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On a correspondence between blocks of finite groups induced from the Isaacs character correspondence

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Abstract. We show that the Isaacs character correspondence induces a block correspondence of finite groups and that the corresponding blocks have common invariants.

Key words: finite groups, modular representations, characters.

Let G be a finite group. Let $(\mathcal{K}, \mathcal{O}, \mathcal{F})$ be a p-modular system and suppose $\mathcal{K}$ is algebraically closed. Let A be a finite group which acts on G with $(|A|, |G|)=1$. We denote by $\mathrm{Irr}(G)$ the set of ordinary irreducible characters of G and by $\mathrm{Irr}_A(G)$ the set of A-invariant ordinary irreducible characters of G. If A is a solvable group, there exists the Glauberman correspondence between $\mathrm{Irr}_A(G)$ and $\mathrm{Irr}(\mathrm{C}_G(A))$ and Watanabe [Wa] showed that this correspondence induces perfect isometries between A-invariant blocks of G and blocks of $\mathrm{C}_G(A)$ under some conditions. On the other hand, if A is not solvable, then G is solvable with odd order by the Feit-Thompson Theorem and there exists the Isaacs correspondence between $\mathrm{Irr}_A(G)$ and $\mathrm{Irr}(\mathrm{C}_G(A))$. In this paper we show the Isaacs correspondence also induces perfect isometries between blocks. Thus we may assume that G is a solvable group with odd order. The Isaacs correspondence are given by the following. Starting from $G_0=G$, we define subgroups $G_{i+1}=[G_i, A]'C_G(A)$ of G for any $i \geqq 0$ inductively. Since G is solvable, we have

$$G=G_0>G_1>G_2>\cdots>G_k=\mathrm{C}_G(A)$$

for some k ([I1, p.633]) and there exists a unique $\xi \in \mathrm{Irr}_A(G_{i+1})$ such that $2 \nmid [\chi_{G_{i+1}}, \xi]$ for each $\chi \in \mathrm{Irr}_A(G_i)$ by [I1, Corollary 10.7] for each i, $0 \leqq i \leqq k-1$. Then we have character correspondences $\sigma(G_i, G_{i+1})$: $\mathrm{Irr}_A(G_i) \longrightarrow \mathrm{Irr}_A(G_{i+1})$ and the *Isaacs correspondence*

$$\begin{aligned}&\sigma(G, \mathrm{C}_G(A))\\ &\quad =\sigma(G_{k-1}, G_k)\cdots\sigma(G_0, G_1) : \mathrm{Irr}_A(G) \longrightarrow \mathrm{Irr}(\mathrm{C}_G(A))\end{aligned}$$

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by their composition. The aim of this paper is to show the following theorem.

Theorem 1 *Let A act on G with $(|A|,|G|)=1$ and assume $|G|$ is odd. Let D be a p-subgroup of $\mathrm{C}_G(A)$. We denote by $\mathrm{Bl}_A(G|D)$ the set of A-invariant blocks of G with defect group D. Then the Isaacs correspondence induces a correspondence between $\mathrm{Bl}_A(G|D)$ and a set of $\mathrm{Bl}(\mathrm{C}_G(A)|D)$ which satisfies the following conditions. Let $B \in \mathrm{Bl}_A(G|D)$ correspond to $b \in \mathrm{Bl}(\mathrm{C}_G(A)|D)$.*

(a) *There exists a perfect isometry σ between B and b induced from the Isaacs correspondence $\sigma(G, \mathrm{C}_G(A))$.*

(b) *If $\alpha \in \mathrm{IBr}(B)$, then $\sigma(\alpha) \in \mathrm{IBr}(b)$ where $\mathrm{IBr}(B)$ is the set of irreducible Brauer characters of B. This correspondence is a bijection from $\mathrm{IBr}(B)$ to $\mathrm{IBr}(b)$.*

(c) *$d_{\chi\alpha}=d_{\sigma(\chi)\sigma(\alpha)}$ and $c_{\alpha\alpha'}=c_{\sigma(\alpha)\sigma(\alpha')}$ for all $\chi \in \mathrm{Irr}(B)$ and $\alpha, \alpha' \in \mathrm{IBr}(B)$, where $d_{\chi\alpha}$ are decomposition numbers and $c_{\alpha\alpha'}$ are Cartan invariants.*

