0$ for $t \in J$.*

Proof. Assume $x'(t_0) = 0$ for some $t_0 \in J$. Then

$$a[x(x(t_0)) - x(t_0)] + 1 = 0. \quad (8)$$

Consider the IVP

$$w' = a(x(w) - w) + 1, \quad w(t_0) = x(t_0). \quad (9)$$

Then $x(t)$ is a solution of IVP (9) on J . On the other hand, equality (8) implies that the constant function $w(t) = x(t_0)$ is a solution of IVP (9) on \mathbb{R} . Since $x \in C^1(J)$, the uniqueness theorem for ODEs gives $x(t) = x(t_0)$ for $t \in J$, and so $x' = 0$ which is impossible. Hence $x'(t) \neq 0$ on J . \square

Corollary 1 *Let $x \in \mathcal{A}^+$ be defined on J . Then either $0 < x'(t) < 1$ or $x'(t) < 0$ for $t \in J$.*

Set

$$\mathcal{A}_1^+ = \left\{ x : x \in \mathcal{A}^+, 0 < x' < 1 \right\}, \quad \mathcal{A}_2^+ = \left\{ x : x \in \mathcal{A}^+, x' < 0 \right\}.$$

Remark 4 By Corollary 1, $\mathcal{A}^+ = \mathcal{A}_1^+ \cup \mathcal{A}_2^+$.

Theorem 1 \mathcal{A}_2^+ is an empty set.

Proof. Assume $x \in \mathcal{A}_2^+$ is defined on J . Then

$$x'(t) = ax'(\xi)(x(t) - t) + 1 \tag{10}$$

for $t \in J$, where ξ lies between t and $x(t)$. Since $x(t) - t < 0$ for $t \in J$ by Lemma 7 and $x'(\xi) < 0$, we have (cf. (10)) $x'(t) > 1$ for $t \in J$, a contradiction. \square

Lemma 9 *Let $x \in \mathcal{A}_1^+$ be defined on J . Then $x''(t) < 0$ for $t \in J$.*

Proof. Since (cf. Remark 2)

$$x''(t) = ax'(t)(x'(x(t)) - 1) \tag{11}$$

for $t \in J$ and $ax'(t) > 0$, $x'(x(t)) - 1 < 0$ on J , it follows from (11) that $x''(t) < 0$ for $t \in J$. \square

Theorem 2 *Let $x \in \mathcal{A}_1^+$ be defined on J . Then $J = \mathbb{R}$.*

Proof. We first prove that $J = \bar{J}$, where \bar{J} stands for the closure of J (in \mathbb{R}). Assume $J \neq \bar{J}$. Then there exists $\xi \in \bar{J} - J$. Let $\xi < t$ for $t \in J$. Then, by Lemma 2 and Lemma 7, $\xi < x(t) < t$ for $t \in J$ and therefore $\lim_{t \rightarrow \xi^+} x(t) = \xi$. Define $z \in C^0(J \cup \{\xi\})$ by

$$z(t) = \begin{cases} x(t) & \text{for } t \in J, \\ \xi & \text{for } t = \xi. \end{cases}$$

We shall show that $z(t)$ is a solution of (5) on the interval $J_1 = J \cup \{\xi\}$. Since $z(t)$ is a solution of (5) on J , it is sufficient to verify that $z \in C^1(J_1)$ and

$$z'(\xi) = a[z(z(\xi)) - z(\xi)] + 1 = a(\xi - \xi) + 1 = 1.$$

From the equality

$$z(t) - z(\xi) = \int_{\xi}^t \left(a[x(x(s)) - x(s)] + 1 \right) ds, \quad t \in J_1,$$

we deduce that

$$\begin{aligned} \frac{z(t) - z(\xi)}{t - \xi} - 1 &= \frac{a}{t - \xi} \int_{\xi}^t [x(x(s)) - x(s)] ds \\ &= \frac{a}{t - \xi} \int_{\xi}^t [z(z(s)) - z(s)] ds \end{aligned} \quad (12)$$

for $t > \xi$, $t \in J$. The function $z(z(t)) - z(t)$ is continuous on J_1 , $z(z(\xi)) - z(\xi) = 0$, and consequently for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z(z(t)) - z(t)| < \frac{\varepsilon}{a} \quad \text{for } \xi \leq t < \xi + \delta.$$

Hence (cf. (12))

$$\left| \frac{z(t) - z(\xi)}{t - \xi} - 1 \right| < \frac{a}{t - \xi} \frac{\varepsilon}{a} (t - \xi) = \varepsilon \quad \text{for } \xi < t < \xi + \delta,$$

which yields $z'(\xi) = 1$. We have proved that $z(t)$ is a solution of (5) on J_1 , and so z is a left continuation of x , a contradiction.

