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Volterra integral equations: the singular case

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Abstract. Positive solutions are established for the Volterra integral equation $y(t) = \int_0^t k(t,s)\, f(s, y(s))\, ds$, $t \in [0, T]$. Our nonlinearity may be singular at $y = 0$.

Key words: Volterra integral equation, singular, lower type inequalities, positive solution.

1. Introduction

This paper discusses the singular Volterra equation

$$y(t) = \int_0^t k(t,s)\, f(s, y(s))\, ds \quad \text{for} \quad t \in [0,T], \quad T > 0 \ \text{fixed}. \qquad (1.1)$$

Our nonlinearity f may not be a Carathéodory function because of the singular behavior of the y variable i.e. f may be singular at $y = 0$. In the literature (see [3, 4] and the references therein) almost all results concern the case when f is a L^∞-Carathéodory function; to our knowledge only one paper [1] has discussed (1.1), in its full generality, when f is singular at $y = 0$. We also note that only a handful of papers (see [2, Chapter 1]) have discussed the initial value problem (which is a special case of (1.1)),

$$\begin{cases} y^{(n)} = \phi(t)\, f(t, y) \quad \text{for} \quad t \in [0,T] \\ y^{(i)}(0) = 0, \quad 0 \le i \le n-1, \quad n \ge 1 \end{cases}$$

when f is singular at $y = 0$. This paper presents new results for (1.1). In particular new "lower type inequalities" on solutions to (1.1) are presented. Also by exploiting the monotonicity of the kernel we are able to relax some of the assumptions in [1]. For example if we consider the initial value problem

$$\begin{cases} y'' = [y(t)]^{-a} + A[y(t)]^b \quad \text{for} \quad t \in [0,T] \\ y(0) = y'(0) = 0, \quad A > 0, \quad 0 \le b \le 1, \quad a > 0, \end{cases} \qquad (1.2)$$

then the results in [1] guarantee that (1.2) has a solution if $a \in \left(0, \frac{1}{2}\right)$ whereas the results in this paper guarantee that (1.2) has a solution if

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$a \in (0,1)$. Moreover in this paper we can consider (1.2) with $A = 0$ (see Remark 2.2). We were unable to discuss the case $A = 0$ in [1] since (2.7) in [1] is not satisfied.

The theory in Section 2 makes use of the following well known existence principle from the literature [4].

Theorem 1.1 *Suppose the following conditions hold:*

$$h \in C[0,T] \tag{1.3}$$

$$\begin{cases} F : [0,T] \times \mathbf{R} \to \mathbf{R} \ \ \textit{is a} \ \ L^\infty\textit{-Carathéodory function.} \\ \textit{By this we mean:} \\ \text{(i)} \ \ \textit{the map} \ \ y \mapsto F(t,y) \ \ \textit{is continuous for almost all} \\ \qquad t \ \ \textit{in} \ \ [0,T], \\ \text{(ii)} \ \ \textit{the map} \ \ t \mapsto F(t,y) \ \ \textit{is measurable for all} \ \ y \ \ \textit{in} \ \ \mathbf{R}, \\ \text{(iii)} \ \ \textit{for any} \ \ r > 0 \ \ \textit{there exists} \ \ \mu_r \in L^\infty[0,T] \ \ \textit{such that} \\ \qquad |y| \le r \ \ \textit{implies} \ |F(t,y)| \le \mu_r(t) \ \ \textit{for almost all} \ t \ \ \textit{in} \ \ [0,T] \end{cases} \tag{1.4}$$

$$k_t(s) = k(t,s) \in L^1[0,t] \quad \textit{for each} \ \ t \in [0,T] \tag{1.5}$$

and

$$\begin{cases} \textit{for any} \ \ t,\, t' \in [0,T], \displaystyle\int_0^{t^\star} |k_t(s) - k_{t'}(s)|\, ds \to 0 \\ \textit{as} \ \ t \to t'; \ \ \textit{here} \ \ t^\star = \min\{t,t'\}. \end{cases} \tag{1.6}$$

