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An extremal class of 3-dimensional elliptic affine spheres

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Abstract. In analogy to an inequality of Chen [Che93], Scharlach, Simon, Verstraelen and Vrancken [SSVV97] have found a new inequality for (equi-) affine spheres. This inequality is optimal and in this paper we classify those 3-dimensional elliptic affine spheres for which the corresponding equality is assumed. This is achieved through reducing the problem to the problem of classifying those 2-dimensional minimal surfaces in S_3^5 whose ellipses of curvature are circles. We end with the investigation of 2-dimensional minimal surfaces in S_3^5 with positive definite induced metric whose ellipses of curvature are circles.

Key words: affine differential geometry, Chen's equality, 1-dimensional nullity distribution, affine spheres.

1. Introduction

Consider an immersed hypersurface with relative normalization, i.e., an immersion $f\colon M \to \mathbb{R}^{n+1}$ together with a transverse vector field ξ such that $\mathrm{D}\xi$ has its image in f_*T_xM. The *relative hypersurface* (f,ξ) is a *sphere* if its normal lines $\mathbb{R}\xi(x)$ have a common intersection. From this definition one immediately has the existence of a function $\epsilon\colon M \to \mathbb{R}$ with $0 = \mathrm{D}(f(x) + \epsilon(x)\xi(x)) = \mathrm{D}f(x) + \epsilon(x)\mathrm{D}\xi + \mathrm{d}\epsilon(x) \otimes \xi$ which in turn implies that $\epsilon \neq 0$ is constant. Moreover, the Weingarten map is a multiple of the identity. Since ϵ is constant we can always rescale f to obtain $\epsilon = \pm 1$. In the following we will only be concerned with such *unit spheres*. Whether an immersed hypersurface constitutes a sphere depends crucially on the chosen normalization ξ. For instance, in Euclidean hypersurface theory (ξ is the outer normal to $f(M)$ with respect to a fixed scalar product in $\mathbb{R}^{n+1}$), a sphere $f(M)$ is necessarily isometric to the round sphere $S^n(r)$ of radius r. The other extreme occurs if we choose the centro-affine normalization where we take for ξ simply $-f$. Clearly, in this theory any hypersurface is a unit sphere. The classification of unit spheres is a highly non-trivial problem for the equi-affine theory of immersed hypersurfaces introduced by

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Blaschke (see, for instance, [Bla23]). Spheres with respect to this theory are often called *affine spheres.*

While in Euclidean hypersurface theory an immersion $f: M \to \mathbb{R}^{n+1}$ inherits a scalar product from Euclidean space, in equi-affine hypersurface theory one only has a constant determinant function det of $\mathbb{R}^{n+1}$. Denote by det* the dual volume form in $(\mathbb{R}^{n+1})^*$ and by $Y: M \to (\mathbb{R}^{n+1})^*$ the *conormal* defined by $Y(\xi) = 1$ and $Y_{|f_*TM} = 0$. For each choice of transverse vector field ξ, the volume forms det and det* define n-forms $\omega := (-1)^n f^*(\det(\xi, \cdot, \dots, \cdot))$ and $\omega^* = (-1)^n Y^*(\det^*(Y, \cdot, \dots, \cdot))$. The *equi-affine normalization* ξ is the (up to orientation) unique relative normalization which is invariant under the unimodular group and satisfies $\omega = \pm\omega^*$ (see, for instance, [LSZ93]). The induced equi-affine connection ∇ is given by $\mathrm{D}_U V = \nabla_U V + h(U, V)\xi$, where h is called the *equi-affine metric* or *Blaschke metric.* (See [Bla23, §§39–40] for a somewhat different introduction.)

In the case that h is definite we can fix the orientation of the equi-affine normal $\xi(= \pm f)$ such that h is positive definite. Then the sign of ϵ in the definition of spheres is an invariant, and M is called an *elliptic* affine unit sphere if $\epsilon = 1$ and a *hyperbolic* affine unit sphere if $\epsilon = -1$.[1]

The abundance of affine unit spheres dwarfs any attempts at a complete classification. In order to obtain detailed information one has therefore to revert to sub-classes such as the class of complete affine unit spheres [LSZ93]. Various authors have also imposed curvature conditions but even in the case of constant curvature (with respect to h) only a partial classification has been achieved yet [VLS91, MR92, KV97]. In analogy to work by Chen [Che93], Scharlach, Simon, Verstraelen and Vrancken [SSVV97] have found a new curvature invariant for (equi-) affine spheres. They also gave a lower bound, depending only on ϵ and the dimension of the sphere. In this paper we will classify those 3-dimensional, elliptic affine unit spheres which extremize the new curvature invariant. In Section 2 we will show that they admit a preferred ruling. Thus the geometry is determined by 2-dimensional submanifolds transverse to this ruling. This will be used in Section 3 to find a reduction to the problem of investigating 2-dimensional minimal submanifolds (i.e. the mean curvature vector field vanishes) of S^5_3

[1]In this terminology an ellipsoid is an elliptic and a hyperboloid a hyperbolic affine sphere.

whose ellipses of curvature are non-degenerate circles. Here $S^5_3 \subset \mathbb{R}^6$ is the 5-dimensional (pseudo)sphere of index 3. Finally, 2-dimensional minimal submanifolds of S^5_3 with positive definite induced metric whose ellipses of curvature are circles are studied in Section 4.

2. Affine unit 3-spheres which satisfy Chen's equality

Let $f\colon M \to \mathbb{R}^{n+1}$ be an affine unit sphere. Denote the Levi-Civita connection with respect to h by $\widehat{\nabla}$ and the normalized scalar curvature and sectional curvature by $\hat{\kappa}$ and $\hat{K}$, respectively. Furthermore, $G_2(T_xM)$ denotes the Grassmannian of 2-dimensional subspaces of T_xM. Following [Che93], Scharlach, Simon et al. [SSVV97] have shown that the inequality

$$\frac{n(n-1)}{2}\hat{\kappa}(p) - \sup_{\Pi \in G_2(T_xM)} \hat{K}_p(\Pi) \geq \epsilon\frac{1}{2}(n+1)(n-2)$$

holds.[2] We call this inequality *Chen's inequality* and the corresponding equality *Chen's equality.* In this paper, we will classify those elliptic 3-dimensional affine unit spheres which realize Chen's equality. We will start by chosing a frame adapted to the problem.

Lemma 1 ([SSVV97]) *Let M be an n-dimensional affine unit sphere which realizes Chen's equality, $x \in M$ and $K(U,V) = \nabla_U V - \widehat{\nabla}_U V$. If $K_x \neq 0$ then there exists an h-orthonormal frame $\{E_1, \dots, E_n\}$ and a function λ in a neighbourhood of x such that*

$$\begin{aligned} &K(E_1,E_1) = \lambda E_1, && K(E_1,E_i) = 0, && K(E_1,E_2) = -\lambda E_2, && \\ &K(E_2,E_i) = 0, && K(E_2,E_2) = -\lambda E_1, && K(E_i,E_j) = 0, && (i,j > 2). \end{aligned}$$

The Christoffel symbols of the Levi-Civita connection $\widehat{\nabla}$ defined by $\widehat{\nabla}_{E_b} E_c = \sum_{a=1}^n \widehat{\Gamma}^a_{bc} E_a$ satisfy

$$\widehat{\Gamma}^i_{11} = \widehat{\Gamma}^i_{22}, \;\; \widehat{\Gamma}^i_{12} = -\widehat{\Gamma}^i_{21}, \;\; \widehat{\Gamma}^1_{ij} = \widehat{\Gamma}^2_{ij} = 0, \;\; \widehat{\Gamma}^2_{i1} = -\frac{1}{3}\widehat{\Gamma}^i_{12}, \;\; (i,j > 2).$$

Remark 1 Here span$\{E_3, \dots, E_n\}$ is an $(n-2)$-dimensional integrable distribution which spans at each point x with $K_x \neq 0$ the nullity space

[2] Scharlach, Simon et al. have actually studied the much more general case of arbitrary centro-affine hypersurfaces but proved that equality can only hold for centro-affine hypersurfaces which at the same time are affine unit spheres.

of K_x.

Let us now specialize to 3-dimensional, definite affine unit spheres which satisfy Chen's equality.

Lemma 2 *Let $f = E_0 \colon M \to \mathbb{R}^4$ be a 3-dimensional, definite affine unit sphere with positive definite equi-affine metric and $x \in M$ with $K_x \neq 0$. Then in a neighbourhood of x there are functions α, β, γ, δ and h-orthonormal 1-forms ω^1, ω^2, ω^3 such that*

$$\mathrm{D}\begin{pmatrix} E_0 \\ E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega^1 & \omega^2 & \omega^3 \\ -\epsilon\omega^1 & \lambda\omega^1 & -\gamma\omega^1 + (\delta - \lambda)\omega^2 - \frac{\beta}{3}\omega^3 & \alpha\omega^1 - \beta\omega^2 \\ -\epsilon\omega^2 & \gamma\omega^1 - (\delta + \lambda)\omega^2 + \frac{\beta}{3}\omega^3 & -\lambda\omega^1 & \beta\omega^1 + \alpha\omega^2 \\ -\epsilon\omega^3 & -\alpha\omega^1 + \beta\omega^2 & -\beta\omega^1 - \alpha\omega^2 & 0 \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \\ E_2 \\ E_3 \end{pmatrix} \tag{1}$$

holds, where E_1, E_2, E_3 are chosen as in Lemma 1.