Assume $\xi > t$ for $t \in J$. Since $0 < x'(t) < 1$, $x''(t) < 0$ and $x(t) < t$ for $t \in J$, there exists the finite limit $\lim_{t \rightarrow \xi^-} x(t) = C$ and $C < \xi$. Set

$$u(t) = \begin{cases} x(t) & \text{for } t \in J, \\ C & \text{for } t = \xi. \end{cases}$$

Then one can prove that $u(t)$ is a solution of (5) on J_1 . Consequently, u is a right continuation of x , a contradiction.

Hence $J = \bar{J}$ and therefore either $J = [A, B]$ or $J = [A, \infty)$ or $J = (-\infty, B]$ or $J = \mathbb{R}$, where $A, B \in \mathbb{R}$. Assume either $J = [A, B]$ or $J = [A, \infty)$. Then $x(A) \geq A$ by Lemma 2. On the other hand $x(t) < t$ for $t \in J$ by Lemma 7, and consequently $x(A) < A$, a contradiction. Let $J = (-\infty, B]$. Then $x(B) < B$ by Lemma 7. Consider the IVP

$$w' = a(x(w) - w) + 1, \quad w(B) = x(B). \quad (13)$$

Since $x \in C^1(J)$ and $x(B) < B$, IVP (13) has a unique solution in a neighbourhood \mathcal{U} of the point $t = \xi$, say $w(t)$. On the other hand $x(t)$ is

a solution of IVP (13) on $J = (-\infty, B]$. So $x(t) = w(t)$ for $t \in J \cap \mathcal{U}$ and then $w(t) \leq B$ for $t \in \mathcal{U}$. Define

$$y(t) = \begin{cases} x(t) & \text{for } t \in J, \\ w(t) & \text{for } t \in \mathcal{U} - J. \end{cases}$$

Then $y \in C^1(J \cup \mathcal{U})$, $y'(t) = w'(t) = a[x(w(t)) - w(t)] + 1 = a[x(y(t)) - y(t)] + 1 = a[y(y(t)) - y(t)] + 1$ for $t \in \mathcal{U} - J$; hence $y(t)$ is a right continuation of x on the interval $J \cup \mathcal{U}$, a contradiction. This proves $J = \mathbb{R}$. \square

Lemma 10 *Let $x \in \mathcal{A}_1^+$. Then there exists the finite limit $\lim_{t \rightarrow \infty} x(t) = T$ and*

$$x(T) = T - \frac{1}{a}.$$

Moreover, $x'(T) < \frac{1}{2}$,

$$\lim_{t \rightarrow -\infty} (x(t) - t)^{(i)} = 0, \quad i = 0, 1, 2$$

and

$$\lim_{t \rightarrow \infty} x'(t) = 0, \quad \lim_{t \rightarrow \infty} x''(t) = 0.$$

Proof. Since $0 < x'(t) < 1$ and $x''(t) < 0$ for $t \in \mathbb{R}$, there exist finite limits $\lim_{t \rightarrow -\infty} x'(t) = K$ and $\lim_{t \rightarrow \infty} x'(t) = L$, $1 \geq K > L \geq 0$. Assume $L > 0$. Then $\lim_{t \rightarrow \infty} x(t) = \infty$, and consequently $\lim_{t \rightarrow \infty} x''(t) = \lim_{t \rightarrow \infty} ax'(t)(x'(x(t)) - 1) = aL(L - 1) < 0$, which contradicts $\lim_{t \rightarrow \infty} x'(t) = L$. Assume $K < 1$. Using the equality $\lim_{t \rightarrow -\infty} x(t) = -\infty$ which follows from the inequality $x(t) < t$ for $t \in \mathbb{R}$, we have $\lim_{t \rightarrow -\infty} x''(t) = \lim_{t \rightarrow -\infty} ax'(t)(x'(x(t)) - 1) = aK(K - 1) < 0$ and then $\lim_{t \rightarrow -\infty} x'(t) = -\infty$, a contradiction. Hence $K = 1$, $L = 0$ and therefore $\lim_{t \rightarrow \pm\infty} x''(t) = \lim_{t \rightarrow \pm\infty} x'(t)(x'(x(t)) - 1) = 0$.