Let G and H be finite groups. Let $\mathfrak{B}$ (resp. $\mathfrak{b}$) be a set of blocks of G (resp. H) and σ an isometry between the $\mathbb{Z}$-linear space with bases $\bigcup_{B\in\mathfrak{B}} \mathrm{Irr}(B)$ and $\bigcup_{b\in\mathfrak{b}} \mathrm{Irr}(b)$. Then we can see that σ is an isometry between the $\mathcal{K}$-linear spaces with bases $\bigcup_{B\in\mathfrak{B}} \mathrm{Irr}(B)$ and $\bigcup_{b\in\mathfrak{b}} \mathrm{Irr}(b)$ by the extension of coefficients. We define the generalized character μ of $G \times H$ by $\mu(g,h)=\sum_{B\in\mathfrak{B}}\sum_{\chi\in B}\chi(g)\sigma(\chi)(h)$ for all $g \in G$ and $h \in H$. If σ is bijective and satisfies the following conditions, then σ is called a *perfect isometry* between $\mathfrak{B}$ and $\mathfrak{b}$ ([B, Definition 1.1]).

(a) For all $g \in G$ and $h \in H$, $(\mu(g,h)/|\mathrm{C}_G(g)|) \in \mathcal{O}$ and $(\mu(g,h)/|\mathrm{C}_H(h)|) \in \mathcal{O}$.

(b) If $\mu(g,h) \neq 0$, then g is p-regular if and only if h is p-regular.

If there exists a perfect isometry σ between $\mathfrak{B}$ and $\mathfrak{b}$, then we have the cardinality of $\mathfrak{B}$ is equal to that of $\mathfrak{b}$ and there exists a perfect isometry σ_B between B and $s(B)$ for each $B \in \mathfrak{B}$ such that $\sigma=\sum_{B\in\mathfrak{B}}\sigma_B$ where s is a bijection between $\mathfrak{B}$ and $\mathfrak{b}$ ([B, Theorem 1.5]). We can see a Brauer character as a function over G mapping all p-singular elements to 0. Then σ_B induces a natural linear map between the $\mathcal{K}$-linear space with bases $\mathrm{IBr}(B)$ and the $\mathcal{K}$-linear space with bases $\mathrm{IBr}(s(B))$ for any $B \in \mathfrak{B}$ ([B, Proposition 4.1]).

We assume that the reader is familiar with the definitions and theorems from [I2], [N] and [NT].

1. Fully ramified correspondence

In this section, we show that the fully ramified correspondence induces perfect isometries. Let $H \trianglelefteq G$ and $\theta \in \mathrm{Irr}(H)$. We denote by $\mathrm{Irr}(G|\theta) = \{\chi \in \mathrm{Irr}(G) \mid [\chi_H, \theta] \neq 0\}$. If H is a p'-group, then we denote by $\mathrm{Bl}(G|\theta)$ the set of blocks of G which covers θ. Let K and L be normal subgroups of G with $K \leq L$ and K/L abelian. Let $\theta \in \mathrm{Irr}(K)$ and $\phi \in \mathrm{Irr}(L)$. If ϕ (and θ) is G-invariant and fully ramified with respect to K/L, that is, (ϕ is K-invariant and) θ is the unique irreducible constituent of ϕ^K, then (G, K, L, θ, ϕ) is called a *character five*.

Theorem 2 ([I1, Theorem 9.1]) *Let (G, K, L, θ, ϕ) be a character five. Assume that either $|G : K|$ or $|K : L|$ is odd. Let ψ be the character of G as in* [I1, Theorem 9.1]. *Then there exists $U \leq G$ such that*

(a) *$UK = G$ and $U \cap K = L$,*
(b) *U^a is G-conjugate to U for all $a \in \mathrm{Aut}(G)$ such that $K^a = K$, $L^a = L$ and $\phi^a = \phi$,*
(c) *the equation, $\chi_U = \psi_U \xi$, for $\chi \in \mathrm{Irr}(G|\theta)$ and $\xi \in \mathrm{Irr}(U|\phi)$ defines a one-to-one correspondence between these sets of characters,*
(d) *if $g \in G$ is not G-conjugate to an element of U then $\chi(g) = 0$ for all $\chi \in \mathrm{Irr}(G|\theta)$,*
(e) *if $|G : L|$ is odd, then χ and ξ are corresponding each other as above, if and only if $2 \nmid [\chi_U, \xi]$,*
(f) *$|\psi(g)|^2 = |\mathrm{C}_{K/L}(g)|$ for all $g \in G$.* □

The character correspondence in Theorem 2 (c) is called the *fully ramified correspondence* with respect to (G, K, L, θ, ϕ).