In addition suppose there is a constant $M > 0$, independent of λ, with $|y|_0 = \sup_{[0,T]} |y(t)| \neq M$ for any solution $y \in C[0,T]$ to

$$y(t) = h(t) + \lambda \int_0^t k(t,s)\, F(s,y(s))\, ds, \quad t \in [0,T],$$

for each $\lambda \in (0,1)$. Then

$$y(t) = h(t) + \int_0^t k(t,s)\, F(s,y(s))\, ds, \quad t \in [0,T]$$

has at least one solution in $C[0,T]$.

2. Singular problems

In this section we consider the singular integral equation

$$y(t) = \int_0^t k(t,s)\,[g(y(s)) + h(y(s))]\,ds \quad \text{for} \quad t \in [0,T]; \tag{2.1}$$

here $T > 0$ is fixed.

For our main result we will assume the following conditions are satisfied:

$$g > 0 \;\text{ is continuous and nonincreasing on }\; (0,\infty) \tag{2.2}$$

$$\begin{cases} h \geq 0 \;\text{ is continuous and nondecreasing} \\ \text{on } \;[0,\infty)\; \text{ with } \; h > 0 \; \text{ on } \; (0,\infty) \end{cases} \tag{2.3}$$

$$k_t(s) = k(t,s) \in L^1[0,t] \quad \text{for each} \quad t \in [0,T] \tag{2.4}$$

$$\begin{cases} \text{for any } \; t,\, t' \in [0,T], \; \displaystyle\int_0^{t^\star} |k_t(s) - k_{t'}(s)|\,ds \to 0 \\ \text{as } \; t \to t'; \; \text{ here } \; t^\star = \min\{t,t'\} \end{cases} \tag{2.5}$$

$$\text{for each } \; t \in [0,T], \; k(t,s) \geq 0 \quad \text{for a.e. } \; s \in [0,t] \tag{2.6}$$

$$\begin{cases} \text{for } \; t_1,\, t_2 \in (0,T) \; \text{ with } \; t_1 < t_2 \; \text{ we have} \\ k(t_1,s) \leq k(t_2,s) \quad \text{for a.e. } \; s \in [0,t_1] \end{cases} \tag{2.7}$$

$$\begin{cases} \exists\, a \in L^1[0,T], \; a > 0 \; \text{ on } \; (0,T] \; \text{ with } \; k(t,s) \leq a(s) \\ \text{for a.e. } \; s \in [0,t], \quad \text{for each } \; t \in [0,T] \end{cases} \tag{2.8}$$

$$\begin{cases} \displaystyle\int_0^T a(s)\,g(\alpha(s))\,ds < \infty \; \text{ where } \; \alpha(s) = G^{-1}\Big(\int_0^s k(s,x)\,dx\Big) \\ \text{for } \; s \in [0,T] \; \text{ and } \; G(z) = \dfrac{z}{g(z)} \; \text{ for } \; z > 0 \end{cases} \tag{2.9}$$

$$\begin{cases} \exists\, r \in C[0,T] \; \text{ with } \; \displaystyle\int_0^t |k(x,s) - k(t,s)|\,g(\alpha(s))\,ds \\ \qquad\qquad\qquad\qquad\qquad \leq |r(x) - r(t)| \\ \text{for } \; t,\, x \in [0,T] \; \text{ with } \; t < x \end{cases} \tag{2.10}$$

and

$$\int_\theta^\infty \frac{dx}{h(x)} = \infty \quad \text{for any} \quad \theta > 0. \tag{2.11}$$

Remark 2.1 Notice (2.2) guarantees that G is an increasing function.

Theorem 2.1 *Suppose* (2.2)–(2.11) *hold. Then* (2.1) *has a solution* $y \in C[0,T]$ *with* $y(t) \geq \alpha(t)$ *for* $t \in [0,T]$ (*here* α *is as in* (2.9)).