Proof. Since $\{E_1, E_2, E_3\}$ is an orthonormal frame we have $\widehat{\Gamma}^a_{bc} = -\widehat{\Gamma}^c_{ba}$. Hence in view of Lemma 1 we have $\alpha := \widehat{\Gamma}^3_{11} = \widehat{\Gamma}^3_{22}$, $\beta := \widehat{\Gamma}^3_{12} = -\widehat{\Gamma}^3_{21} = -3\widehat{\Gamma}^2_{31}$, $\gamma := \widehat{\Gamma}^1_{12}$, $\delta := \widehat{\Gamma}^2_{21}$, $\widehat{\Gamma}^1_{33} = \widehat{\Gamma}^2_{33} = 0$. Now the assertion follows from the equation $\mathrm{D}_{E_b}E_c = (K^a_{bc} + \widehat{\Gamma}^a_{bc})E_a - \epsilon\delta_{bc}E_0$ and the form of K^a_{bc}. □

In [SSVV97] all affine unit spheres (of arbitrary dimension n) which satisfy Chen's equality and for which the distribution $\operatorname{span}\{E_1, E_2\}$ is integrable, have been classified. For $n = 3$, we have $h([E_1, E_2], E_3) = 2\beta$ which implies that the classification for the case $\beta = 0$ is known. We will give an existence and uniqueness result for elliptic 3-dimensional affine unit spheres satisfying Chen's equality which rests on the following observation.

Corollary 1 *Let M be an affine unit sphere which satisfies Chen's equality and $x \in M$. If $K_x \neq 0$ then there is a neighbourhood $\mathcal{U}$ of x such that $f(\mathcal{U})$ is ruled by arcs of ellipses if it is elliptic and by arcs of hyperbolas if it is hyperbolic. These ellipses (respectively hyperbolas) are centered at 0.*

Proof. We show that the integral curves of E_3 are centered ellipses if $\epsilon = 1$ and centered hyperbolas if $\epsilon = -1$. Since $\mathrm{D}_{E_3}\mathrm{D}_{E_3}E_0 = \mathrm{D}_{E_3}E_3 = -\epsilon E_0$ the integral curves of E_3 satisfy $\ddot{\gamma} = -\epsilon\gamma$. It follows that there exist vectors $A, B \in \mathbb{R}^4$ such that $\gamma(t) = A\cos(t) + B\sin(t)$ in the elliptic case ($\epsilon = 1$) and $\gamma(t) = A\cosh(t) + B\sinh(t)$ in the hyperbolic case ($\epsilon = -1$). □

In view of Corollary 1 the geometry of M is determined by 2-dimensional submanifolds transverse to this elliptic/hyperbolic ruling. In the next section we will give an explicit reduction in the elliptic case.

3. The correspondence of affine unit 3-spheres which satisfy Chen's equality and minimal immersions whose ellipses of curvature are circles

The following lemma is a consequence of the fact that $\mathrm{SL}(4,\mathbb{R})$ and $\mathrm{SO}(3,3)$ are locally isomorphic [Hel78].

Lemma 3 *There is a natural local diffeomeorphism $\iota\colon \mathrm{SL}(4,\mathbb{R}) \to \mathrm{SO}(3,3)$ such that each $\iota(E)$ has exactly two pre-images.*

Proof. Let $E = \{E_0, E_1, E_2, E_3\}$ be a basis with $\det(E_0, E_1, E_2, E_3) = 1$ and define

$$(\iota(E))_0 = \frac{1}{2}\left(E_0 \wedge E_3 + E_1 \wedge E_2\right), \quad (\iota(E))_5 = \frac{1}{2}\left(E_0 \wedge E_3 - E_1 \wedge E_2\right),$$

$$(\iota(E))_1 = \frac{1}{2}\left(E_0 \wedge E_1 + E_2 \wedge E_3\right), \quad (\iota(E))_3 = \frac{1}{2}\left(E_0 \wedge E_1 - E_2 \wedge E_3\right),$$

$$(\iota(E))_2 = \frac{1}{2}\left(E_0 \wedge E_2 + E_3 \wedge E_1\right), \quad (\iota(E))_4 = \frac{1}{2}\left(E_0 \wedge E_2 - E_3 \wedge E_1\right).$$

In the 6-dimensional linear space $\operatorname{span}\{(\iota(E))_0, \ldots, (\iota(E))_5\}$ we define a scalar product via

$$\langle X, Y\rangle E_0 \wedge E_1 \wedge E_2 \wedge E_3 = 2X \wedge Y.$$

It is easy to see that this scalar product has signature $(+,+,+,-,-,-)$ and that $\{(\iota(E))_0, \ldots, (\iota(E))_5\}$ is an orthonormal basis with respect to this scalar product. The map ι is clearly smooth. Assume that there exist $E, \tilde{E} \in \mathrm{SL}(4,\mathbb{R})$ with $\iota(E) = \iota(\tilde{E})$. It then follows that for all i, j we have $E_i \wedge E_j = \tilde{E}_i \wedge \tilde{E}_j$ and therefore $\tilde{E}_i = +E_i$ for all i or $\tilde{E}_i = -E_i$ for all i. Hence $\iota^{-1}(A)$ is empty or has exactly two elements for any $A \in \mathrm{SO}(3,3)$ and our claim follows from the fact that $\mathrm{SO}(3,3)$ and $\mathrm{SL}(4,\mathbb{R})$ have the same dimension. □

Let $E = \{E_0, E_1, E_2, E_3\}\colon M \to \mathrm{SL}(4,\mathbb{R})$ be the adapted frame introduced in Lemma 1. We will now use this diffeomorphism to lift E to $\mathrm{SO}(3,3)$.

Lemma 4 *Let E be chosen as in Lemma 2 and let $F = \iota(E)$. Then F satisfies the structure equations $\mathrm{d}F = T_\epsilon F$, where*

$$T_- = \begin{pmatrix} 0 & -\alpha\omega^1+\beta\omega^2 & -\beta\omega^1-\alpha\omega^2 & -\omega^2 & \omega^1 & 0 \\ \alpha\omega^1-\beta\omega^2 & 0 & -\gamma\omega^1+\delta\omega^2-\beta/3\omega^3 & \lambda\omega^1 & -\lambda\omega^2-\omega^3 & \omega^2 \\ \beta\omega^1+\alpha\omega^2 & \gamma\omega^1-\delta\omega^2+\beta/3\omega^3 & 0 & -\lambda\omega^2+\omega^3 & -\lambda\omega^1 & -\omega^1 \\ -\omega^2 & \lambda\omega^1 & -\lambda\omega^2+\omega^3 & 0 & -\gamma\omega^1+\delta\omega^2-\beta/3\omega^3 & \alpha\omega^1-\beta\omega^2 \\ \omega^1 & -\lambda\omega^2-\omega^3 & -\lambda\omega^1 & \gamma\omega^1-\delta\omega^2+\beta/3\omega^3 & 0 & \beta\omega^1+\alpha\omega^2 \\ 0 & \omega^2 & -\omega^1 & -\alpha\omega^1+\beta\omega^2 & -\beta\omega^1-\alpha\omega^2 & 0 \end{pmatrix}$$

and

$$T_+ = \begin{pmatrix} 0 & -\alpha\omega^1+(1+\beta)\omega^2 & -(1+\beta)\omega^1-\alpha\omega^2 & 0 & 0 & 0 \\ \alpha\omega^1-(1+\beta)\omega^2 & 0 & -\gamma\omega^1+\delta\omega^2+(1-\beta/3)\omega^3 & \lambda\omega^1 & -\lambda\omega^2 & 0 \\ (1+\beta)\omega^1+\alpha\omega^2 & \gamma\omega^1-\delta\omega^2-(1-\beta/3)\omega^3 & 0 & -\lambda\omega^2 & -\lambda\omega^1 & 0 \\ 0 & \lambda\omega^1 & -\lambda\omega^2 & 0 & -\gamma\omega^1+\delta\omega^2-(1+\beta/3)\omega^3 & \alpha\omega^1+(1-\beta)\omega^2 \\ 0 & -\lambda\omega^2 & -\lambda\omega^1 & \gamma\omega^1-\delta\omega^2+(1+\beta/3)\omega^3 & 0 & -(1-\beta)\omega^1+\alpha\omega^2 \\ 0 & 0 & 0 & -\alpha\omega^1-(1-\beta)\omega^2 & (1-\beta)\omega^1-\alpha\omega^2 & 0 \end{pmatrix}$$

Proof. This is a direct calculation using equation (1). We use the notations: $F_{1+} = F_0$, $F_{1-} = F_5$, $F_{2+} = F_1$, $F_{2-} = F_3$, $F_{3+} = F_2$, $F_{3-} = F_4$.