Proposition 3 *Assume the situation of Theorem 2 and that K is a p'-group. Then the fully ramified correspondence induces a correspondence between $\mathrm{Bl}(G|\theta)$ and $\mathrm{Bl}(U|\phi)$ which satisfies the following conditions. Let $B \in \mathrm{Bl}(G|\theta)$ correspond to $b \in \mathrm{Bl}(U|\phi)$.*

(a) *There exists a perfect isometry σ between B and b induced from the fully ramified correspondence.*
(b) *B and b have a common defect group.*

Proof. Let B be a block of G which covers θ and put $\mathrm{Irr}(B) = \{\chi_1, \ldots, \chi_k\}$.

Let $\chi_{iU} = \psi_U \xi_i$ with $\xi_i \in \mathrm{Irr}(U|\theta)$ as in Theorem 2 (c) and set $\mu = \sum_{i=1}^k \chi_i \times \xi_i$. We suppose that $\mu(g,u) \neq 0$ for $g \in G$ and $u \in U$. Since $\mu(g,u) = \psi(u)^{-1} \sum_{i=1}^k \chi_i(g)\chi_i(u)$ by Theorem 2 (c), we have $g_p =_G u_p^{-1}$ by the second orthogonality relation for blocks. If $g \in U$, then $\mu(g,u)/\psi(g) = \sum_{i=1}^k \xi_i(g)\xi_i(u)$. Thus if $g \in U_{p'}$ and $u \in U \backslash U_{p'}$, $\sum_{i=1}^k \xi_i(g)\xi_i(u) = 0$. By [O, Theorem 3] $\{\xi_1, \ldots, \xi_k\}$ is the set of irreducible characters of blocks $\{b_s\}_s$ of U. We claim that $\mu(g,u)/|\mathrm{C}_G(g)| \in \mathcal{O}$ and $\mu(g,u)/|\mathrm{C}_U(u)| \in \mathcal{O}$ for any $g \in G$ and $u \in U$. By Theorem 2 (d) we may assume $g \in U$. Since $\mathrm{C}_{K/L}(g)$ is a p'-group, $\psi(u)$ and $\psi(g)$ are invertible elements of $\mathcal{O}$ by Theorem 2 (f). Then we have

$$\frac{\mu(g,u)}{|\mathrm{C}_G(g)|} = \frac{1}{\psi(u)} \frac{1}{|\mathrm{C}_G(g)|} \sum_{i=1}^k \chi_i(g)\chi_i(u) \in \mathcal{O},$$

$$\frac{\mu(g,u)}{|\mathrm{C}_U(u)|} = \psi(g) \frac{1}{|\mathrm{C}_U(u)|} \sum_{i=1}^k \xi_i(g)\xi_i(u) \in \mathcal{O}.$$

Therefore μ induces a perfect isometry between B and $\{b_s\}_s$. By [B, Theorem 1.5 (1)] $\mathrm{Irr}(b) = \{\xi_1, \ldots, \xi_k\}$ for some $b \in \mathrm{Bl}(U)$. Thus the fully ramified correspondence induces a blockwise correspondence and there exists a perfect isometry between corresponding blocks.

We put simply $\chi = \chi_1$ and $\xi = \xi_1$. Let C_b be a defect class of b and C be the conjugacy class of G which contains C_b. Since K is a p'-group, we have $\psi(u)$ is an invertible element of $\mathcal{O}$ for any $u \in C_b$ by Theorem 2 (f). Since $\omega_\xi(\widehat{C_b})$ is an invertible element of $\mathcal{O}$ and $p \nmid |K : L| = |G : U|$, $(\psi(u)/\psi(1))|G : U|\omega_\xi(\widehat{C_b})$ is an invertible element of $\mathcal{O}$. Since we have

$$\omega_\chi(\widehat{C}) = \frac{1}{|\mathrm{C}_G(u) : \mathrm{C}_U(u)|} \left(\frac{\psi(u)}{\psi(1)} |G : U| \omega_\xi(\widehat{C_b}) \right),$$

$|\mathrm{C}_G(u) : \mathrm{C}_U(u)|^{-1}$ is an element of $\mathcal{O}$ and $|\mathrm{C}_G(u) : \mathrm{C}_U(u)|$ does not divide by p where $u \in C_b$. Thus C and C_b have a common defect group. Since $\omega_\chi(\widehat{C}) \not\equiv 0$, a defect group of b contains some defect group of B. Since B and b have a same defect by [B, Theorem 1.5 (2)], B and b have a common defect group. □

In the situation of Proposition 3, $\alpha_{U_{p'}} = \sigma(\alpha)\psi_{U_{p'}}$ for any $\alpha \in \mathrm{IBr}(B)$. Then $\sigma(\alpha)$ is also an irreducible Brauer character for each $\alpha \in \mathrm{IBr}(B)$ by the following proposition.

