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Aluthge transformations and invariant subspaces of p-hyponormal operators

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Abstract. It is unknown at present whether every hyponormal operator has a nontrivial invariant subspace. Many authors presented conditions for a hyponormal operator to have nontrivial invariant subspaces. In this paper, we give a p-hyponormal version of Nakamura's result [7] by using the principal functions.

Key words: hyponormal operator, p-hyponormal operator, invariant subspace.

1. Introduction

An (bounded linear) operator T on a Hilbert space $\mathcal{H}$ is said to be p-hyponormal, if $(TT^*)^p \leq (T^*T)^p$ for a positive number p. If $p = 1$, then T is said to be hyponormal, and if $p = \frac{1}{2}$, then T is said to be semi-hyponormal. We assume that $0 < p \leq \frac{1}{2}$. An operator T is called pure if it has no nontrivial reducing subspace on which it is normal.

It is unknown at present whether every hyponormal operator has a nontrivial invariant subspace. Putnam [8] and Apostol and Clancey [2] presented some conditions for a hyponormal operator to have invariant subspaces. Nakamura [7] improved these results. In this paper, we give a p-hyponormal version of Nakamura's result.

Let $T = X + iY$ be a pure hyponormal operator, where X and Y are self-adjoint. Then it is known that X and Y are absolutely continuous (see [4, Chap. 2, Th. 3.2]). For a self-adjoint operator Z, let $Z = \int t dG(t)$ be the spectral resolution of Z. Then the absolutely continuous support E_Z of Z is defined as a Borel subset of the real line (determined uniquely up to a null set) having the least Lebesgue measure and satisfying $G(E_Z) = I$. Then Nakamura's results are as follows.

Theorem A ([7], Theorem 1) *Let T be a pure hyponormal operator and $T = X + iY$ be the Cartesian decomposition of T. Suppose that there exists*

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a real μ_0 such that the spectrum of T has non-empty intersection with each of the open half-planes $\{z : \operatorname{Re} z < \mu_0\}$ and $\{z : \operatorname{Re} z > \mu_0\}$, and

$$\int_{E_X} \frac{F(x)}{(x-\mu_0)^2}\, dx < \infty$$

where $F(x)$ is the linear measure of the vertical cross $\sigma(T) \cap \{z : \operatorname{Re} z = x\}$. Then T has a nontrivial invariant subspace.

Theorem B ([7], Theorem 2) *In Theorem 1, the existence of a nontrivial invariant subspace is also guaranteed if the integrability condition is replaced by*

$$\int_{E_X} \frac{1}{|x-\mu_0|}\, dx < \infty.$$

Let $T = U|T|$ be the polar decomposition of T. Put $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Let $\tilde{T} = V|\tilde{T}|$ denote the polar decomposition of $\tilde{T}$. Put $\hat{T} = |\tilde{T}|^{\frac{1}{2}}V|\tilde{T}|^{\frac{1}{2}}$. Then $\tilde{T}$ and $\hat{T}$ are called the Aluthge transformation and the second Aluthge transformation of T, respectively. It is well known that if T is p-hyponormal, then $\hat{T}$ is hyponormal by [1]. Also it is well known that $\sigma(T) = \sigma(\tilde{T}) = \sigma(\hat{T})$.

The main results in this paper are the following:

Theorem 1 *Let T be a pure p-hyponormal operator with dense range. For the second Aluthge transformation $\hat{T}$ of T, let $\hat{T} = X_2 + iY_2$ denote the Cartesian decomposition of $\hat{T}$. Suppose that there exists a real μ_0 such that the spectrum of T has non-empty intersection with each of the open half-planes $\{z : \operatorname{Re} z < \mu_0\}$ and $\{z : \operatorname{Re} z > \mu_0\}$, and*

$$\int_{E_{X_2}} \frac{F(x)}{(x-\mu_0)^2}\, dx < \infty$$

where $F(x)$ is the linear measure of the vertical cross $\sigma(T) \cap \{z : \operatorname{Re} z = x\}$. Then T has a nontrivial invariant subspace.

Theorem 2 *In Theorem 1, the existence of a nontrivial invariant subspace is also guaranteed if the integrability condition is replaced by*

$$\int_{E_{X_2}} \frac{1}{|x-\mu_0|}\, dx < \infty.$$

2. Aluthge transformation

Lemma 3 *Let $T = U|T|$ be an operator with* $\ker|T| = \{0\}$. *If T has a cyclic vector, then the Aluthge transformation $\tilde{T}$ has also a cyclic vector and satisfies* $\ker|\tilde{T}| = \{0\}$.