$$
\begin{aligned}
&2\mathrm{D}F_{1\pm}\\
&\quad = \mathrm{D}(E_0 \wedge E_3 \pm E_1 \wedge E_2)\\
&\quad = (\omega^1 \otimes E_1 + \omega^2 \otimes E_2 + \omega^3 \otimes E_3) \wedge E_3\\
&\qquad + E_0 \wedge (-\epsilon\omega^3 \otimes E_0 + (-\alpha\omega^1 + \beta\omega^2) \otimes E_1 - (\beta\omega^1 + \alpha\omega^2) \otimes E_2)\\
&\qquad \pm \Big(-\epsilon\omega^1 \otimes E_0 + \lambda\omega^1 \otimes E_1 + \Big(-\gamma\omega^1 + (\delta - \lambda)\omega^2 - \frac{\beta}{3}\omega^3\Big) \otimes E_2\\
&\qquad\qquad + (\alpha\omega^1 - \beta\omega^2) \otimes E_3\Big) \wedge E_2\\
&\qquad \pm E_1 \wedge \Big(-\epsilon\omega^2 \otimes E_0 + \Big(\gamma\omega^1 - (\lambda + \delta)\omega^2 + \frac{\beta}{3}\omega^3\Big) \otimes E_1 - \lambda\omega^1 \otimes E_2\\
&\qquad\qquad + (\beta\omega^1 + \alpha\omega^2) \otimes E_3\Big)\\
&\quad = (-\alpha\omega^1 + \beta\omega^2 \pm \epsilon\omega^2) \otimes E_0 \wedge E_1 + (\omega^2 \mp (\alpha\omega^1 - \beta\omega^2)) \otimes E_2 \wedge E_3\\
&\qquad + (-\beta\omega^1 - \alpha\omega^2 \mp \epsilon\omega^1) \otimes E_0 \wedge E_2 + (-\omega^1 \mp \beta\omega^1 \mp \alpha\omega^2) \otimes E_3 \wedge E_1\\
&\quad = ((1 \pm 1)(-\alpha\omega^1 + \beta\omega^2) + (1 \pm \epsilon)\omega^2) \otimes F_{2+}\\
&\qquad + ((1 \mp 1)(-\alpha\omega^1 + \beta\omega^2) - (1 \mp \epsilon)\omega^2) \otimes F_{2-}\\
&\qquad + (-(1 \pm \epsilon)\omega^1 - (1 \pm 1)(\beta\omega^1 + \alpha\omega^2)) \otimes F_{3+}\\
&\qquad + ((1 \mp \epsilon)\omega^1 - (1 \mp 1)(\beta\omega^1 + \alpha\omega^2)) \otimes F_{3-},
\end{aligned}
$$

$$
\begin{aligned}
&2\mathrm{D}F_{2\pm}\\
&\quad = \mathrm{D}(E_0 \wedge E_1 \pm E_2 \wedge E_3)\\
&\quad = (\omega^1 \otimes E_1 + \omega^2 \otimes E_2 + \omega^3 \otimes E_3) \wedge E_1\\
&\qquad + E_0 \wedge \Big(-\epsilon\omega^1 \otimes E_0 + \lambda\omega^1 \otimes E_1 + \Big(-\gamma\omega^1 + (\delta - \lambda)\omega^2 - \frac{\beta}{3}\omega^3\Big) \otimes E_2\\
&\qquad\qquad + (\alpha\omega^1 - \beta\omega^2) \otimes E_3\Big)\\
&\qquad \pm \Big(-\epsilon\omega^2 \otimes E_0 + \Big(\gamma\omega^1 - (\lambda + \delta)\omega^2 + \frac{\beta}{3}\omega^3\Big) \otimes E_1 - \lambda\omega^1 \otimes E_2\\
&\qquad\qquad + (\beta\omega^1 + \alpha\omega^2) \otimes E_3\Big) \wedge E_3\\
&\qquad \pm E_2 \wedge (-\epsilon\omega^3 \otimes E_0 + (-\alpha\omega^1 + \beta\omega^2) \otimes E_1 - (\beta\omega^1 + \alpha\omega^2) \otimes E_2)\\
&\quad = (\alpha\omega^1 - \beta\omega^2 \mp \epsilon\omega^2) \otimes E_0 \wedge E_3 + (-\omega^2 \pm \alpha\omega^1 \mp \beta\omega^2) \otimes E_1 \wedge E_2\\
&\qquad + (\lambda\omega^1) \otimes E_0 \wedge E_1 + (\mp\lambda\omega^1) \otimes E_2 \wedge E_3\\
&\qquad + \Big((\delta - \lambda)\omega^2 - \gamma\omega^1 - \frac{\beta}{3}\omega^3 \pm \epsilon\omega^3\Big) \otimes E_0 \wedge E_2
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \Big(\omega^3 \pm (\delta+\lambda)\omega^2 \mp \gamma\omega^1 \mp \frac{\beta}{3}\omega^3\Big) \otimes E_3 \wedge E_1 \\
&= ((1\pm 1)(\alpha\omega^1 - \beta\omega^2) - (1\pm\epsilon)\omega^2) \otimes F_{1+} \\
&\quad + ((1\mp 1)(\alpha\omega^1 - \beta\omega^2) + (1\mp\epsilon)\omega^2) \otimes F_{1-} \\
&\quad + (1\mp 1)\lambda\omega^1 \otimes F_{2+} + (1\pm 1)\lambda\omega^1 \otimes F_{2-} \\
&\quad + \Big((1\pm\epsilon)\omega^3 - (1\mp 1)\lambda\omega^2 + (1\pm 1)(-\gamma\omega^1 + \delta\omega^2 - \frac{\beta}{3}\omega^3)\Big) \otimes F_{3+} \\
&\quad + \Big(-(1\mp\epsilon)\omega^3 - (1\pm 1)\lambda\omega^2 + (1\mp 1)\Big(-\gamma\omega^1 + \delta\omega^2 - \frac{\beta}{3}\omega^3\Big)\Big) \otimes F_{3-},
\end{aligned}
$$

$$
\begin{aligned}
&2\mathrm{D}F_{3\pm} \\
&= \mathrm{D}(E_0 \wedge E_2 \pm E_3 \wedge E_1) \\
&= (\omega^1 \otimes E_1 + \omega^2 \otimes E_2 + \omega^3 \otimes E_3) \wedge E_2 \\
&\quad + E_0 \wedge \Big(-\epsilon\omega^2 \otimes E_0 + \Big(\gamma\omega^1 - (\lambda+\delta)\omega^2 + \frac{\beta}{3}\omega^3\Big) \otimes E_1 - \lambda\omega^1 \otimes E_2 \\
&\qquad\qquad + (\beta\omega^1 + \alpha\omega^2) \otimes E_3\Big) \\
&\quad \pm (-\epsilon\omega^3 \otimes E_0 + (-\alpha\omega^1 + \beta\omega^2) \otimes E_1 - (\beta\omega^1 + \alpha\omega^2) \otimes E_2) \wedge E_1 \\
&\quad \pm E_3 \wedge \Big(-\epsilon\omega^1 \otimes E_0 + \lambda\omega^1 \otimes E_1 + \Big(-\gamma\omega^1 + (\delta-\lambda)\omega^2 - \frac{\beta}{3}\omega^3\Big) \otimes E_2 \\
&\qquad\qquad + (\alpha\omega^1 - \beta\omega^2) \otimes E_3\Big) \\
&= (\beta\omega^1 + \alpha\omega^2 \pm \epsilon\omega^1) \otimes E_0 \wedge E_3 + (\omega^1 \pm \beta\omega^1 \pm \alpha\omega^2) \otimes E_1 \wedge E_2 \\
&\quad + \Big(\gamma\omega^1 - (\lambda+\delta)\omega^2 + \frac{\beta}{3}\omega^3 \mp \epsilon\omega^3\Big) \otimes E_0 \wedge E_1 \\
&\quad + \Big(-\omega^3 \mp \Big(-\gamma\omega^1 + (\delta-\lambda)\omega^2 - \frac{\beta}{3}\omega^3\Big)\Big) \otimes E_2 \wedge E_3 \\
&\quad + (-\lambda\omega^1) \otimes E_0 \wedge E_2 + (\pm\lambda\omega^1) \otimes E_3 \wedge E_1 \\
&= ((1\pm 1)(\beta\omega^1 + \alpha\omega^2) + (1\pm\epsilon)\omega^1) \otimes F_{1+} \\
&\quad + ((1\mp 1)(\beta\omega^1 + \alpha\omega^2) - (1\mp\epsilon)\omega^1) \otimes F_{1-} \\
&\quad + \Big((1\pm 1)(\gamma\omega^1 - \delta\omega^2 + \frac{\beta}{3}\omega^3) - (1\mp 1)\lambda\omega^2 - (1\pm\epsilon)\omega^3\Big) \otimes F_{2+} \\
&\quad + \Big((1\mp 1)(\gamma\omega^1 - \delta\omega^2 + \frac{\beta}{3}\omega^3) - (1\pm 1)\lambda\omega^2 + (1\mp\epsilon)\omega^3\Big) \otimes F_{2-} \\
&\quad + (-(1\mp 1)\lambda\omega^1) \otimes F_{3+} + (-(1\pm 1)\lambda\omega^1) \otimes F_{3-}.
\end{aligned}
$$

$\square$

We are now in a position to describe the reduction referred to in the last section. We will show how the frame given in Lemma 4 in the elliptic case can be geometrically interpreted. Unfortunately such an interpretation does not seem to exist in the hyperbolic case. We denote by S^5_3 the unit pseudosphere in R^6_3, i.e. S^5_3 has dimension 5 and index 3. In the following the notion of a minimal immersion $g: N \to S^5_3$ means that the mean curvature vector field of g vanishes. This is not equivalent to $g(N)$ spanning locally minimal area.