Proposition 4 *Assume the situation of Proposition* 3 *and that* G *is a* p*-solvable group. If* $B \in \mathrm{Bl}(G|\theta)$ *corresponds to* $b \in \mathrm{Bl}(U|\phi)$*, then we have the following.*

(a) *If* $\alpha \in \mathrm{IBr}(B)$*, then* $\sigma(\alpha) \in \mathrm{IBr}(b)$*. This correspondence is a bijection between* $\mathrm{IBr}(B)$ *and* $\mathrm{IBr}(b)$*.*

(b) $d_{\chi\alpha} = d_{\sigma(\chi)\sigma(\alpha)}$ *and* $c_{\alpha\alpha'} = c_{\sigma(\alpha)\sigma(\alpha')}$ *for all* $\chi \in \mathrm{Irr}(B)$ *and* $\alpha, \alpha' \in \mathrm{IBr}(B)$*.*

Proof. We use the notations in the proof of the above proposition. Let $\mathrm{IBr}(b) = \{\beta_1, \ldots, \beta_l\}$. By the Fong-Swan Theorem we may assume that $\beta_j = \xi_{jU_{p'}}$ for all j, $1 \leqq j \leqq l$. For all i, $1 \leqq i \leqq k$, $\chi_{iU_{p'}} = \psi_{U_{p'}}\xi_{iU_{p'}} = \psi_{U_{p'}} \sum_{j=1}^{l} d_{\xi_i\beta_j}\beta_j = \sum_{j=1}^{l} d_{\xi_i\beta_j}\chi_{jU_{p'}}$. By Theorem 2 (d),

$$\chi_{iG_{p'}} = \sum_{j=1}^{l} d_{\xi_i\beta_j}\chi_{jG_{p'}}.$$

Let $\alpha \in \mathrm{IBr}(B)$. By the Fong-Swan Theorem $\alpha = \chi_{iG_{p'}}$ for some i, $1 \leqq i \leqq k$. Since $d_{\xi_i\beta_j} \geqq 0$, $\alpha = \chi_{iG_{p'}} = \chi_{jG_{p'}}$ for some j, $1 \leqq j \leqq l$. On the other hand, since $|\mathrm{IBr}(B)| = l$, we have $\mathrm{IBr}(B) = \{\chi_{1G_{p'}}, \ldots, \chi_{lG_{p'}}\}$. We put $\alpha_j = \chi_{jG_{p'}}$ for all j, $1 \leqq j \leqq l$. We note $\sigma(\alpha_j) = \beta_j$. This completes the proof of (a). Since $\sum_{j=1}^{l} d_{\xi_i\beta_j}\alpha_j = \chi_{iG_{p'}} = \sum_{j=1}^{l} d_{\chi_i\alpha_j}\alpha_j$, we have $d_{\chi_i\alpha_j} = d_{\xi_i\beta_j}$ for all i, $1 \leqq i \leqq k$ and for all j, $1 \leqq j \leqq l$. This completes the proof. □

2. Proof of Theorem 1

We often use the following lemma in this section.

Lemma 5 ([Glauberman; see [I2, Lemma 13.8]) *Let* A *act on* G *with* $(|A|, |G|) = 1$*. Suppose that* A *and* G *act on a set* S*, such that* G *transitive on* S*. Assume that* $(s \cdot g) \cdot a = (s \cdot a) \cdot g^a$ *for all* $s \in S$*,* $g \in G$ *and* $a \in A$*. Then*

(i) *A fixes an element of* S*.*

(ii) $\mathrm{C}_G(A)$ *acts on the set of* A*-fixed elements of* S *transitively.* □

For example, let A act on G with $(|A|, |G|) = 1$ and N be an A-stable normal subgroup of G. We suppose A acts on G/N trivially. Then A acts on each N-coset gN of G and N acts on gN transitively by multiplication.

