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On the nilpotent complex of simple groups of Lie type

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Abstract. In this paper we describe the connected components of $Nl(G)$, the partially ordered set of nilpotent subgroups of a finite simple group of Lie type.

Key words: prime graphs, simplicial complexes, nilpotent subgroups, simple groups.

1. Introduction

The prime graph $\Gamma(G)$ of a finite group G has been studied by various authors. Its connected components have been described in [8], [9], [11] and [16]. Its diameter has been calculated in [12] and the groups in which the prime graph is a tree have been investigated in [13].

More recently some generalisations of the prime graph have been introduced by Abe and Iiyori, in [1], as follows. In the prime graph $\Gamma(G)$ vertices p and q are defined to be joined when there exists an element x of G whose order is pq. This condition can be interpreted by the property that G contains a cyclic subgroup of order pq. This suggests to define the Ξ-graph of a group G, $\Gamma_{\Xi}(G)$, as follows: the vertices are the primes dividing the order of G and two vertices p, q are joined if G contains a Ξ-subgroup of order pq (here Ξ is a group theoretical property). They define the cyclic graph, $\Gamma_{\mathrm{cycl}}(G)=\Gamma(G)$, the abelian and nilpotent graph, denoted respectively by $\Gamma_{\mathrm{abel}}(G)$ and $\Gamma_{\mathrm{nilp}}(G)$. They observe that $\Gamma_{\mathrm{cycl}}(G)=\Gamma_{\mathrm{abel}}(G)=\Gamma_{\mathrm{nilp}}(G)$ and investigate the soluble graph $\Gamma_{\mathrm{sol}}(G)$.

However these graphs defined over a group G cannot be equipped with a G-structure. Therefore, instead of considering just the order of Ξ-subgroups, we can investigate the poset of all non trivial Ξ-subgroups of G. Then G acts by conjugation over these Ξ-posets. Moreover there is a covariant functor from the category of finite posets to the category of finite simplicial complexes. This allows to associate combinatorial or topological concepts and terminology to the posets.

This procedure has been applied to different classes of Ξ-subgroups of G.

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The most interesting instance is the so-called Brown complex, introduced by K. Brown in connection to cohomological questions; it is the complex associated to the poset of non trivial p-subgroups of G. By now there is a rich literature on this subject; see for instance the seminal paper by Quillen [14] or the more recent paper by Aschbacher and Smith [2].

As for the graphs defined in [1], the corresponding posets have been defined in [10]. In fact in that paper the simplicial complexes related to the following posets have been studied:

$$K(G) = \{\text{non trivial cyclic subgroups of } G\},$$
$$Ab(G) = \{\text{non trivial abelian subgroups of } G\},$$
$$Nl(G) = \{\text{non trivial nilpotent subgroups of } G\}.$$

In Proposition 1.2 of [10] it is proved that they are all G-homotopy equivalent. In that paper it is also proved the following:

Proposition 1 (Proposition 2.1 of [10]) *$Nl(G)$ is not connected if and only if $\Gamma(G)$ is not connected.*

However the connected components of $\Gamma(G)$ are not in correspondance with those of $Nl(G)$. In the paper [10], we determine the connected components of $Nl(G)$ in the case in which G is a soluble group. In this paper we determine the connected components of $Nl(G)$, where G is a simple non abelian group of Lie type. We consider separately the case in which G admits a partition.

Proposition 2 *Let G be a simple non abelian group of Lie type, defined over the field with $q = p^f$ elements, except for $A_1(q)$, ${}^2B_2(q)$ and $A_2(4)$. Let U be a p-Sylow subgroup of G. Then the connected components of $Nl(G)$ are $[U]$ and those described in Tables* 1a, 1b.

If $G \cong A_1(q)$, ${}^2B_2(q)$ or $A_2(4)$, then G admits a partition and the connected components are $\{Nl(R) : R$ is a subgroup of the partition$\}$.

Moreover we describe the action of G over $Nl(G)$:

Corollary 2 *Let G be a finite simple group of Lie type with $t(G) \geq 2$. Then the number of G-orbits of $Nl(G)$ is exactly $t(G)$, the number of connected components of $\Gamma(G)$. Moreover, if G has not a partition, then there is only one G-orbit, $[U]$, fixed by G.*

2. Notation

We first briefly recall how to construct a simplicial complex from a poset. Let X be a finite partially ordered set. To X it is associated an abstract simplicial complex $|X|$, the simplicial realisation of X, by taking the elements of X as vertices (0-simplices) and, as n-simplices the chains of $(n+1)$ elements of X, for $n \geq 0$. Furthermore, any map of posets $f : (X_1, \leq_1) \longrightarrow (X_2, \leq_2)$ yields a simplicial map between $|X_1|$ and $|X_2|$. Thus, if a finite group acts on a poset X, that is X is a G-poset, then G will act on $|X|$ simplicially.