Theorem 1 *Let $f: M \to \mathbb{R}^4$ be a 3-dimensional elliptic affine unit sphere which satisfies Chen's equality. Assume that $K_x \neq 0$ for all $x \in M$ and let N be a 2-dimensional submanifold of M which is transverse to the elliptic ruling defined by E_3. Then $g: N \to S^5_3$, $x \mapsto g(x) = F_0(x) = \frac{1}{2}(E_0(x) \wedge E_3(x) + E_1(x) \wedge E_2(x))$ is a minimal immersion with positive definite induced metric whose ellipses of curvature are circles.*

Proof. Since N is transverse to the flow lines of E_3, F_0 restricted to N is an immersion into S^5_3. If $\{f_1, f_2\}$ is the basis dual to the pullback $\{(\omega^1)_{TN}, (\omega^2)_{TN}\}$ of (ω^1, ω^2) to N then the basis

$$\begin{aligned} e_1 &= \frac{1}{\alpha^2 + (1+\beta)^2}(-\alpha f_1 + (1+\beta) f_2), \\ e_2 &= \frac{1}{\alpha^2 + (1+\beta)^2}(-(1+\beta) f_1 - \alpha f_2) \end{aligned}$$

satisfies $\mathrm{d}F_0(e_i) = F_i$ and is therefore an orthonormal basis of N equipped with the metric induced by F_0. The shape tensor is given by

$$\begin{aligned} I\!I(e_1, e_1) &= \frac{\lambda}{\alpha^2 + (1+\beta)^2}(-\alpha F_3 - (1+\beta) F_4), \\ I\!I(e_1, e_2) &= \frac{\lambda}{\alpha^2 + (1+\beta)^2}(-(1+\beta) F_3 + \alpha F_4), \\ I\!I(e_2, e_2) &= \frac{\lambda}{\alpha^2 + (1+\beta)^2}(\alpha F_3 + (1+\beta) F_4). \end{aligned}$$

Hence F_0 must be a minimal immersion. Since

$$\begin{aligned} &I\!I(\cos(t)e_1 + \sin(t)e_2, \cos(t)e_1 + \sin(t)e_2) \\ &= \frac{\lambda}{\alpha^2 + (1+\beta)^2}\big(\cos^2(t)(-\alpha F_3 - (1+\beta) F_4) \\ &\qquad + 2\sin(t)\cos(t)(-(1+\beta) F_3 + \alpha F_4) \end{aligned}$$

$$
\begin{aligned}
&+\sin^2(t)(\alpha F_3+(1+\beta)F_4))\\
&=\frac{\lambda}{\alpha^2+(1+\beta)^2}\big(\cos(2t)(-\alpha F_3-(1+\beta)F_4)\\
&\quad+\sin(2t)(-(1+\beta)F_3+\alpha F_4)\big)
\end{aligned}
$$

the ellipses of curvature are circles. □

In the rest of this section we will elaborate on the relation between this class of minimal immersions and elliptic affine unit 3-spheres which satisfy Chen's equality. We will show that we can find an orthonormal frame adapted to the minimal immersion such that the frame map satisfies $dF = T_+F$.

Let N be a 2-dimensional manifold, let $g\colon N \to S^5_3$ be an immersion with positive definite induced metric and without totally geodesic points. Then, taking an arbitrary orthonormal basis $\{e_1, e_2\}$ at a point p and writing a unit vector v as $v = \cos\theta e_1 + \sin\theta e_2$, we see that

$$
\begin{aligned}
I\!I(v,v) &= \cos^2\theta I\!I(e_1,e_1)+2\sin\theta\cos\theta I\!I(e_1,e_2)+\sin^2\theta I\!I(e_2,e_2)\\
&= \frac{1}{2}(I\!I(e_1,e_1)+I\!I(e_2,e_2))+\frac{1}{2}\cos 2\theta(I\!I(e_1,e_1)\\
&\qquad - I\!I(e_2,e_2))+\sin 2\theta I\!I(e_1,e_2).
\end{aligned}
$$

From this, we see that the image of $\{I\!I(v,v)|\langle v,v\rangle = 1\}$ is at every point an ellipse. This ellipse is a circle centered at the origin if and only if

$$I\!I(e_1,e_1)+I\!I(e_2,e_2)=0, \tag{2}$$

$$\langle I\!I(e_1,e_1)-I\!I(e_2,e_2), I\!I(e_1,e_2)\rangle = 0, \tag{3}$$

$$
\begin{aligned}
&\Big\langle \frac{1}{2}(I\!I(e_1,e_1)-I\!I(e_2,e_2)), \frac{1}{2}(I\!I(e_1,e_1)-I\!I(e_2,e_2))\Big\rangle\\
&\quad= \langle I\!I(e_1,e_2), I\!I(e_1,e_2)\rangle .
\end{aligned} \tag{4}
$$

In particular, we see that an immersion with ellipse of curvature centered at the origin is minimal and has the property that $\|I\!I(v,v)\| = \sqrt{-\langle I\!I(v,v), I\!I(v,v)\rangle}$ is independent of the unit length vector at the point p. Conversely, it is also clear that all minimal immersions with that property have ellipse of curvature a circle centered at the origin.

Assume now that g has no totally geodesic points and has ellipse of curvature a circle. Let $\{e_1, e_2\}$ be an orthonormal frame. Then $M = \|I\!I(e,e)\| > 0$ for any unit vector e and independent of the choice of unit vector at a point p and we can associate with g a *second order frame*

$$G: N \to \mathrm{SO}(3,3),$$
$$x \mapsto G(x) = (G_0(x)\ G_1(x)\ G_2(x)\ G_3(x)\ G_4(x)\ G_5(x))^\top$$

such that $G_0 = g$, $G_1 = g_*(e_1)$, $G_2 = g_*(e_2)$, $G_3 = \frac{1}{M} I\!I(e_1, e_2)$, $G_4 = \frac{1}{M} I\!I(e_1, e_1)$. G_5 is uniquely defined by $G \in \mathrm{SO}(3,3)$.

Since $g_*TN = \mathrm{span}\{G_1, G_2\}$ and the image of $I\!I$ is given by span $\{G_3, G_4\}$, we have for each $x \in N$ a splitting $\mathbb{R}^6 = \mathrm{span}\{G_0\} \oplus$ span $\{G_1, G_2\} \oplus \mathrm{span}\{G_3, G_4\} \oplus \mathrm{span}\{G_5\}$ which is invariant with respect to the orthonormal frame $\{e_1, e_2\}$. We denote the orthogonal projection $\mathbb{R}^6 \to \mathrm{span}\{G_5\}$ by pr_5 and we let $\nabla_U I\!I(V, W) = D_U^\perp(I\!I(V, W)) - I\!I(\nabla_U V, W) - I\!I(V, \nabla_U W)$, where $D_U^\perp$ denotes the normal component to the surface, but tangential to the sphere, of $\mathrm{D}_U I\!I(V, W)$.

Lemma 5 *Let $g: N \to S_3^5$ be a minimal immersion whose ellipses of curvature are non-degenerate circles. Then there exist a local orthonormal frame $\{e_1, e_2\}$ with dual basis $\{\mu^1, \mu^2\}$, one-forms ω_2^1 and S_4^3, and a function N_4 such that the corresponding frame G satisfies*

$$\mathrm{d}G = \begin{pmatrix} 0 & \mu^1 & \mu^2 & 0 & 0 & 0 \\ -\mu^1 & 0 & -\omega_2^1 & M\mu^2 & M\mu^1 & 0 \\ -\mu^2 & \omega_2^1 & 0 & M\mu^1 & -M\mu^2 & 0 \\ 0 & M\mu^2 & M\mu^1 & 0 & -S_4^3 & N_4\mu^1 \\ 0 & M\mu^1 & -M\mu^2 & S_4^3 & 0 & N_4\mu^2 \\ 0 & 0 & 0 & -N_4\mu^1 & -N_4\mu^2 & 0 \end{pmatrix} G. \qquad (5)$$