We have

$$((gn_0)\cdot n)\cdot a = (gn_0n)^a = (gn_0)^a n^a = ((gn_0)\cdot a)\cdot n^a$$

for all $gn_0 \in gN$, $a \in A$ and $n \in N$. Thus there exists an A-fixed element in gN by Lemma 5. Therefore we have a set of the representatives of G/N whose member is A-fixed and we have $G = N\mathrm{C}_G(A)$.

The following proposition gives the blockwise correspondence σ between G_i and G_{i+1} for each i in the introduction.

Proposition 6 *Let A be a finite group which acts on G with $(|A|,|G|)=1$ and assume $|G|$ is odd. Let L and K be A-stable normal p'-subgroups of G with $L \leq K$ and K/L abelian. Let $H/L = \mathrm{C}_{G/L}(A)$ and assume $HK = G$ and $H \cap K = L$. Let $\theta \in \mathrm{Irr}_A(L)$ and we define $\overline{\mathrm{Bl}}_A(G|\theta) = \{B \in \mathrm{Bl}(G) \mid \mathrm{Irr}(B) \subseteq \mathrm{Irr}_A(G|\theta)\}$. Then there exists a correspondence between $\overline{\mathrm{Bl}}_A(G|\theta)$ and $\overline{\mathrm{Bl}}_A(H|\theta)$ which satisfies the following conditions. Let $B \in \overline{\mathrm{Bl}}_A(G|\theta)$ correspond to $b \in \overline{\mathrm{Bl}}_A(H|\theta)$.*

(a) *If $\chi \in \mathrm{Irr}(B)$, there exists a unique constituent $\xi \in \mathrm{Irr}(H|\theta)$ of χ_H such that $2 \nmid [\chi_H, \xi]$. This correspondence is a bijection between* $\mathrm{Irr}(B)$ *and* $\mathrm{Irr}(b)$.

(b) *There exists a perfect isometry σ between B and b induced from the correspondence in* (a).

(c) *B and b have a common defect group.*

(d) *If $\alpha \in \mathrm{IBr}(B)$, then $\sigma(\alpha) \in \mathrm{IBr}(b)$. This correspondence is a bijection between* $\mathrm{IBr}(B)$ *and* $\mathrm{IBr}(b)$.

(e) *$d_{\chi\alpha} = d_{\sigma(\chi)\sigma(\alpha)}$ and $c_{\alpha\alpha'} = c_{\sigma(\alpha)\sigma(\alpha')}$ for all $\chi \in \mathrm{Irr}(B)$ and $\alpha, \alpha' \in$* $\mathrm{IBr}(B)$.

Proof. We have already known that there exists a correspondence between $\mathrm{Irr}_A(G|\theta)$ and $\mathrm{Irr}_A(H|\theta)$ by [I1, Theorem 10.6]. In order to show this correspondence is a blockwise correspondence and satisfies the conditions, we follow the proof of [I1, Theorem 10.6].

Let T be the inertial group of θ in G. Since θ is A-invariant, A stabilizes T. We put $K_0 = K \cap T$ and $H_0 = H \cap T$. Then $H_0/L = \mathrm{C}_{T/L}(A)$ and A acts trivially on T/K_0. Since $(|A|, |T:L|) = 1$, we have $T/L = K_0/L\cdot\mathrm{C}_{T/L}(A) = K_0/L\cdot H_0/L$ by Lemma 5 and $T = K_0H_0$. Thus T satisfies the hypotheses of the theorem with respect to the subgroups L, K_0 and H_0. If $T < G$, then by induction, there exists the correspondence between $\overline{\mathrm{Bl}}_A(T|\theta)$ and $\overline{\mathrm{Bl}}_A(H_0|\theta)$. Let $B \in \overline{\mathrm{Bl}}_A(G|\theta)$. Take $\widetilde{B} \in \mathrm{Bl}(T)$ which covers