Proof. Let x be a cyclic vector for T. For any positive integer n,

$$(\tilde{T})^n|T|^{\frac{1}{2}} = \left(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \cdots |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right)|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}T^n.$$

Let y be a vector such that $\left((\tilde{T})^n|T|^{\frac{1}{2}}x, y\right) = 0$ for $n = 0, 1, 2, \ldots$. Then

$$\left(T^n x, |T|^{\frac{1}{2}}y\right) = 0.$$

Since x is a cyclic vector for T, $|T|^{\frac{1}{2}}y = 0$, so that $|T|y = 0$. Hence by the assumption we have $y = 0$. This implies that $|T|^{\frac{1}{2}}x$ is a cyclic vector for $\tilde{T}$.

Next we show $\ker|\tilde{T}| = \{0\}$. Since $\ker|T| = \{0\}$, we may assume that U is isometry. Let $\tilde{T}w = 0$, so that $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}w = 0$. Since $\ker|T| = \{0\}$ and U is isometry, we have $|T|^{\frac{1}{2}}w = 0$, that is, $w = 0$. Therefore, $\ker\tilde{T} = \{0\}$. Since $\ker\tilde{T} = \ker|\tilde{T}|$, we have $\ker|\tilde{T}| = \{0\}$. □

The following lemma improves [3, Lemma 2].

Lemma 4 *Let $T = U|T|$ be a pure p-hyponormal operator with dense range. Then the Aluthge transformation $\tilde{T}$ is pure $\left(p + \frac{1}{2}\right)$-hyponormal.*

Proof. It is well known that $\tilde{T}$ is $\left(p + \frac{1}{2}\right)$-hyponormal. Hence we may only prove that $\tilde{T}$ is pure. Since T is p-hyponormal and has a dense range, $\ker T = \ker|T| = \ker T^* = \{0\}$. Hence U is unitary. Let $\mathcal{X}$ be a reducing subspace of $\tilde{T}$ such that $\tilde{T}$ is normal on $\mathcal{X}$. Then for $x \in \mathcal{X}$,

$$\tilde{T}(\mathcal{X}) \subseteq \mathcal{X}, \ (\tilde{T})^*(\mathcal{X}) \subseteq \mathcal{X} \quad \text{and} \quad (\tilde{T})^*\tilde{T}x = \tilde{T}(\tilde{T})^*x. \tag{1}$$

If $((\tilde{T})^*\tilde{T})^n x = (\tilde{T}(\tilde{T})^*)^n x$ for $x \in \mathcal{X}$, then by (1) we have

$$\begin{aligned}((\tilde{T})^*\tilde{T})^{n+1}x &= ((\tilde{T})^*\tilde{T})^n((\tilde{T})^*\tilde{T}x) = (\tilde{T}(\tilde{T})^*)^n((\tilde{T})^*\tilde{T}x) \\ &= (\tilde{T}(\tilde{T})^*)^n(\tilde{T}(\tilde{T})^*x) = (\tilde{T}(\tilde{T})^*)^{n+1}x.\end{aligned}$$

Hence we have $((\tilde{T})^*\tilde{T})^n x = (\tilde{T}(\tilde{T})^*)^n x$ for every non-negative integer n and $x \in \mathcal{X}$. Since it holds that $f((\tilde{T})^*\tilde{T})x = f(\tilde{T}(\tilde{T})^*)x$ for every polynomial f, we have

$$\left(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}\right)^{p+\frac{1}{2}}x = \left(|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}}\right)^{p+\frac{1}{2}}x. \tag{2}$$

By Aluthge's result [1], we have

$$\left(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}\right)^{p+\frac{1}{2}} \geq |T|^{2(p+\frac{1}{2})} \geq \left(|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}}\right)^{p+\frac{1}{2}}.$$

Put $A = \left(|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}\right)^{p+\frac{1}{2}}$, $B = |T|^{2(p+\frac{1}{2})}$ and $C = \left(|T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}}\right)^{p+\frac{1}{2}}$.