We know that $x(t)$ is increasing on \mathbb{R} . Hence either $\lim_{t \rightarrow \infty} x(t) = \infty$ or $\lim_{t \rightarrow \infty} x(t) = T$. Assume $\lim_{t \rightarrow \infty} x(t) = \infty$. Since $a[x(x(t)) - x(t)] + 1 > 0$ on \mathbb{R} and $\lim_{t \rightarrow -\infty} x(t) = -\infty$ (see, e.g., Lemma 7), we have $a(x(t) - t) + 1 > 0$ for $t \in \mathbb{R}$ and therefore $x(t) > t - \frac{1}{a}$ on \mathbb{R} . Let $A \in \mathbb{R}$ be a number such that $x'(t) \leq \frac{1}{2}$ for $t \in [A, \infty)$. The existence of A follows from the equality

$\lim_{t \rightarrow \infty} x'(t) = 0$. Then

$$x(t) - x(A) = \int_A^t x'(s) ds \leq \frac{1}{2}(t - A)$$

and therefore

$$t - \frac{1}{a} < x(t) \leq x(A) + \frac{1}{2}(t - A), \quad t \in [A, \infty),$$

which is impossible. Hence $\lim_{t \rightarrow \infty} x(t) = T$. From the equality

$$x'(t) = -ax(t) + (ax(x(t)) + 1), \quad t \in \mathbb{R}$$

we obtain (for $t \in \mathbb{R}$)

$$x(t) = e^{-at} \left(x(0) + \int_0^t e^{as} (ax(x(s)) + 1) ds \right)$$

and using the L'Hospital rule

$$\begin{aligned} T &= \lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} \frac{x(0) + \int_0^t e^{as} (ax(x(s)) + 1) ds}{e^{at}} \\ &= \frac{1}{a} \lim_{t \rightarrow \infty} (ax(x(t)) + 1) = x(T) + \frac{1}{a}. \end{aligned}$$

Hence $x(T) = T - \frac{1}{a}$. Then $x'(T) = a[x(T - \frac{1}{a}) - x(T)] + 1 = -x'(\nu) + 1$ for some $\nu \in (T - \frac{1}{a}, T)$. By Lemma 9, x' is decreasing, and so $x'(\nu) > x'(T)$. Consequently, $x'(T) < -x'(T) + 1$ which implies $x'(T) < \frac{1}{2}$.

By the Taylor formula,

$$(x(t) - t)' = a[x(x(t)) - x(t)] = ax'(\xi)(x(t) - t), \quad (14)$$

where ξ lies between $x(t)$ and t , and therefore $\xi < t$, which gives $x'(\xi) > x'(t)$ using the fact that x' is decreasing on \mathbb{R} . Then

$$(x(t) - t)' < ax'(t)(x(t) - t), \quad t \in \mathbb{R}.$$

Applying differential inequalities (see, e.g., [2]) we have

$$x(t) - t \geq x(0)e^{a(x(t)-x(0))}$$

for $t \in (-\infty, 0]$. Since $\lim_{t \rightarrow -\infty} x(t) = -\infty$ and $x(t) - t < 0$ for $t \in \mathbb{R}$, we obtain $\lim_{t \rightarrow -\infty} (x(t) - t) = 0$. Hence the lemma is proved. \square

Theorem 3 For each $(T, \xi) \in \mathbb{R}^2$, $0 < T - \xi \leq \frac{1}{a}$, there exists $x \in \mathcal{A}_1^+$

such that

$$x(T) = \xi.$$

The *proof* of Theorem 3 is based on the following two lemmas, where $\mathcal{K}_{(T,\xi)}$ will be denoted, for each $(T, \xi) \in \mathbb{R}^2$, $0 < T - \xi \leq \frac{1}{a}$, the set

$$\mathcal{K}_{(T,\xi)} = \left\{ x : x \in C^1((-\infty, T]), x(T) = \xi, t - T + \xi \leq x(t) \leq t, \right. \\ \left. 0 \leq x'(t) \leq 1 \text{ for } t \in (-\infty, T] \right\}.$$