Proof. Let $N_0 = \{1, 2, \ldots\}$. We first show the nonsingular problem

$$y(t) = \frac{1}{m} + \int_0^t k(t,s)\,[g^\star(y(s)) + h(y(s))]\,ds, \;\; t \in [0,T] \qquad (2.12)^m$$

has a solution for each $m \in N_0$; here

$$g^\star(u) = \begin{cases} g(u), & u \geq \dfrac{1}{m} \\[2mm] g\left(\dfrac{1}{m}\right), & u \leq \dfrac{1}{m}. \end{cases}$$

Fix $m \in N_0$. To show $(2.12)^m$ has a solution we will use Theorem 1.1, so as a result we consider the family of problems

$$y(t) = \frac{1}{m} + \lambda \int_0^t k(t,s)\,[g^\star(y(s)) + h(y(s))]\,ds, \;\; t \in [0,T] \qquad (2.13)^m_\lambda$$

for $0 < \lambda < 1$. Let $y \in C[0,T]$ be any solution of $(2.13)^m_\lambda$. Then $y(t) \geq \frac{1}{m}$ for $t \in [0,T]$. Also for $t \in [0,T]$ we have from (2.8) that

$$y(t) \leq \frac{1}{m} + \int_0^t a(s)\left[g\left(\frac{1}{m}\right) + h(y(s))\right] ds,$$

so

$$y(t) \leq K_m + \int_0^t a(s)\,h(y(s))\,ds \quad \text{for} \quad t \in [0,T]; \tag{2.14}$$

here $K_m = \frac{1}{m} + g\left(\frac{1}{m}\right) \int_0^T a(s)\,ds$. Let

$$u(t) = K_m + \int_0^t a(s)\,h(y(s))\,ds \quad \text{for} \quad t \in [0,T].$$

Then $u'(t) = a(t)\,h(y(t)) \le a(t)\,h(u(t))$ for $t \in (0,T)$, so as a result

$$\int_{K_m}^{u(t)} \frac{du}{h(u)} \le \int_0^T a(x)\,dx \quad \text{for} \quad t \in [0,T]. \tag{2.15}$$

Let

$$J_m(z) = \int_{K_m}^{z} \frac{du}{h(u)} \quad \text{for} \quad z \ge K_m,$$

so (2.14) and (2.15) imply

$$\frac{1}{m} \le y(t) \le u(t) \le J_m^{-1}\left(\int_0^T a(x)\,dx\right) \quad \text{for} \quad t \in [0,T].$$

Theorem 1.1 guarantees that $(2.12)^m$ has a solution $y_m \in C[0,T]$ with $y_m(t) \ge \frac{1}{m}$ for $t \in [0,T]$, and of course y_m is a solution of

$$y(t) = \frac{1}{m} + \int_0^t k(t,s)\,[g(y(s)) + h(y(s))]\,ds \ \text{ for } \ t \in [0,T]. \tag{2.16}$$

We will now obtain a solution to (2.1) by means of the Arzela-Ascoli Theorem, as a limit of solutions of $(2.12)^m$. To this end we will show

$$\{y_m\}_{m \in N_0} \ \text{ is a bounded, equicontinuous family on } \ [0,T]. \tag{2.17}$$

However before we prove (2.17) we will show

$$y_m(t) \ge G^{-1}\Big(\int_0^t k(t,s)\,ds\Big) \equiv \alpha(t) \quad \text{for} \quad t \in [0,T], \tag{2.18}$$

for each $m \in N_0$. Fix $m \in N_0$ and t, $x \in [0,T]$ with $t < x$. Then (2.7) implies

$$\begin{aligned} y_m(x) - y_m(t) &= \int_0^t [k(x,s) - k(t,s)]\,[g(y_m(s)) + h(y_m(s))]\,ds \\ &\quad + \int_t^x k(x,s)\,[g(y_m(s)) + h(y_m(s))]\,ds \\ &\ge 0, \end{aligned}$$