θ and corresponds to B by the Clifford correspondence. Since θ and B are A-invariant, $\widetilde{B}$ is A-invariant. Since all ordinary irreducible characters of B are A-invariant, those characters of $\widetilde{B}$ are also A-invariant and so $\widetilde{B} \in \overline{\mathrm{Bl}}_A(T|\theta)$. By the induction, let $\widetilde{b} \in \overline{\mathrm{Bl}}_A(H_0|\theta)$ be the corresponding block of $\widetilde{B}$ which satisfies the conditions from (a) to (e) with respect to T and H_0. Then $(\widetilde{b})^H$ is defined in the sense of Brauer and $(\widetilde{b})^H$ corresponds to $\widetilde{b}$ by the Clifford correspondence. We put $b = (\widetilde{b})^H$ and show that B and b satisfy the conditions from (a) to (e) as follows. Let $\chi \in \mathrm{Irr}(B)$ and $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{B})$ with $\chi = (\widetilde{\chi})^G$. By the induction, we can take $\widetilde{\xi}$ which is the corresponding character of $\widetilde{\chi}$ as in (a). We put $\xi = (\widetilde{\xi})^G$. In this way, we have a character correspondence between $\mathrm{Irr}(B)$ and $\mathrm{Irr}(b)$. We claim the multiplicity condition in (a). Since $[\chi_L, \theta] = [\widetilde{\chi}_L, \theta] = [((\widetilde{\chi}_{H_0})^H)_L, \theta]$, we have $\chi_H = (\widetilde{\chi}_{H_0})^H + \eta$ where $[\eta_L, \theta] = 0$ by [I1, Lemma 10.4]. By induction, we have $\widetilde{\chi}_{H_0} = \widetilde{\xi} + 2\widetilde{\delta}$ where $\widetilde{\delta}$ is a character of H_0. Then $\chi_H = \widetilde{\xi}^H + 2\widetilde{\delta}^H + \eta = \xi + 2\widetilde{\delta}^H + \eta$ and ξ is a unique irreducible constituent of χ_H such that $2 \nmid [\chi_H, \xi]$ and $\xi \in \mathrm{Irr}(H|\theta)$. Since the Clifford correspondence induces a perfect isometry, we have (b). Moreover it is well-known that the Clifford correspondence also induces correspondence between irreducible Brauer characters and preserves Cartan invariants, decomposition numbers and defect groups, so we have (c), (d) and (e).

Thus we may suppose that θ is invariant in G. Let $R = K^{\perp}$ which is computed in (K, L, θ) ([I1, §2]). Then A stabilizes R and $R \trianglelefteq G$. Moreover θ is extendible to R by [I1, Theorem 2.7]. Let C be the group of linear characters of R/L and let $\mathcal{S}$ be the set of extensions of θ to R. Then C acts transitively on $\mathcal{S}$ by multiplication. Since $(|A|, |C|) = 1$, it follows from Lemma 5 that A fixes some $\phi \in \mathcal{S}$. Since $\mathrm{C}_C(A) = 1$, ϕ is a unique A-fixed element of $\mathcal{S}$ by Lemma 5 (ii). We claim that ϕ is invariant in G. Since ϕ is a character of $R = K^{\perp}$, ϕ is invariant in K by [I1, Lemma 2.4]. Let $h \in H$ and $a \in A$. Since $H/L = \mathrm{C}_{G/L}(A)$, we have $h^a = lh$ for some $l \in L$. Then $(\phi^h)^a = \phi^{h^a} = \phi^{lh} = \phi^h$ and ϕ^h is an A-fixed element of $\mathcal{S}$. Thus we have $\phi = \phi^h$ for all $h \in H$ by the uniqueness of ϕ. Since $G = HK$, ϕ is invariant in G.

Let $M = RH$. The restriction map induces a correspondence between $\mathrm{Irr}(M|\phi)$ and $\mathrm{Irr}(H|\theta)$ by [I1, Lemma 10.5]. Then there exists a correspondence between $\mathrm{Bl}(M|\phi)$ and $\mathrm{Bl}(H|\theta)$ such that the corresponding blocks are isomorphic. In particular, there exists a perfect isometry between corresponding blocks and the restriction induces a correspondence of Brauer characters. Moreover, the corresponding blocks have a common defect group and same decomposition numbers as well-known (see [HK, Theorem 4.1]). Now, if $\psi \in \mathrm{Irr}_A(M|\theta)$, then all irreducible constituents of ψ_R lie in $\mathcal{S}$. By Lemma 5, one of these is A-fixed. Therefore ϕ is a constituent of ψ_R and $\mathrm{Irr}_A(M|\theta) \subseteq \mathrm{Irr}(M|\phi)$. Since the restriction maps from $\mathrm{Irr}_A(M|\theta)$ to $\mathrm{Irr}_A(H|\theta)$, each block in $\overline{\mathrm{Bl}}_A(M|\theta)$ corresponds to a block in $\overline{\mathrm{Bl}}_A(H|\theta)$.