The set $\mathcal{K}_{(T,\xi)} \neq \emptyset$ since $u_c \in \mathcal{K}_{(T,\xi)}$ for each $c \in [0, \frac{1}{T-\xi}]$ where

$$u_c(t) = t - (T - \xi)e^{c(t-T)}, \quad t \in (-\infty, T].$$

Lemma 11 *Let $(T, \xi) \in \mathbb{R}^2$, $0 < T - \xi \leq \frac{1}{a}$ and $x \in \mathcal{K}_{(T,\xi)}$. Then there exists a unique solution y of the IVP*

$$y' = a(x(y) - x(t)) + 1, \tag{15}$$

$$y(T) = \xi \tag{16}$$

on $(-\infty, T]$ and, moreover, $y \in \mathcal{K}_{(T,\xi)}$.

Proof. Since $x \in C^1((-\infty, T])$, there exists the unique maximal solution $y(t)$ of IVP (15), (16) on an interval I . We shall show that $I = (-\infty, T]$. Assume $I \neq (-\infty, T]$. Let $y(t_0) = t_0$ for a $t_0 \in I$. Consider equation (15) together with the initial condition

$$y(t_0) = t_0. \tag{17}$$

Then $y(t)$ is a solution of IVP (15), (17) on I and since the function $u(t) \equiv t$ is also a solution of this IVP on $(-\infty, T]$, we have $y(t) = u(t) = t$ for $t \in I$ by the uniqueness theorem for ODEs, which contradicts $y(T) = \xi$. Hence

$$y(t) < t \quad \text{for } t \in I. \tag{18}$$

Let $t_1 \in I$. Then (cf. (18)) $x(y(t_1)) - x(t_1) = x'(\xi)(y(t_1) - t_1) \leq 0$ where $y(t_1) < \xi < t_1$, and therefore $y'(t_1) = ax'(\xi)(y(t_1) - t_1) + 1 \leq 1$ which proves

$$y'(t) \leq 1 \quad \text{for } t \in I. \tag{19}$$

Assume $y(t_2) < t_2 - T + \xi$ for some $t_2 \in I$. Then (cf. (16)) there exists $t_3 \in I$ such that $y'(t_3) > 1$ which contradicts (19). Hence

$$t - T + \xi \leq y(t) \quad \text{for } t \in I. \quad (20)$$

By our assumption y is the maximal solution of IVP (15), (16) defined on I , $I \neq (-\infty, T]$. Thus $I = (A, T]$, $-\infty < A < T$ and $\limsup_{t \rightarrow A^+} |y(t)| = \infty$ which contradicts (18) and (20); hence $I = (-\infty, T]$.

Assume $y'(t_4) < 0$ where $t_4 \in (-\infty, T]$. Since (cf. (20))

$$a[x(y(t_4)) - x(t_4)] = ax'(\varepsilon)(y(t_4) - t_4) \geq -x'(\varepsilon)$$

for some $\varepsilon \in (y(t_4), t_4)$, we have

$$0 > y'(t_4) = a[x(y(t_4)) - x(t_4)] + 1 \geq -x'(\varepsilon) + 1.$$

Then $x'(\varepsilon) > 1$ which contradicts $x \in \mathcal{K}_{(T, \xi)}$. Thus

$$0 \leq y'(t), \quad t \in (-\infty, T]. \quad (21)$$

From (16), (18)–(21) and $I = (-\infty, T]$ it follows that $y \in \mathcal{K}_{(T, \xi)}$. \square

Let $(T, \xi) \in \mathbb{R}^2$, $0 < T - \xi \leq \frac{1}{a}$. By Lemma 11, for each $x \in \mathcal{K}_{(T, \xi)}$ there exists a unique maximal solution y_x of IVP (15), (16) and $y_x \in \mathcal{K}_{(T, \xi)}$. Define the operator $P_{(T, \xi)}$ by $P_{(T, \xi)}(x) = y_x$ for $x \in \mathcal{K}_{(T, \xi)}$. Then

$$P_{(T, \xi)} : \mathcal{K}_{(T, \xi)} \rightarrow \mathcal{K}_{(T, \xi)}. \quad (22)$$

Let \mathbf{X}_T be the Fréchet space of C^1 -functions on $(-\infty, T]$ with the topology of locally uniform convergence of the functions and their derivatives on $(-\infty, T]$.