Proof. We start with a second order frame $G: N \to \mathrm{SO}(3,3)$. Since the Codazzi equation is given by $\nabla_U I\!I(V, W) = \nabla_V I\!I(U, W)$ and G_5 is perpendicular to $I\!I(V, W)$ for all $V, W \in TM$, we have

$$\begin{aligned} \langle \mathrm{D}_{e_1} G_3, G_5 \rangle &= \Big\langle \frac{1}{M} \mathrm{D}_{e_1}(I\!I(e_1, e_2)), G_5 \Big\rangle = \Big\langle \frac{1}{M} \nabla_{e_1} I\!I(e_1, e_2), G_5 \Big\rangle \\ &= \Big\langle \frac{1}{M} \nabla_{e_2} I\!I(e_1, e_1), G_5) \Big\rangle = \langle \mathrm{D}_{e_2} G_4, G_5 \rangle \end{aligned}$$

and similarly

$$\begin{aligned} \langle \mathrm{D}_{e_2} G_3, G_5 \rangle &= \Big\langle \frac{1}{M} \mathrm{D}_{e_2}(I\!I(e_1, e_2)), G_5 \Big\rangle = \Big\langle \frac{1}{M} \nabla_{e_1} I\!I(e_2, e_2), G_5 \Big\rangle \\ &= -\Big\langle \frac{1}{M} \nabla_{e_1} I\!I(e_1, e_1), G_5 \Big\rangle = -\langle \mathrm{D}_{e_1} G_4, G_5 \rangle, \end{aligned}$$

where we have used that $I\!I(e_1,e_1)+I\!I(e_2,e_2)=0$. This implies that $S := \mathrm{d}GG^{-1}\in \mathrm{SO}(3,3)$ is of the following form:

$$\mathrm{d}G = \begin{pmatrix} 0 & \mu^1 & \mu^2 & 0 & 0 & 0 \\ -\mu^1 & 0 & -\omega_2^1 & M\mu^2 & M\mu^1 & 0 \\ -\mu^2 & \omega_2^1 & 0 & M\mu^1 & -M\mu^2 & 0 \\ 0 & M\mu^2 & M\mu^1 & 0 & -S_4^3 & N_4\mu^1 - N_3\mu^2 \\ 0 & M\mu^1 & -M\mu^2 & S_4^3 & 0 & N_3\mu^1+N_4\mu^2 \\ 0 & 0 & 0 & -N_4\mu^1+N_3\mu^2 & -N_3\mu^1-N_4\mu^2 & 0 \end{pmatrix} G. \tag{6}$$

By a rotation of $\{e_1,e_2\}$ we can obtain $N_3=0$. (Here we have redefined G_1, G_2 accordingly so that $g_*e_i=G_i$). To see this note first that $\langle \mathrm{D}_{e_1}G_3, \mathrm{D}_{e_1}G_4\rangle = -N_3N_4$ and $\langle \mathrm{D}_{e_2}G_3, \mathrm{D}_{e_2}G_4\rangle = N_3N_4$. For $\psi\in[0,2\pi)$ we define $\tilde e_1=\cos\psi e_1+\sin\psi e_2$, $\tilde e_2=-\sin\psi e_1+\cos\psi e_2$. Since $I\!I(e_2,e_2)=-MG_4$ we have

$$\tilde G_3 = \frac{1}{M} I\!I(\tilde e_1,\tilde e_2) = (\cos^2\psi-\sin^2\psi)G_3 - 2\cos\psi\sin\psi G_4$$

and

$$\tilde G_4 = \frac{1}{M} I\!I(\tilde e_1,\tilde e_1) = 2\cos\psi\sin\psi G_3 + (\cos^2\psi-\sin^2\psi)G_4.$$

Now a short calculation gives for $\psi=0$

$$\langle \mathrm{D}_{\tilde e_1}\tilde G_3, \mathrm{D}_{\tilde e_1}\tilde G_4\rangle = \langle \mathrm{D}_{e_1}G_3, \mathrm{D}_{e_1}G_4\rangle = -N_3N_4$$

and for $\psi=\pi/2$

$$\langle \mathrm{D}_{\tilde e_1}\tilde G_3, \mathrm{D}_{\tilde e_1}\tilde G_4\rangle = \langle \mathrm{D}_{e_2}G_3, \mathrm{D}_{e_2}G_4\rangle = N_3N_4.$$

Hence there is a $\psi\in[0,\pi/2]$ with $-\tilde N_3\tilde N_4=0$. Without loss of generality we can assume that $N_3=0$. Choosing the corresponding orthonormal frame $\{\tilde e_1,\tilde e_2\}$ and dropping the ˜ proves our assertion. □

If we rotate the pair of vectors (G_1,G_2) and (G_3,G_4) independently by two angles θ and φ so that

$$\tilde G_1 = \cos\theta G_1+\sin\theta G_2,\quad \tilde G_2 = -\sin\theta G_1+\cos\theta G_2,$$
$$\tilde G_3 = \cos\varphi G_3+\sin\varphi G_4,\quad \tilde G_4 = -\sin\varphi G_3+\cos\varphi G_4,$$

then we obtain the 1-form valued matrix

$$\tilde{S} = \begin{pmatrix} 0 & \cos\theta\mu^1 + \sin\theta\mu^2 & -\sin\theta\mu^1 + \cos\theta\mu^2 & 0 & 0 & 0 \\ -\cos\theta\mu^1 - \sin\theta\mu^2 & 0 & -\omega^1_2 + \mathrm{d}\theta & M(\sin(\theta+\varphi)\mu^1 + \cos(\theta+\varphi)\mu^2) & M(\cos(\theta+\varphi)\mu^1 - \sin(\theta+\varphi)\mu^2) & 0 \\ \sin\theta\mu^1 - \cos\theta\mu^2 & \omega^1_2 - \mathrm{d}\theta & 0 & M(\cos(\theta+\varphi)\mu^1 - \sin(\theta+\varphi)\mu^2) & -M(\sin(\theta+\varphi)\mu^1 + \cos(\theta+\varphi)\mu^2) & 0 \\ 0 & M(\sin(\theta+\varphi)\mu^1 + \cos(\theta+\varphi)\mu^2) & M(\cos(\theta+\varphi)\mu^1 - \sin(\theta+\varphi)\mu^2) & 0 & -S^3_4 + \mathrm{d}\varphi & N_4(\cos\varphi\mu^1 + \sin\varphi\mu^2) \\ 0 & M(\cos(\theta+\varphi)\mu^1 - \sin(\theta+\varphi)\mu^2) & -M(\sin(\theta+\varphi)\mu^1 + \cos(\theta+\varphi)\mu^2) & S^3_4 - \mathrm{d}\varphi & 0 & N_4(-\sin\varphi\mu^1 + \cos\varphi\mu^2) \\ 0 & 0 & 0 & -N_4(\cos\varphi\mu^1 + \sin\varphi\mu^2) & -N_4(-\sin\varphi\mu^1 + \cos\varphi\mu^2) & 0 \end{pmatrix}.$$

Lemma 6 *Let N be a 2-dimensional manifold. For any frame $G\colon N \to \mathrm{SO}(3,3)$ satisfying* (5) *there exist uniquely defined functions $\varphi, \lambda, \alpha, \beta, \gamma, \delta\colon \mathbb{R} \times N \to \mathbb{R}$ and 1-forms ω^1, ω^2, ω^3 such that the map $(\theta, x^1, x^2) \mapsto \tilde{G}$ satisfies $\mathrm{d}\tilde{G} = T_+\tilde{G}$.*

Proof. We will use the freedom to choose $\varphi(x^1, x^2, \theta)$ so that $\tilde{S} = T_+$.

Since

$$(T_+)^1_0 + (T_+)^5_3 + \frac{2}{\lambda}(T_+)^4_1 = 0, \quad (T_+)^2_0 + (T_+)^5_4 + \frac{2}{\lambda}(T_+)^3_1 = 0,$$

we have to satisfy

$$\tilde{S}^1_0 + \tilde{S}^5_3 + \frac{2}{\lambda}\tilde{S}^4_1 = 0, \quad \tilde{S}^2_0 + \tilde{S}^5_4 + \frac{2}{\lambda}\tilde{S}^3_1 = 0,$$

and therefore

$$\begin{aligned} &\cos\theta\mu^1 + \sin\theta\mu^2 + N_4(\cos\varphi\mu^1 + \sin\varphi\mu^2) \\ &\qquad = \frac{2M}{\lambda}(-\cos(\theta+\varphi)\mu^1 + \sin(\theta+\varphi)\mu^2), \\ &\sin\theta\mu^1 - \cos\theta\mu^2 + N_4(\sin\varphi\mu^1 - \cos\varphi\mu^2) \end{aligned}$$

$$= \frac{2M}{\lambda}(\sin(\theta+\varphi)\mu^1 + \cos(\theta+\varphi)\mu^2).$$

By interchanging $\mu^1 \to -\mu^2$ and $\mu^2 \to \mu^1$ it is clear that both equations are equivalent. From the first equation the 1-forms $\cos\theta\mu^1 + \sin\theta\mu^2 + N_4(\cos\varphi\mu^1 + \sin\varphi\mu^2)$ and $-\cos(\theta+\varphi)\mu^1 + \sin(\theta+\varphi)\mu^2$ are linearly dependent which is equivalent to

$$\sin(2\theta+\varphi) + N_4\sin(\theta+2\varphi) = 0. \tag{7}$$

From the same equation we get then

$$\lambda = -\frac{2M\cos(\theta+\varphi)}{\cos\theta + N_4\cos\varphi}. \tag{8}$$

The equations $(T_+)_1^3 = \tilde{S}_1^3$ and $(T_+)_1^4 = \tilde{S}_1^4$ imply

$$\omega^1 = \frac{M}{\lambda}(\sin(\theta+\varphi)\mu^1 + \cos(\theta+\varphi)\mu^2), \tag{9}$$

$$\omega^2 = -\frac{M}{\lambda}(\cos(\theta+\varphi)\mu^1 - \sin(\theta+\varphi)\mu^2). \tag{10}$$

Since $(T_+)_0^1 - (T_+)_3^5 = \tilde{S}_0^1 - \tilde{S}_3^5$ is equivalent to

$$-2\alpha\omega^1 + 2\beta\omega^2 = (\cos\theta - N_4\cos\varphi)\mu^1 + (\sin\theta - N_4\sin\varphi)\mu^2,$$

we obtain

$$\alpha = -\frac{\lambda}{2M}(\sin(2\theta+\varphi) - N_4\sin(\theta+2\varphi)), \tag{11}$$

$$\beta = -\frac{\lambda}{2M}(\cos(2\theta+\varphi) - N_4\cos(\theta+2\varphi)). \tag{12}$$