Now, we construct a correspondence between $\overline{\mathrm{Bl}}_A(G|\theta)$ and $\overline{\mathrm{Bl}}_A(M|\theta)$. ϕ is fully ramified with respect to K/R by [I1, Theorem 2.7]. Let τ be the unique irreducible constituent of ϕ^K so that (G, K, R, τ, ϕ) is a character five. Since $|K : L|$ is odd, the hypothesis of Theorem 2 is satisfied for (G, K, R, τ, ϕ). Let $U \subseteq G$ be as in [I1, Theorem 2.7] and let $\mathcal{T}$ be the G-conjugacy class of U. Then A acts on $\mathcal{T}$, and K/R acts on $\mathcal{T}$ transitively. By Lemma 5 we may assume that A stabilizes U. Then we have $[U, A] \subseteq U \cap K = R$ and A acts trivially on U/R. Since $H/L = \mathrm{C}_{G/L}(A)$, it follows that $U \subseteq RH = M$ and $U = M$. Thus Theorem 2 provides a correspondence between $\mathrm{Irr}(G|\tau) = \mathrm{Irr}(G|\phi)$ and $\mathrm{Irr}(M|\phi)$. Since ψ is determined by ϕ and the action of G on K/R ([I1, p.619, Theorem 6.3]), ψ is A-invariant. Therefore A-invariant characters correspond to A-invariant characters by Theorem 2 (c). Since $\mathrm{Irr}_A(G|\theta) \subseteq \mathrm{Irr}(G|\phi)$ by the above argument, it follows from Proposition 3 that there exists a correspondence between $\overline{\mathrm{Bl}}_A(G|\theta)$ and $\overline{\mathrm{Bl}}_A(M|\theta)$.

Then the composition of the two correspondences gives a correspondence between $\overline{\mathrm{Bl}}_A(G|\theta)$ and $\overline{\mathrm{Bl}}_A(H|\theta)$ with (b) and (c) by Proposition 3, (d) and (e) by Proposition 4. We claim that the correspondence satisfies the multiplicity condition in (a). Let B correspond to b and $\chi \in \mathrm{Irr}(B)$. Then we have $\chi_M = \xi + 2\alpha$ for a character α of M and an irreducible character ξ of M by Theorem 2 (e). Since there exists the character correspondence between B and b by mapping χ to ξ_H as shown as above, we have that ξ_H is a unique constituent of χ_H in $\mathrm{Irr}(H|\theta)$ with odd multiplicity. $\square$

Let B be an A-invariant block of G and D be a defect group of B. If $D \leq \mathrm{C}_G(A)$, then all irreducible characters of B are A-invariant ([Wa, §2 Proposition 1]). Now, we are ready to prove Theorem 1.

Proof of Theorem 1. First we prove the theorem when D is a Sylow p-subgroup of G. Let $K = [G, A]$, $L = [G, A]'$ and $U = [G, A]'\mathrm{C}_G(A)$. Then we note $G_1 = U$ in the introduction. It suffices to establish the correspondence between $\mathrm{Bl}_A(G|D)$ and $\mathrm{Bl}_A(U|D)$ and to show that it satisfies the following conditions. Let $B \in \mathrm{Bl}_A(G|D)$ correspond to $b \in \mathrm{Bl}_A(U|D)$.

(a') $\sigma(G,U)$ induces a perfect isometry σ between B and b.

(b') If $\alpha \in \mathrm{IBr}(B)$, then $\sigma(\alpha) \in \mathrm{IBr}(b)$. This correspondence is a bijection from $\mathrm{IBr}(B)$ to $\mathrm{IBr}(b)$.

(c') $d_{\chi\alpha} = d_{\sigma(\chi)\sigma(\alpha)}$ and $c_{\alpha\alpha'} = c_{\sigma(\alpha)\sigma(\alpha')}$ for all $\chi \in \mathrm{Irr}(B)$ and $\alpha, \alpha' \in \mathrm{IBr}(B)$.