Lemma 12 *Let $(T, \xi) \in \mathbb{R}^2$, $0 < T - \xi \leq \frac{1}{a}$. Then the operator $P_{(T, \xi)} : \mathcal{K}_{(T, \xi)} \subset \mathbf{X}_T \rightarrow \mathbf{X}_T$ is compact.*

Proof. We first prove that $P_{(T, \xi)}$ is a continuous operator. Let $\{x_n\} \subset \mathcal{K}_{(T, \xi)}$ be a convergent sequence (in \mathbf{X}_T), $x_n \rightarrow x$. Then $\lim_{n \rightarrow \infty} x_n^{(i)}(t) = x^{(i)}(t)$ locally uniformly on $(-\infty, T]$ for $i = 0, 1$. Set

$$y_n = P_{(T, \xi)}(x_n), \quad y = P_{(T, \xi)}(x), \quad n \in \mathbb{N}.$$

Then

$$y_n(T) = \xi, \quad y_n'(t) = a[x_n(y_n(t)) - x_n(t)] + 1 \quad (23)$$

for $t \in (-\infty, T]$, $n \in \mathbb{N}$ and

$$y(T) = \xi, \quad y'(t) = a[x(y(t)) - x(t)] + 1, \quad t \in (-\infty, T].$$

We now show that $\lim_{n \rightarrow \infty} y_n^{(i)}(t) = y^{(i)}(t)$ locally uniformly on $(-\infty, T]$ ($i = 0, 1$). Since (cf. (22)) $\{y_n\} \subset \mathcal{K}_{(T, \xi)}$, we have $0 \leq y_n'(t) \leq 1$, $t - T + \xi \leq y_n(t) \leq t$ for $t \in (-\infty, T]$ and $n \in \mathbb{N}$. Let $\{y_{k_n}\}$ be a subsequence of $\{y_n\}$. By the Cauchy diagonal process and the Arzelà-Ascoli theorem, there exists a subsequence $\{y_{k_{j_n}}(t)\}$ of $\{y_{k_n}(t)\}$ locally uniformly convergent on $(-\infty, T]$. Set

$$z(t) = \lim_{n \rightarrow \infty} y_{k_{j_n}}(t), \quad t \in (-\infty, T].$$

Applying the Lebesgue dominated convergence theorem as $n \rightarrow \infty$ in the equalities

$$y_{k_{j_n}}(t) = \xi + \int_T^t \left(a[x_{k_{j_n}}(y_{k_{j_n}}(s)) - x_{k_{j_n}}(s)] + 1 \right) ds, \\ t \in (-\infty, T], \quad n \in \mathbb{N},$$

we get

$$z(t) = \xi + \int_T^t \left(a[x(z(s)) - x(s)] + 1 \right) ds, \quad t \in (-\infty, T],$$

and so $z(t)$ is a solution of IVP (15), (16) on $(-\infty, T]$. We know that this IVP has a unique solution and that $y(t)$ is a solution of this problem, and consequently $y(t) = z(t)$ for $t \in (-\infty, T]$. We have proved that any subsequence $\{y_{k_n}\}$ of $\{y_n\}$ has in turn a subsequence $\{y_{k_{j_n}}\}$ such that $\lim_{n \rightarrow \infty} y_{k_{j_n}}(t) = y(t)$ locally uniformly on $(-\infty, T]$. Hence $\lim_{n \rightarrow \infty} y_n(t) = y(t)$ locally uniformly on $(-\infty, T]$ and then

$$\lim_{n \rightarrow \infty} (y_n'(t) - y'(t)) = \lim_{n \rightarrow \infty} a[x_n(y_n(t)) - x_n(t) - x(y(t)) + x(t)] = 0$$

locally uniformly on $(-\infty, T]$. Thus $\lim_{n \rightarrow \infty} y_n = y$ in \mathbf{X}_T , and so

$$\lim_{n \rightarrow \infty} P_{(T, \xi)}(x_n) = \lim_{n \rightarrow \infty} y_n = y = P_{(T, \xi)}(x),$$

which proves that $P_{(T, \xi)}$ is a continuous operator.