The only independent relations left is the pair of equations $2\omega_3 = (T_+)_1^2 - (T_+)_3^4 = \tilde{S}_1^2 + \tilde{S}_3^4 = S_4^3 - \omega_2^1 + \mathrm{d}(\theta-\varphi)$, $2(-\gamma\omega^1 + \delta\omega^2 - \frac{\beta}{3}\omega^3) = (T_+)_1^2 + (T_+)_3^4 = \tilde{S}_1^2 + \tilde{S}_3^4 = -S_4^3 - \omega_2^1 + \mathrm{d}(\theta+\varphi)$. Denote the basis dual to $\{\mu^1, \mu^2, \mathrm{d}\theta\}$ by $\{e_1, e_2, e_\theta\}$ and write $\omega^3 = (\omega^3)_1\mu^1 + (\omega^3)_2\mu^2 + (\omega^3)_\theta\mathrm{d}\theta$. Then the first equation is equivalent to

$$(\omega^3)_1 = \frac{1}{2}(S_4^3 - \omega_2^1 - \mathrm{d}\varphi)(e_1), \tag{13}$$

$$(\omega^3)_2 = \frac{1}{2}(S_4^3 - \omega_2^1 - \mathrm{d}\varphi)(e_2), \tag{14}$$

$$(\omega^3)_\theta = \frac{1}{2}(1 - \mathrm{d}\varphi(e_\theta)). \tag{15}$$

The second equation gives

$$\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = -\frac{\lambda}{2M}\begin{pmatrix}\sin(\theta+\varphi) & \cos(\theta+\varphi)\\ \cos(\theta+\varphi) & -\sin(\theta+\varphi)\end{pmatrix} \begin{pmatrix}\left((\frac{\beta}{3}-1)(S^3_4-d\varphi)-(\frac{\beta}{3}+1)\omega^1_2\right)(e_1)\\ \left((\frac{\beta}{3}-1)(S^3_4-d\varphi)-(\frac{\beta}{3}+1)\omega^1_2\right)(e_2)\end{pmatrix}, \tag{16}$$

$$(\omega^3)_\theta = -\frac{3}{2\beta}\left(1+\mathrm{d}\varphi(e_\theta)\right). \tag{17}$$

Thus the functions φ, λ, α, β, γ, δ and the 1-forms ω^1, ω^2, ω^3 are all determined. Therefore we only need to show that the equation (17) is a consequence of equations (7)–(16). Inserting equations (9), (10) into $(T_+)^1_0 = \tilde{S}^1_0$ we obtain $-\frac{M}{\lambda}(1+\beta) = \cos(2\theta+\varphi)$. Using equation (7) and equation (8) after some computations we obtain $\lambda = -2M/(\cos(2\theta+\varphi)+N_4\cos(\theta+2\varphi))$. Taking the θ-derivative of equation (7) gives

$$\begin{aligned} 0 &= (\cos(2\theta+\varphi)+N_4\cos(\theta+2\varphi))(1+2\mathrm{d}\varphi(e_\theta))\\ &\qquad +\cos(2\theta+\varphi)(1-\mathrm{d}\varphi(e_\theta))\\ &= \frac{-2M}{\lambda}(1+2\mathrm{d}\varphi(e_\theta)) - (1-\mathrm{d}\varphi(e_\theta))\frac{(1+\beta)M}{\lambda}. \end{aligned}$$

This is equivalent to $0 = -3-\beta-(3-\beta)\mathrm{d}\varphi(e_\theta)$ which together with equation (15) implies equation (17). □

Theorem 2 *Let $\tilde{N}$ be a two-dimensional manifold and let $\tilde{g}\colon \tilde{N} \to S^5_3$ be a minimal immersion with positive definite induced metric whose ellipses of curvature are non-degenerate circles. Then there is a 3-dimensional elliptic affine unit sphere $f\colon M \to \mathbb{R}^4$ which satisfies Chen's equality and a natural immersion $j\colon \tilde{N} \to M$ such that $j(\tilde{N})$ can be identified with the submanifold N given in Theorem 1.*

Proof. Let $\{e_1, e_2\}$ be the orthornormal frame of $T\tilde{N}$ provided by Lemma 5 and denote by $\{\mu^1, \mu^2\}$ the dual frame. Let (x^1, x^2) be a coordinate system of $\tilde{N}$, and denote an additional coordinate by θ. We now can define functions φ, α, β, γ, δ, λ and 1-forms ω^1, ω^2, ω^3 via equations (7)–(16). The frame map

$$F\colon \mathcal{U} \subset \mathbb{R}^3 \to \mathrm{SO}(3,3),$$

$$x \mapsto \begin{pmatrix} g \\ \cos(\theta)\mathrm{d}g(e_1) + \sin(\theta)\mathrm{d}g(e_2) \\ -\sin(\theta)\mathrm{d}g(e_1) + \cos(\theta)\mathrm{d}g(e_2) \\ \frac{1}{M}\left(\cos(\varphi)I\!I(e_1,e_2) + \sin(\varphi)I\!I(e_1,e_1)\right) \\ \frac{1}{M}\left(-\sin(\varphi)I\!I(e_1,e_2) + \cos(\varphi)I\!I(e_1,e_1)\right) \\ F_5 \end{pmatrix},$$

where F_5 is uniquely determined by the requirement $F \in \mathrm{SO}(3,3)$, satisfies $\mathrm{d}F = T_+F$. Since $\iota\colon \mathrm{SL}(4,\mathbb{R}) \to \mathrm{SO}(3,3)$ is a covering map, it has a local inverse ι^{-1}. By the form of T_+, the immersion $f(x^1,x^2,x^3) = (\iota^{-1}(F))_0$ defines a 3-dimensional ellipic affine unit sphere which satisfies Chen's equality. $\square$

Remark 2 Let (x^1,x^2) be coordinates of $N \subset M$. It would then be natural to choose a third coordinate x^3 of M such that it is aligned with the invariant vector field E_3.

Let $x^3 = \frac{1}{2}(\theta - \varphi)$, $\rho = \frac{1}{2}(\theta + \varphi)$. Then we have $E_3 = \partial_{x^3}$ and the system (7)–(17) is equivalent to

$$0 = \sin(x^3 + 3\rho) + N_4 \sin(-x^3 + 3\rho), \tag{18}$$

$$\lambda = \frac{-2M\cos(2\rho)}{\cos(x^3+\rho) + N_4\cos(-x^3+\rho)}, \tag{19}$$

$$\omega^1 = -\frac{\cos(x^3+\rho) + N_4\cos(-x^3+\rho)}{2\cos(2\rho)}(\sin(2\rho)\mu^1 + \cos(2\rho)\mu^2), \tag{20}$$

$$\omega^2 = \frac{\cos(x^3+\rho) + N_4\cos(-x^3+\rho)}{2\cos(2\rho)}(\cos(2\rho)\mu^1 - \sin(2\rho)\mu^2), \tag{21}$$

$$\omega^3 = \frac{1}{2}(S_4^3 - \omega_2^1) + \mathrm{d}x^3, \tag{22}$$

$$\begin{aligned}(\alpha,\beta) = \cos(2\rho)\Bigg(&\frac{\sin(x^3+3\,\rho) - N_4\sin(3\,\rho - x^3)}{\cos(x^3+\rho) + N_4\cos(-x^3+\rho)}, \\ &\frac{\cos(x^3+3\,\rho) - N_4\cos(3\,\rho - x^3)}{\cos(x^3+\rho) + N_4\cos(-x^3+\rho)}\Bigg),\end{aligned} \tag{23}$$

$$\begin{aligned}\begin{pmatrix}\gamma\\ \delta\end{pmatrix} &= -\frac{\lambda}{2M}\begin{pmatrix}\sin(2\rho) & \cos(2\rho)\\ \cos(2\rho) & -\sin(2\rho)\end{pmatrix}\\ &\begin{pmatrix}\left((\frac{\beta}{3}-1)(S_4^3+d(x^3-\rho))-(\frac{\beta}{3}+1)\omega_2^1\right)(e_1)\\ \left((\frac{\beta}{3}-1)(S_4^3-d(x^3-\rho))-(\frac{\beta}{3}+1)\omega_2^1\right)(e_2)\end{pmatrix}.\end{aligned} \tag{24}$$

Remark 3 The frame $\{e_1, e_2\}$ used in Theorem 2 is uniquely (up to finitely many choices) defined unless $\mathrm{d}I\!I(e_1,e_1)(TN) \subset \mathrm{span}\{\mathrm{d}g(e_1), \mathrm{d}g(e_2), I\!I(e_1,e_1), I\!I(e_1,e_2)\}$. Hence in this (generic) case, to each minimal immersion $g\colon N \to S_3^5$ whose ellipses of curvature are non-degenerate circles we can uniquely (up to finitely many choices) assign an affine unit sphere $f\colon M \to \mathbb{R}^4$ which satisfies Chen's equality and vice versa.