Then for each $x \in \mathcal{X}$, by (2) we have

$$Ax = Bx = Cx. \tag{3}$$

We assume that $A^n y = B^n y = C^n y$ for each $y \in \mathcal{X}$. Since $Bx \in \mathcal{X}$,

$$\begin{aligned} A^{n+1}x &= A^n Ax = B^n Ax = B^n Bx \ (= B^{n+1}x) \\ &= C^n Bx = C^n Cx = C^{n+1}x. \end{aligned}$$

Hence by (3) we have $A^n x = B^n x = C^n x$ for every non-negative integer n and $x \in \mathcal{X}$. From the above, since we have $f(A)x = f(B)x = f(C)x$ for every polynomial f and $x \in \mathcal{X}$, similarly we obtain

$$|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}x = |T|^2 x = |T|^{\frac{1}{2}}U|T|U^*|T|^{\frac{1}{2}}x \in \mathcal{X}. \tag{4}$$

From (4) we have

$$\ker \tilde{T} \cap \mathcal{X} = \ker(\tilde{T})^* \cap \mathcal{X} = \ker |T| \cap \mathcal{X} = \{0\}.$$

Since $\mathcal{X}$ is a reducing subspace of $\tilde{T}$ and $|T|$,

$$\overline{\tilde{T}(\mathcal{X})} = \overline{(\tilde{T})^*(\mathcal{X})} = \mathcal{X} = \overline{|T|(\mathcal{X})} = \overline{|T|^{\frac{1}{2}}(\mathcal{X})}. \tag{5}$$

Since $\ker |T| = \{0\}$, from (4), we have

$$U^*|T|^{\frac{1}{2}}\tilde{T}x = |T|^{\frac{3}{2}}x = U|T|^{\frac{1}{2}}(\tilde{T})^*x. \tag{6}$$

From (5) and (6), it holds that

$$U^*|T|^{\frac{1}{2}}(\mathcal{X}) \subseteq \mathcal{X} \quad \text{and} \quad U|T|^{\frac{1}{2}}(\mathcal{X}) \subseteq \mathcal{X},$$

so that $U^*(\mathcal{X}) \subseteq \mathcal{X}$ and $U(\mathcal{X}) \subseteq \mathcal{X}$. Hence $\mathcal{X}$ is a reducing subspace of $|T|$ and U. From (5), we have

$$|T||T|^{\frac{1}{2}}x = U|T|U^*|T|^{\frac{1}{2}}x.$$

By (5), we have $|T|y = U|T|U^*y$ for $y \in \mathcal{X}$. Therefore, $\mathcal{X}$ is a reducing subspace of T such that T is normal on $\mathcal{X}$. This completes the proof. □

3. Proofs of theorems

In order to give proofs of the main results, we need the following theorem.

Theorem C ([6, Theorem 1.15]) *For an operator $T = U|T|$, T has a nontrivial invariant subspace if and only if so does $\tilde{T}$.*

Proof of Theorem 1. If T has no cyclic vectors, then T has a nontrivial invariant subspaces. Hence we may assume that T has a cyclic vector and is pure. By Lemma 3, all T, $\tilde{T}$ and $\hat{T}$ have dense ranges.

By Aluthge's result, $\tilde{T}$ is a semi-hyponormal operator. It follows from Lemma 4 that $\hat{T}$ is a pure hyponormal operator. Since $\sigma(\hat{T}) = \sigma(T)$, $\hat{T}$ is a hyponormal operator satisfying Theorem A. Hence $\hat{T}$ has a nontrivial invariant subspace. Therefore by Theorem C, T has a nontrivial invariant subspace. □

A similar argument implies Theorem 2.

References

[1] Aluthge A., *On p-hyponormal operator for* $0 < p < 1$. Integr. Equat. Oper. Th. **13** (1990), 307–315.

[2] Apostol C. and Clancey K., *Local resolvents of operators with one-dimensional self-commutator*. Proc. Amer. Math. Soc. **58** (1976), 158–162.

[3] Chō M., Huruya T. and Yamazaki T., *Mosaic and principal functions of log-hyponormal operators*. J. Math. Soc. Japan **55** (2003), 255–268.

[4] Clancey K., *Seminormal operators*. Lecture Notes in Math. Vol. 742, Springer-Verlag, New York, 1979.

[5] Duggal B.P., *p-hyponormal operators and invariant subspaces*. Acta Sci. Math. (Szeged) **64** (1998), 249–257.

[6] Jung I.B., Ko E. and Pearcy C., *Aluthge transforms of operators*. Integr. Equat. Oper. Th. **37** (2000), 437–448.

[7] Nakamura Y., *Principal functions and invariant subspaces of hyponormal operators*. Hokkaido Math. J. **12** (1983), 1–9.

[8] Putnam C.R., *Invariant subspaces of operators having nearly disconnected spectra*. Operator theory and functional analysis, Res. Notes in Math. **38**, Pitman, 1979.

[9] Xia D., *Spectral theory of hyponormal operators*. Birkhäuser Verlag, Basel (1983).

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