It remains to show that $P_{(T, \xi)}(\mathcal{K}_{(T, \xi)})$ is a relatively compact subset of \mathbf{X}_T . Let $\{y_n\} \subset P_{(T, \xi)}(\mathcal{K}_{(T, \xi)})$. Then there exists a sequence $\{x_n\} \subset \mathcal{K}_{(T, \xi)}$ such that $y_n = P_{(T, \xi)}(x_n)$, $n \in \mathbb{N}$, and therefore equalities (23) are satisfied. Using the Cauchy diagonal process and the Arzelà-Ascoli theorem

we can assume, without loss of generality, that $\{x_n(t)\}$ and $\{y_n(t)\}$ are locally uniformly convergent on $(-\infty, T]$ and let $\lim_{n \rightarrow \infty} x_n(t) = x(t)$, $\lim_{n \rightarrow \infty} y_n = y(t)$. Then (23) implies that $\{y'_n(t)\}$ is locally uniformly convergent on $(-\infty, T]$ and we have $\lim_{n \rightarrow \infty} y'_n(t) = y'(t)$. Hence $\{y_n\}$ is a convergent sequence in \mathbf{X}_T , and consequently $P_{(T,\xi)}(\mathcal{K}_{(T,\xi)})$ is relatively compact subset of \mathbf{X}_T . This completes the proof. \square

Proof of Theorem 3. Fix $(T, \xi) \in \mathbb{R}^2$, $0 < T - \xi \leq \frac{1}{a}$. Since $\mathcal{K}_{(T,\xi)}$ is a bounded convex closed subset of the Fréchet space \mathbf{X}_T and $P_{(T,\xi)} : \mathcal{K}_{(T,\xi)} \rightarrow \mathcal{K}_{(T,\xi)}$ is a compact operator by Lemma 12, we can apply Tychonoff-Schauder fixed point theorem to the operator $P_{(T,\xi)}$. Hence there exists a fixed point $y \in \mathcal{K}_{(T,\xi)}$ of $P_{(T,\xi)}$. Of course, $0 \leq y'(t) \leq 1$,

$$y'(t) = a[y(y(t)) - y(t)] + 1, \quad t \in (-\infty, T]$$

and $y(T) = \xi$. Then y is a solution of (5) on $(-\infty, T]$. By Theorem 2, any $z \in \mathcal{A}_1^+$ is defined on \mathbb{R} , and consequently there exists a right continuation of y on \mathbb{R} , say x . The solution x satisfies the conclusions of Theorem 3. \square

4. Set \mathcal{B}^+

Theorem 4 \mathcal{B}^+ is an empty set.

Proof. Assume $\mathcal{B}^+ \neq \emptyset$. Then there exists $x \in \mathcal{B}^+$, and let x be defined on an interval J . Since $x'(t) > 1$ for $t \in J$, the equality $x''(t) = ax'(t)[x'(x(t)) - 1]$ implies $x''(t) > 0$ on J , and consequently $x'(t)$ and $(0 <) x(t) - t$ are increasing on J . By Lemma 2, $x(t) \in J$ for each $t \in J$ and therefore $[t, \infty) \subset J$ for each $t \in J$. From the Taylor formula we get $x'(t) = ax'(\varepsilon)[x(t) - t] + 1$ for $t \in J$ where $t < \varepsilon (= \varepsilon(t)) < x(t)$. Then $x'(t) < x'(\varepsilon)$ since $x'' > 0$, and so $x'(t) > ax'(t)[x(t) - t] + 1$ and

$$x'(t)[1 - a(x(t) - t)] > 1 \tag{24}$$

for $t \in J$. We know that $\lim_{t \rightarrow \infty} (x(t) - t) = \infty$. Hence $\lim_{t \rightarrow \infty} x'(t)[1 - a(x(t) - t)] = -\infty$ which contradicts (24). \square

5. Survey of main results

Set $\varepsilon = \text{sign } a$ and $\nu = \text{sign } b$.

Theorem 5 For each $(T, \xi) \in \mathbb{R}^2$, $0 < \varepsilon(T - \xi) \leq \frac{1}{|a|}$, there exists a maximal solution x of (5) on \mathbb{R} such that $x(T) = \xi$.