so y_m is nondecreasing on $(0,T)$. As a result for $t \in [0,T]$ we have

$$y_m(t) \ge \int_0^t k(t,s)\,g(y_m(s))\,ds \ge g(y_m(t)) \int_0^t k(t,s)\,ds.$$

That is

$$G(y_m(t)) = \frac{y_m(t)}{g(y_m(t))} \geq \int_0^t k(t,s)\,ds \quad \text{for} \quad t \in [0,T],$$

so (2.18) holds; note G is an increasing function since g is nonincreasing. Next we show $\{y_m\}_{m\in N_0}$ is a bounded family on $[0,T]$. Fix $m \in N_0$. For $t \in [0,T]$ we have from (2.18) that

$$\begin{aligned} y_m(t) &= \frac{1}{m} + \int_0^t k(t,s)\,[g(y_m(s)) + h(y_m(s))]\,ds \\ &\leq 1 + \int_0^t a(s)\,[g(\alpha(s)) + h(y_m(s))]\,ds, \end{aligned}$$

so

$$y_m(t) \leq K + \int_0^t a(s)\,h(y_m(s))\,ds \quad \text{for} \quad t \in [0,T];$$

here $K = 1 + \int_0^T a(s)\,g(\alpha(s))\,ds$. Let

$$w(t) = K + \int_0^t a(s)\,h(y_m(s))\,ds \quad \text{for} \quad t \in [0,T].$$

Notice $w'(t) = a(t)\,h(y_m(t)) \leq a(t)\,h(w(t))$ for $t \in (0,T)$, so

$$0 \leq y_m(t) \leq w(t) \leq J^{-1}\left(\int_0^T a(x)\,dx\right) \equiv M \quad \text{for} \quad t \in [0,T];$$

here

$$J(z) = \int_K^z \frac{du}{h(u)} \quad \text{for} \quad z \geq K.$$

Thus $|y_m|_0 = \sup_{t\in[0,T]} |y_m(t)| \leq M$ for $m \in N_0$. To show the second part of (2.17) fix $m \in N_0$, and note for t, $x \in [0,T]$ with $t < x$ that

$$\begin{aligned} 0 \leq y_m(x) - y_m(t) &= \int_0^t [k(x,s) - k(t,s)]\,[g(y_m(s)) + h(y_m(s))]\,ds \\ &\quad + \int_t^x k(x,s)\,[g(y_m(s)) + h(y_m(s))]\,ds \\ &\leq \int_0^t [k(x,s) - k(t,s)]\,g(\alpha(s))\,ds \\ &\quad + h(M)\int_0^t [k(x,s) - k(t,s)]\,ds \end{aligned}$$

$$
\begin{aligned}
&+\int_t^x a(s)\,[g(\alpha(s))+h(M)]\,ds\\
&\le |r(x)-r(t)|+h(M)\int_0^t [k(x,s)-k(t,s)]\,ds\\
&+\int_t^x a(s)\,[g(\alpha(s))+h(M)]\,ds;
\end{aligned}
$$

here we used (2.10). Now this together with (2.5), (2.9) and (2.10) implies $\{y_m\}_{m\in N_0}$ is a equicontinuous family on $[0,T]$.

The Arzela-Ascoli theorem guarantees the existence of a subsequence N of N_0 and a function $y\in C[0,T]$ with y_m converging uniformly on $[0,T]$ to y as $m\to\infty$ through N. In addition $\alpha(t)\le y(t)\le M$ for $t\in[0,T]$. Next fix $t\in[0,T]$. Then

$$
y_m(t)=\frac{1}{m}+\int_0^t k(t,s)\,[g(y_m(s))+h(y_m(s))]\,ds.
$$

Let $m\to\infty$ through N, and use the Lebesgue dominated convergence theorem with (2.9), to obtain

$$
y(t)=\int_0^t k(t,s)\,[g(y(s))+h(y(s))]\,ds.
$$

We can do this argument for each $t\in[0,T]$. □

Remark 2.2 If $h\equiv 0$ in (2.1), then the result in Theorem 2.1 is again true with (2.3) and (2.11) removed.