If $\mathrm{d}I\!I(e_1,e_1)(TN) \subset \mathrm{span}\{\mathrm{d}g(e_1), \mathrm{d}g(e_2), I\!I(e_1,e_1), I\!I(e_1,e_2)\}$, then $N_3 = N_4 = 0$ and we can reduce to an immersion $\hat{g}\colon N \to S_2^4$. Then the 1-form valued matrix T_+ is determined by

$$\varphi(\theta) = -2\theta + k\pi, \quad k \in \mathbb{Z}, \quad \lambda = \pm 2M, \quad \alpha = 0, \quad \beta = 1,$$

$$\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = -\frac{2}{3}\begin{pmatrix}-\sin(\theta) & \cos(\theta)\\ \cos(\theta) & \sin(\theta)\end{pmatrix}\begin{pmatrix}(S_4^3+2\omega_2^1)(e_1)\\ (S_4^3+2\omega_2^1)(e_2)\end{pmatrix},$$

$$\omega^1 = \frac{1}{2}(\sin\theta\mu^1 - \cos\theta\mu^2), \quad \omega^2 = \frac{1}{2}(\cos\theta\mu^1 + \sin\theta\mu^2),$$

$$\omega^3 = \frac{1}{2}\left(S_4^3 - \omega_2^1 + 3\mathrm{d}\theta\right),$$

which follow directly from equations (7)–(16) and $N_4 = 0$.

Remark 4 The sub-case previously classified in [SSVV97] is the case where $\beta = 0$ (cf. the discussion following Lemma 2). By equations (7), (12), this reduces to $N_4 = \pm 1$. The function φ again depends only on θ. In the next section we will see that in this case the integrability conditions have an especially simple form.

4. Minimal immersions $g\colon N \to S_3^5$ whose ellipses of curvature are circles

After we have reduced the problem to the classification of minimal immersions whose ellipses of curvature are circles, an investigation of this class will be done in the following. Consider a minimal immersion $g\colon N \to S_3^5$

with positive definite induced metric whose ellipses of curvature are non-degenerate circles. We can choose isothermal coordinates (u, v) and a frame G as defined in equation (6) such that $\mu^1 = \sigma \mathrm{d}u$, $\mu^2 = \sigma \mathrm{d}v$. Writing $\mathrm{d}G = SG$ the integrability conditions $\mathrm{dd}G = 0$ are equivalent to $\mathrm{d}S - S \wedge S = 0$ and reduce to

$$\begin{aligned}
&{}^1_0\colon\ 0 = \mathrm{d}\mu^1 - \mu^2 \wedge \omega^1_2 \\
&{}^2_0\colon\ 0 = \mathrm{d}\mu^2 + \mu^1 \wedge \omega^1_2 \\
&{}^2_1\colon\ 0 = -\mathrm{d}\omega^1_2 + \mu^1 \wedge \mu^2 + 2M^2 \mu^1 \wedge \mu^2 \\
&{}^3_1\colon\ 0 = \mathrm{d}(M\mu^2) - M\mu^1 \wedge (\omega^1_2 + S^3_4) \\
&{}^4_1\colon\ 0 = \mathrm{d}(M\mu^1) - M(\omega^1_2 + S^3_4) \wedge \mu^2 \\
&{}^4_3\colon\ 0 = -\mathrm{d}S^3_4 + (2M^2 + (N_3)^2 + (N_4)^2)\mu^1 \wedge \mu^2 \\
&{}^5_3\colon\ 0 = \mathrm{d}(N_4\mu^1 - N_3\mu^2) + S^3_4 \wedge (N_3\mu^1 + N_4\mu^2) \\
&{}^5_4\colon\ 0 = \mathrm{d}(N_3\mu^1 + N_4\mu^2) - S^3_4 \wedge (N_4\mu^1 - N_3\mu^2).
\end{aligned}$$

Since $M \neq 0$, the integrability conditions can be re-expressed as

$$\begin{aligned}
&\omega^1_2 = (\ln\sigma)_v \mathrm{d}u - (\ln\sigma)_u \mathrm{d}v, \quad S^3_4 = a\mathrm{d}u + b\mathrm{d}v, \quad (\ln(\sigma^2 M))_u = b, \\
&(\ln(\sigma^2 M))_v = -a, \quad (\ln\sigma)_{uu} + (\ln\sigma)_{vv} = -(1 + 2M^2)\sigma^2, \\
&-a_v + b_u = (2M^2 + (N_3)^2 + (N_4)^2)\sigma^2, \\
&(N_3)_u + (N_4)_v = -N_3(\ln(\sigma^3 M))_u - N_4(\ln(\sigma^3 M))_v, \\
&(N_3)_v - (N_4)_u = -N_3(\ln(\sigma^3 M))_v + N_4(\ln(\sigma^3 M))_u.
\end{aligned}$$

Setting $\sigma^3 M = A$, the system of differential equations is equivalent to the following determined system:

$$(\ln A)_{uu} + (\ln A)_{vv} = \sigma^2((N_3)^2 + (N_4)^2 - 1),$$

$$(\ln\sigma)_{uu} + (\ln\sigma)_{vv} = -\Big(\sigma^2 + \frac{2}{\sigma^4}A^2\Big), \tag{25}$$

$$(AN_3)_u + (AN_4)_v = 0, \quad (AN_3)_v - (AN_4)_u = 0. \tag{26}$$

Lemma 7 *Let $g\colon N \to S^5_3$ be a minimal immersion with positive definite induced metric whose ellipse of curvature is a non-degenerate circle. Then either N_3 and N_4 vanish identically, in which case the image of N is contained in a totally geodesic S^4_2, or except at isolated points there exist isothermic coordinates with $\sigma^3 M N_4 = 1$ and $N_3 = 0$.*

Proof. Let $(\tilde{u}, \tilde{v})$ be any system of isothermal coordinates and define the complex coordinate $\tilde{z}$ by $\tilde{z} = \tilde{u} + \mathrm{i}\tilde{v}$. Equations (26) are satisfied if and only if $\tilde{A}(\tilde{N}_3 - \mathrm{i}\tilde{N}_4)$ is a holomorphic function with respect to $\tilde{z}$. It is straight forward to show that $\frac{1}{2}\tilde{A}(\tilde{N}_3 - \mathrm{i}\tilde{N}_4)$ is just the G_5 component of $\nabla \mathit{II}(\partial_{\tilde{z}}, \partial_{\tilde{z}}, \partial_{\tilde{z}})$. In fact, note first that

$$\begin{aligned} \mathit{II}(\partial_{\tilde{z}}, \partial_{\tilde{z}}) &= \frac{1}{4}\left(\mathit{II}(\partial_{\tilde{u}}, \partial_{\tilde{u}}) - \mathit{II}(\partial_{\tilde{v}}, \partial_{\tilde{v}}) - 2\mathrm{i}\mathit{II}(\partial_{\tilde{u}}, \partial_{\tilde{v}})\right) \\ &= \frac{1}{2}\tilde{\sigma}^2 M\left(G_4 - \mathrm{i}G_3\right). \end{aligned}$$

Since $\nabla \mathit{II}(\partial_{\tilde{z}}, \partial_{\tilde{z}}, \partial_{\tilde{z}}) = D^{\perp}_{\frac{\partial}{\partial \tilde{z}}}(\mathit{II}(\partial_{\tilde{z}}, \partial_{\tilde{z}})) - 2\mathit{II}(\nabla_{\partial_{\tilde{z}}}\partial_{\tilde{z}}, \partial_{\tilde{z}})$, where $\mathit{II}(\nabla_{\partial_{\tilde{z}}}\partial_{\tilde{z}}, \partial_{\tilde{z}} \in$ span$\{G_3, G_4\}$, the projection pr_5 of this vector to span$\{G_5\}$ is given by

$$\begin{aligned} \mathrm{pr}_5(\nabla \mathit{II}(\partial_{\tilde{z}}, \partial_{\tilde{z}}, \partial_{\tilde{z}})) &= \mathrm{pr}_5(D^{\perp}_{\frac{\partial}{\partial \tilde{z}}}(\mathit{II}(\partial_{\tilde{z}}, \partial_{\tilde{z}}))) \\ &= \frac{\tilde{\sigma}^2 M}{4}\mathrm{pr}_5(D^{\perp}_{\partial_{\tilde{u}} - \mathrm{i}\partial_{\tilde{v}}}(G_4 - \mathrm{i}G_3)) \\ &= \frac{1}{2}\tilde{A}(\tilde{N}_3 - \mathrm{i}\tilde{N}_4)G_5. \end{aligned}$$

Observe that in $T_x S^5_3$ span$\{(G_5)_x\} = (g_* T_x N \oplus \mathrm{Image}(\mathit{II}_x))^{\perp}$ which implies that $\pm G_5$ is invariant under coordinate transformations $(\tilde{u}, \tilde{v}) \mapsto (u, v)$. Since $\tilde{A}(\tilde{N}_3 - i\tilde{N}_4)$ is a holomorphic function, it vanishes either identically or only at isolated points. In the first case, it follows immediately that G_5 is a constant vector and hence N is contained in a totally geodesic S^4_2. In the other case, at a point where $(N_3, N_4) \neq 0$, it is now clear that there is a holomorphic coordinate z with respect to which

$$\begin{aligned} \frac{1}{2}A(N_3 - \mathrm{i}N_4)G_5 &= \mathrm{pr}_5\left(\nabla \mathit{II}(\partial_z, \partial_z, \partial_z)\right) \\ &= \left(\frac{\partial \tilde{z}}{\partial z}\right)^3 \mathrm{pr}_5\left(\nabla \mathit{II}(\partial_{\tilde{z}}, \partial_{\tilde{z}}, \partial_{\tilde{z}})\right) = -\frac{\mathrm{i}}{2}G_5. \end{aligned}$$

□

Corollary 2 *Let $(A, \sigma)\colon N \to \mathbb{R}^2$, $(u, v) \mapsto (A(u, v), \sigma(u, v))$ be positive functions such that $(\ln \sigma)_{uu} + (\ln \sigma)_{vv} = -\sigma^2(1 + 2\sigma^{-6}A^2)$ and either*

(i) $(\ln A)_{uu} + (\ln A)_{vv} = \sigma^2(\frac{1}{A^2} - 1)$ *or*

(ii) $(\ln A)_{uu} + (\ln A)_{vv} = -\sigma^2$.