If $x \neq t$ is a maximal solution of (5) on an interval J , then

- (i) $J = \mathbb{R}$;
- (ii) $\varepsilon(x(t) - t) < 0$, $0 < x'(t) < 1$, $\varepsilon x''(t) < 0$ for $t \in \mathbb{R}$;
- (iii) there exists $T \in \mathbb{R}$ such that $x(T) = T - \frac{1}{a}$ and $\lim_{t \rightarrow \varepsilon\infty} x(t) = T$;
- (iv) $x(t) = t + \alpha(t)$, where $\alpha \in C^\infty(\mathbb{R})$, $\varepsilon\alpha(t) < 0$ for $t \in \mathbb{R}$ and

$$\lim_{t \rightarrow -\varepsilon\infty} \alpha(t) = \lim_{t \rightarrow -\varepsilon\infty} \alpha'(t) = \lim_{t \rightarrow -\varepsilon\infty} \alpha''(t) = 0;$$

- (v) $x(t) = T + \beta(t)$, where $\beta \in C^\infty(\mathbb{R})$, $\varepsilon\beta(t) < 0$ for $t \in \mathbb{R}$ and

$$\lim_{t \rightarrow \varepsilon\infty} \beta(t) = \lim_{t \rightarrow \varepsilon\infty} \beta'(t) = \lim_{t \rightarrow \varepsilon\infty} \beta''(t) = 0.$$

Proof. Let $\varepsilon = 1$. The first statement is Theorem 3. Let x be a maximal solution of (5) on an interval J , $x \neq t$. By Theorem 1 and Theorem 4, $x \in \mathcal{A}_1^+$ and $J = \mathbb{R}$ by Theorem 2. Property (ii) follows from the definition of the set \mathcal{A}_1^+ , Lemma 7 and Lemma 9. Remark 2, Lemma 10 and (ii) imply properties (iii)–(v). For $\varepsilon = -1$, the assertions of our theorem follow from Lemma 3 (cf. Remark 3) and Theorem 5 with $\varepsilon = 1$. \square

Applying Lemma 1 to Theorem 5 we immediately obtain the following properties of maximal solutions for equation (4).

Theorem 6 For each $(T, \xi) \in \mathbb{R}^2$, $0 < \varepsilon(T - \xi) \leq \frac{1}{|a|}$, there exists a maximal solution y of (4) on \mathbb{R} such that $y(T) = \frac{1}{b}(T - \xi)$.

If y is a maximal solution of (4) on an interval J and $y \neq 0$, then

- (j) $J = \mathbb{R}$;
- (jj) $\varepsilon\nu y(t) > 0$, $0 < \nu y'(t) < \frac{1}{|b|}$, $\varepsilon\nu y''(t) > 0$ for $t \in \mathbb{R}$;
- (jjj) there exists $T \in \mathbb{R}$ such that $y(T) = \frac{1}{ab}$ and $\lim_{t \rightarrow \varepsilon\infty} (t - by(t)) = T$;
- (jv) $\lim_{t \rightarrow -\varepsilon\infty} y(t) = \lim_{t \rightarrow -\varepsilon\infty} y'(t) = \lim_{t \rightarrow -\varepsilon\infty} y''(t) = 0$;
- (v) $y(t) = \frac{1}{b}(t - T + \gamma(t))$, where $\gamma \in C^\infty(\mathbb{R})$, $\varepsilon\gamma(t) > 0$ for $t \in \mathbb{R}$ and

$$\lim_{t \rightarrow \varepsilon\infty} \gamma(t) = \lim_{t \rightarrow \varepsilon\infty} \gamma'(t) = \lim_{t \rightarrow \varepsilon\infty} \gamma''(t) = 0.$$

References

- [1] Eder E., *The functional differential equation $x'(t) = x(x(t))$* . J. Differential Equations **54** (1984), 390–400.
- [2] Hartman P., *Ordinary Differential Equations*. John Wiley, New York, 1964.
- [3] Hartung F. and Turi J., *On the asymptotic behavior of the solutions of a state-dependent delay equation*. Differential and Integral Equations **8** (1995), 1867–1872.
- [4] Staněk S., *On global properties of solutions of functional differential equation $x'(t) = x(x(t)) + x(t)$* . Dynamical Systems and Appl. **4** (1995), 263–278.
- [5] Staněk S., *Global properties of decreasing solutions of the equation $x'(t) = x(x(t)) + x(t)$* . Functional Differential Equations **4** (1997), 191–213.
- [6] Staněk S., *Global properties of solutions of iterative-differential equations*. Functional Differential Equations **5** (1998), 463–481.
- [7] Wang Ke, *On the equation $x'(t) = f(x(x(t)))$* . Funkcialaj Ekvacioj **33** (1990), 405–425.

Department of Mathematical Analysis
Faculty of Science
Palacký University
Tomkova 40, 779 00 Olomouc
Czech Republic
E-mail: stanek@risc.upol.cz