Remark 2.3 Suppose there exists p, $1\le p\le\infty$ and q, $\frac{1}{p}+\frac{1}{q}=1$ with

$$
\int_0^t g^q(\alpha(s))\,ds<\infty \tag{2.19}
$$

and

$$
\begin{cases}
\text{for any } \ t,\,t'\in[0,T], \ \displaystyle\int_0^{t^\star} |k_t(s)-k_{t'}(s)|^p\,ds\to 0\\
\text{as } \ t\to t'; \ \text{ here } \ t^\star=\min\{t,t'\}
\end{cases} \tag{2.20}
$$

Then (2.10) (and of course (2.5)) is not needed in Theorem 2.1.

To see this notice (2.10) is needed to show (2.17) in Theorem 2.1. However this is automatically true in this case since if t, $x\in[0,T]$ with $t<x$

then

$$y_m(x) - y_m(t) \le \left(\int_0^t |k_x(s) - k_t(s)|^p \, ds \right)^{\frac{1}{p}} \left(\int_0^T [g(\alpha(s)) + h(M)]^q \, ds \right)^{\frac{1}{q}} + \int_t^x a(s) \, [g(\alpha(s)) + h(M)] \, ds.$$

Remark 2.4 If we replace (2.3), (2.8), (2.9) and (2.11) in Theorem 2.1 by

$$\begin{cases} h \ge 0 \ \text{ is continuous on } \ [0,\infty) \ \text{ and} \\ \dfrac{h}{g} \ \text{ is nondecreasing on } \ [0,\infty) \end{cases} \tag{2.21}$$

$$\sup_{t\in[0,T]} \int_0^t k(t,s) \, g(\alpha(s)) \, ds < \infty \tag{2.22}$$

and

$$\begin{cases} \exists \ \text{constants } \ a, \ b \ \text{ and } \ M \ \text{(which may depend on} \\ a \ \text{and} \ b) \ \text{ such that } \ 0 < z \le a + b \left\{ 1 + \dfrac{h(z)}{g(z)} \right\} \\ \text{implies } \ z \le M, \end{cases} \tag{2.23}$$

then the result in Theorem 2.1 is again true.

To see this notice if $y \in C[0,T]$ is any solution of $(2.13)_\lambda^m$ then

$$|y(t)| \le 1 + \int_0^t k(t,s) \, g(y(s)) \left\{ 1 + \frac{h(y(s))}{g(y(s))} \right\} ds \quad \text{for } \ t \in [0,T],$$

so

$$|y|_0 \le 1 + \left(g\left(\frac{1}{m}\right) \sup_{t\in[0,T]} \int_0^t k(t,s) \, ds \right) \left\{ 1 + \frac{h(|y|_0)}{g(|y|_0)} \right\}.$$

Then there exists a constant M_m (independent of any solution y to $(2.13)_\lambda^m$) with $|y|_0 \le M_m$. Theorem 1.1 guarantees that $(2.12)^m$ has a solution y_m and it is easy to check that $y_m(t) \ge \alpha(t)$ for $t \in [0,T]$ and

$$|y_m|_0 \le 1 + \left\{ 1 + \frac{h(|y_m|_0)}{g(|y_m|_0)} \right\} \sup_{t\in[0,T]} \int_0^t k(t,s) \, g(\alpha(s)) \, ds.$$

Then there exists a constant M (independent of m) with $|y_m|_0 \leq M$. Essentially the same reasoning as in Theorem 2.1 establishes the result.