Let $\mathfrak{H}$ be the set of all p-complements in G. Then A acts on $\mathfrak{H}$ and G acts on $\mathfrak{H}$ transitively by the solvability of G. By Lemma 5, there exists an A-invariant element H of $\mathfrak{H}$. Since $G = DH$, we have $[G,A] = [H,A] \leq H$. In particular $K = [G,A]$ is a p'-group. Moreover we have $U/L = \mathrm{C}_{G/L}(A)$, $UK = G$ and $U \cap K = L$. Let $B \in \mathrm{Bl}_A(G|D)$. Since A acts on the set of all ordinary irreducible characters of L which is covered by B and G acts it transitively, we can choose $\theta \in \mathrm{Irr}_A(L)$ which is covered by B by Lemma 5. Let $\chi \in \mathrm{Irr}(B)$ and $\xi = \sigma(G,U)(\chi) \in \mathrm{Irr}_A(U)$ be the unique character such that $2 \nmid [\chi_U, \xi]$ by [I1, Corollary 10.7]. We assume $\theta_0 \in \mathrm{Irr}_A(L)$ with $[\xi_L, \theta_0] \neq 0$. Then $[\chi_L, \theta_0] \neq 0$ and θ_0 and θ are U-conjugate by Lemma 5. Thus $[\xi_L, \theta] = [\xi_L, \theta_0] \neq 0$ and $\xi \in \mathrm{Irr}_A(U|\theta)$. Let B correspond to b as in Proposition 6. Then $\xi \in \mathrm{Irr}(b)$ and B and b satisfies (a'), (b') and (c') by Proposition 6. The proof of the first cases is complete.

Secondly let $X = \mathrm{O}_{p'}(G)$. Since the set of irreducible characters of X which is covered by B is a G-conjugacy class, there exists an A-invariant character ν of X which is covered by B by Lemma 5. Let T be the inertial group of ν in G and $\widetilde{B} \in \mathrm{Bl}(T)$ be the Clifford correspondent of B. Since ν and B are A-invariant and $\widetilde{B}$ is the unique block of T which covers ν with $(\widetilde{B})^G = B$, $\widetilde{B}$ is A-invariant. Let $\mathfrak{D}_B$ (resp. $\mathfrak{D}_{\widetilde{B}}$) be the set of defect groups of B (resp. $\widetilde{B}$). Since $\mathfrak{D}_{\widetilde{B}}$ is a T-conjugacy class and A acts on it, A stabilizes a defect group $\widetilde{D} \in \mathfrak{D}_{\widetilde{B}}$ by Lemma 5. On the other hand, A acts on $\mathfrak{D}_B$ and G acts on it transitively. Since D and $\widetilde{D}$ are A-fixed elements of $\mathfrak{D}_B$, these are $\mathrm{C}_G(A)$-conjugate by Lemma 5. Since $D \leq \mathrm{C}_G(A)$, $\widetilde{D} \leq \mathrm{C}_G(A) \cap T = \mathrm{C}_T(A)$.

Suppose $G > T$. Let $\nu^* = \sigma(X, \mathrm{C}_X(A))(\nu)$. By [Wo, Lemma 2.5 (b)], $\mathrm{C}_T(A)$ is the inertial group of ν^* in $\mathrm{C}_G(A)$. By the induction we can define the correspondence between $\mathrm{Bl}_A(T|\widetilde{D})$ and a set of $\mathrm{Bl}(\mathrm{C}_T(A)|\widetilde{D})$ and let $\widetilde{B}$ correspond to $\widetilde{b}$. By [Wo, Lemma 2.5 (a)], $\widetilde{b}$ covers ν^*. Let b be the Clifford correspondent of $\widetilde{b}$. By [Wo, Lemma 2.5 (b)], the Clifford correspondence commutes with the Isaacs correspondence. Thus characters of B correspond to characters of b by the Isaacs correspondence, and they satisfy (a), (b) and (c) by the induction and the property of the Clifford correspondence.

We assume that $G = T$, that is, ν is G-invariant. Then a defect group of B is a Sylow p-subgroup of G by [N, Theorem 10.20]. In this case, the statement has been already proved. The proof of Theorem 1 is complete. $\square$

If A centralizes a Sylow p-subgroup of G, then each block of G has a defect group centralized by A. Then we have the following corollary.

Corollary 7 *Let A act on G with $(|A|, |G|) = 1$ and $|G|$ is odd. If A centralizes a Sylow p-subgroup of G, then the Isaacs correspondence induces a correspondence between the set of A-invariant blocks of G and the set of blocks of $\mathrm{C}_G(A)$ which satisfies* (a), (b) *and* (c) *in Theorem* 1.

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