Then there is either a

(i) *minimal immersion $g\colon N \to S^5_3$ or a*

(ii) *minimal immersion $g\colon N \to S^4_2$*

with $g^*\langle\cdot,\cdot\rangle = \sigma^2(\mathrm{d}u^2+\mathrm{d}v^2)$ *whose ellipses of curvature are non-degenerate circles and whose curvature is given by* $k = 1 + 2A^2\sigma^{-6} > 1$.

Proof. Note that in case (i) $(A, \sigma, N_3 = 0,\ N_4 = \frac{1}{A})$ is a solution of the integrability conditions (25)–(26) while in case (ii) $(A, \sigma, N_3 = 0,\ N_4 = 0)$ is a solution of the integrability conditions. □

Theorem 3 (i) *Let* $g\colon N \to S^5_3$ *be a minimal immersion with positive definite induced metric whose ellipses of curvature are non-degenerate circles and denote its curvature by* k. *Then* $k > 1$, $B := \Delta \ln\sqrt{k-1} - 3k + 1 \geq 0$, *and satisfies* $B\Delta B - \|grad\, B\|^2 = -2B^2(B-1)$.

(ii) *Let* $g\colon N \to S^4_2$ *be a minimal immersion whose ellipses of curvature are non-degenerate circles and denote its curvature by* k. *Then* $k > 1$ *and* $\Delta \ln\sqrt{k-1} - 3k + 1 = 0$.

Proof. We only prove the first case (i). The proof of (ii) is analogous.

Let g be such a minimal immersion. Remark that except at isolated points there exist isothermal coordinates with $\sigma^3 M N_4 = 1$. Setting $A = \sigma^3 M$ the integrability conditions (25)–(26) reduce to $(\ln A)_{uu} + (\ln A)_{vv} = \sigma^2(\frac{1}{A^2} - 1)$ and $(\ln\sigma)_{uu} + (\ln\sigma)_{vv} = -\sigma^2(1 + 2\sigma^{-6}A^2)$. We only have to show that $A = 1/\sqrt{\Delta\ln\sqrt{k-1} - 3k + 1}$. This follows from $k = 1 + 2M^2$ and

$$
\begin{aligned}
&(\ln(k-1))_{uu} + (\ln(k-1))_{vv}\\
&\quad = (\ln(2M^2))_{uu} + (\ln(2M^2))_{vv} = 2((\ln M)_{uu} + (\ln M)_{vv})\\
&\quad = 2((\ln(\sigma^3 M))_{uu} + (\ln(\sigma^3 M))_{vv} - 3((\ln\sigma)_{uu} + (\ln\sigma)_{vv}))\\
&\quad = 2\sigma^2\Big(3(1+2M^2) + \frac{1}{(\sigma^3 M)^2} - 1\Big).
\end{aligned}
$$

To avoid the problems at isolated points, we now consider $B := \Delta\ln\sqrt{k-1} - 3k + 1$, which then satisfies $B\Delta B - \|grad\, B\|^2 = -2B^2(B-1)$ □

Theorem 4 (i) *Let* $(N, \mathfrak{g})$ *be a 2-dimensional Riemannian manifold with curvature* $k > 1$ *and* $\Delta\ln\sqrt{k-1} - 3k + 1 > 0$. *If* $A := 1/\sqrt{\Delta\ln\sqrt{k-1} - 3k + 1}$ *satisfies* $\Delta \ln A = \frac{1}{A^2} - 1$, *then* $(N, \mathfrak{g})$ *can be realised as a minimal immersion* $g\colon N \to S^5_3$ *whose ellipses of curvature are non-degenerate circles.*

(ii) *Let* $(N, \mathfrak{g})$ *be a 2-dimensional Riemannian manifold with curvature* $k > 1$ *and* $\Delta\ln\sqrt{k-1} - 3k + 1 = 0$. *Then* $(N, \mathfrak{g})$ *can be realised* *as a minimal immersion $g\colon N \to S^4_2$ whose ellipses of curvature are non-degenerate circles.*

Proof. We only prove the first case (i). The proof of (ii) is analogous.

Let $M = \sqrt{(k-1)/2}$ and $\sigma = \sqrt[3]{A/M}$. Then the assertion follows from Corollary 2 applied to (A, σ). $\square$

Remark 5 We have pointed out in Remark 4 that the integrable case $\beta = 0$ discussed in [SSVV97] is equivalent to $N_4 = \pm 1$ (or $A = \pm 1$). In this case $(N, \mathfrak{g})$ also determines an affine unit 2-sphere of mean curvature 1 with non-zero Pick invariant [SW93, Corollary 2.17]. Conversely, any affine unit 2-sphere of mean curvature 1 satisfies $A = 1$ and therefore determines a minimal immersion $g\colon N \to S^5_3$ whose ellipses of curvature are non-degenerate circles by Theorem 3. Affine unit 2-spheres with mean curvature 1 can therefore be considered as a proper subclass of minimal immersions $g\colon N \to S^5_3$ whose ellipses of curvature are circles.

The following generalizes well known facts about affine unit spheres [LSZ93, paragraph 2.4].

Corollary 3 *Any complete minimal immersion with positive definite induced metric $g\colon N \to S^5_3$ whose ellipses of curvature are circles must be compact.*

Proof. Since $k = 1 + 2M^2 \geq 1$ the assertion follows immediately from the Theorem of Myers. $\square$

Theorem 5 *Let $g\colon N \to S^5_3$ be a compact minimal immersion with positive definite induced metric whose ellipses of curvature are circles. Then there exists a spacelike plane E through 0 such that $g(N) = S^5_3 \cap E$. In particular, the circle of curvature is degenerate everywhere.*

Proof. Clearly we may assume that N is orientable. We then recall the following integral formula from [Ros85]. Let T be a tensor field on a compact manifold N with unit tangent bundle UN and let ∇T be the covariant derivative of T. Then $\int_{UN} \sum_{i=1}^{n} (\nabla_{e_i} T)(e_i, v, v, \ldots, v) = 0$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis at a point p. For simplicity we write $\nabla I\!I(X_1, X_2, X_3) = \nabla_{X_1} I\!I(X_2, X_3)$ and by $\nabla^2 I\!I(X_1, X_2, X_3, X_4) = (\nabla_{X_1}(\nabla I\!I))(X_2, X_3, X_4)$. Applying the above integral formula, to the tensor field

$$T(X_1, X_2, X_3, X_4, X_5) = \langle \nabla_{X_1} I\!I(X_2, X_3), I\!I(X_4, X_5) \rangle,$$

we obtain that

$$\begin{aligned}\int_{UN} &\langle \nabla^2 I\!I(v,v,v,v), I\!I(v,v)\rangle + \langle \nabla^2 I\!I(u,u,v,v), I\!I(v,v)\rangle \\ &+ \langle \nabla I\!I(v,v,v), \nabla I\!I(v,v,v)\rangle + \langle \nabla I\!I(u,v,v), I\!I(u,v,v)\rangle = 0,\end{aligned}$$

where $v \in UN$ and u is the corresponding orthogonal vector such that $\{v,u\}$ are positively oriented. Since the induced metric on the normal space is negative definite it follows that

$$\int_{UN} \langle \nabla^2 I\!I(v,v,v,v), I\!I(v,v)\rangle + \langle \nabla I\!I(u,u,v,v), I\!I(v,v)\rangle \geq 0$$

Applying the Ricci identity for submanifolds in space forms,

$$\begin{aligned}&\nabla^2 I\!I(X,Y,Z,W) - \nabla^2 I\!I(Y,X,Z,W) \\ &\quad = R^{\perp}(X,Y) I\!I(Z,W) - I\!I(R(X,Y)Z,W) - I\!I(Z,R(X,Y)W),\end{aligned}$$

together with the fact that the immersion is minimal, we find that

$$\int_{UN} \langle R^{\perp}(u,v) I\!I(v,u) - I\!I(R(u,v)v,u) - I\!I(v,R(u,v)u)), I\!I(v,v)\rangle \geq 0.$$

Using now that the ellipse of curvature is a circle ((2), (3) and (4)) with radius M centered at the origin together with the Ricci equation, it follows that

$$\int_{UN} -2(M^4 + kM^2) \geq 0. \tag{27}$$

Since by the Gauss equation $k \geq 1$ the above equation implies that $0 = M$. Hence the immersion g is totally geodesic. □

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