Example 2.1 Consider

$$y(t) = \int_0^t k(t,s)\left\{[y(s)]^{-a} + A\,[y(s)]^b\right\} ds \quad \text{for } \; t \in [0,T] \tag{2.24}$$

with $A \geq 0$, $0 \leq b \leq 1$ and $a > 0$. Assume (2.4)–(2.8) hold and in addition suppose the following conditions are satisfied:

$$\int_0^T a(s)\left[\int_0^s k(s,x)\,dx\right]^{-\frac{a}{a+1}} ds < \infty \tag{2.25}$$

and

$$\begin{cases} \exists\, r \in C[0,T] \;\; \text{with} \\ \qquad \displaystyle\int_0^t |k(x,s) - k(t,s)| \left[\int_0^s k(s,u)\,du\right]^{-\frac{a}{a+1}} ds \\ \leq |r(x) - r(t)| \;\; \text{for} \;\; t,\, x \in [0,T] \;\; \text{with} \;\; t < x. \end{cases} \tag{2.26}$$

Then (2.24) has a solution $y \in C[0,T]$ with

$$y(t) \geq \left[\int_0^t k(t,x)\,dx\right]^{\frac{1}{a+1}} \quad \text{for} \;\; t \in [0,T].$$

To see this apply Theorem 2.1 with

$$g(y) = y^{-a}, \;\; h(y) = A\,y^b \;\; \text{and note} \;\; G^{-1}(z) = z^{\frac{1}{a+1}};$$

note if $A = 0$ we can apply Remark 2.2. Clearly (2.11) holds since $0 \leq b \leq 1$.

Example 2.2 Consider

$$y(t) = \int_0^t (t-s)\,\phi(s)\left\{[y(s)]^{-a} + A\,[y(s)]^b\right\} ds \quad \text{for} \;\; t \in [0,T] \tag{2.27}$$

with $A \geq 0$, $0 \leq b \leq 1$ and $a > 0$. Assume the following conditions are satisfied:

$$\phi \in C(0,T] \cap L^1[0,T] \;\; \text{with} \;\; \phi > 0 \;\; \text{on} \;\; (0,T] \tag{2.28}$$

and

$$\int_0^T \phi(s) \left[\int_0^s (s-x)\,\phi(x)\,dx\right]^{-\frac{a}{a+1}} ds < \infty. \tag{2.29}$$

Then (2.27) has a solution $y \in C[0,T]$ with

$$y(t) \geq \left[\int_0^t (t-x)\,\phi(x)\,dx\right]^{\frac{1}{a+1}} \quad \text{for} \quad t \in [0,T].$$

To see this we apply Example 2.1 with $k(t,s) = (t-s)\,\phi(s)$ for $0 \leq s \leq t$. Clearly (2.4)–(2.7) hold and in addition (2.8) is satisfied if we choose $a(s) = (T-s)\,\phi(s)$. Also (2.25) is immediate since

$$\begin{aligned}\int_0^T a(s) &\left[\int_0^s k(s,x)\,dx\right]^{-\frac{a}{a+1}} ds \\ &= \int_0^T (T-s)\,\phi(s) \left[\int_0^s (s-x)\,\phi(x)\,dx\right]^{-\frac{a}{a+1}} ds,\end{aligned}$$

which is finite from (2.29). Finally (2.26) holds with a linear function $r(z)$ of z since if $t,\ x \in [0,T]$ with $t < x$ then

$$\begin{aligned}\int_0^t |k(x,s)-k(t,s)| &\left[\int_0^s k(s,u)\,du\right]^{-\frac{a}{a+1}} ds \\ &= (x-t)\int_0^t \phi(s) \left[\int_0^s (s-u)\,\phi(u)\,du\right]^{-\frac{a}{a+1}} ds.\end{aligned}$$

Remark 2.5 In Example 2.2 if $\phi = 1$ then (2.29) (and automatically (2.28)) is satisfied if $0 < a < 1$ since

$$\int_0^T \phi(s) \left[\int_0^s (s-x)\,\phi(x)\,dx\right]^{-\frac{a}{a+1}} ds = \int_0^T \left(\frac{s^2}{2}\right)^{-\frac{a}{a+1}} ds.$$

Remark 2.6 The results in this paper can easily be extended to the Volterra equation $y(t) = h(t) + \int_0^t k(t,s)\,f(s,y(s))\,ds$ for $t \in [0,T]$.

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