For the notation concerning finite groups of Lie type, we refer to [3]. Let G be a finite group of Lie type defined on a field K with $q = p^f$ elements. We denote by Φ a *system of roots* of the corresponding Lie algebra. We also denote by Π a *fundamental system* for Φ, and an element of Π will be called a *simple root*. If G has Lie rank l, then we denote by $r_1, \ldots, r_l$ the simple roots related to G, following the numbering of roots, using Dynkin diagrams, as in [6].

We denote by U the unipotent subgroup of G, generated by the positive root subgroups, H the diagonal subgroup of G, and by $W(G)$ the Weyl group of G. We also write $N_{W(G)}$ for the subgroup of G generated by H and n_r, with $r \in \Phi$ (see [3], page 101). Then we have $H \triangleleft N_{W(G)}$ and $N_{W(G)}/H \cong W(G)$, moreover if $1 \neq H$, then $N_{W(G)} = N_G(H)$.

We can consider the algebraic closure $\widetilde{K}$ of K and $\widetilde{G}$ a connected reductive group over $\widetilde{K}$, with a Frobenius map $F : \widetilde{G} \longrightarrow \widetilde{G}$, such that the group G is exactly $\widetilde{G}^F$, the F-fixed points subgroup of $\widetilde{G}$.

Then H, the diagonal subgroup of G, is exactly $\widetilde{H}^F$, where $\widetilde{H}$ is a maximally-split F-stable maximal torus of $\widetilde{G}$, and $B = UH = \widetilde{B}^F$, where $\widetilde{B}$ is a Borel subgroup of $\widetilde{G}$.

A *maximal torus* T of G is $T = \widetilde{T}^F$, where $\widetilde{T}$ is an F-stable maximal torus of $\widetilde{G}$.

We denote the connected components of the prime graph $\Gamma(G)$ by $\{\pi_i(G),\ i = 1, 2, \ldots, t(G)\}$ where $t(G)$ is the number of connected components of $\Gamma(G)$ and, if the order of G is even, we denote the component containing 2 by $\pi_1(G)$.

We remember that non-connectedness of $\Gamma(G)$ has relations also with the existence of isolated subgroups of G. A proper subgroup M of G is *isolated* if $M \cap M^g = \{1\}$ or M for every $g \in G$ and $C_G(m) \leq M$ for all

$m \in M \setminus \{1\}$. It was proved in [16] that G has a nilpotent isolated Hall π-subgroup whenever G is non-soluble and $\pi = \pi_i(G)$, $i > 1$.

We also recall some notations from [10] and refer to that paper for the construction of topological spaces from posets.

3. Simple groups of Lie type

The aim of this section is to prove Proposition 2.

We define $Nl(G)$ to be the set of all non-trivial nilpotent subgroups of a group G. We define the following relation in $Nl(G) : R \sim S$ if, for some $n \in \mathbb{N}$, there exist $R_0, R_1, \ldots, R_n$ such that $R_0 = R$, $R_n = S$ and either $R_i \leq R_{i+1}$ or $R_i \geq R_{i+1}$. This is an equivalence relation. A *connected component* of $Nl(G)$ is an equivalence class for the relation $\sim$ and we denote by $[R]$ the connected component of $Nl(G)$ containing R. We observe that if all the Sylow subgroups of G are in the same connected component, then $Nl(G)$ itself is connected.

Lemma 1 *If $G = A_1 \times A_2$, with $A_1 \neq 1 \neq A_2$, then $Nl(G)$ is connected.*

Proof. If $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, with p and q primes not necessarily different, then $P = P_1 \times P_2$ and $Q = Q_1 \times Q_2$, with $P_i, Q_i \leq A_i$, $i = 1, 2$. If $P_1 \neq 1$ and $Q_2 \neq 1$ we have $P \sim P_1 \sim P_1 \times Q_2 \sim Q_2 \sim Q$. Similarly we have $P \sim Q$, if $P_2 \neq 1$ and $Q_1 \neq 1$. If $P_1 = 1 = Q_1$, then for any $R \in Nl(A_1)$, we have: $P \sim P \times R \sim R \sim Q \times R \sim Q$. □

Proposition 3 *If G is a finite group of Lie type and $B = UH$ is a Borel subgroup of G, then $Nl(B)$ is connected, except for $G = A_1(q), A_2(4)$, ${}^2B_2(q)$.*

Proof. If $q = 2$, then $H = 1$ and $B = U$ is a nilpotent group so $Nl(B)$ is contractible, by Remark 2.2 of [10]. So we can suppose that $q \neq 2$, and both $Nl(U)$ and $Nl(H)$ are contractible, because they are both nilpotent. Then by Theorem 2.6 1) of [10], we have that $Nl(B)$ is not connected if and only if B is a Frobenius group. We choose now some elements $1 \neq u \in U$, $1 \neq h \in H$. Here t and λ are elements of the field K on which G is defined.

If G is a Chevalley group and $rk(G) \geq 3$, there exist two simple roots r and s such that $1 \neq x_r(t) = u \in U$ and $1 \neq h_s(\lambda) = h \in H$ with $[x_r(t), h_s(\lambda)] = 1$. We suppose now that $rk(G) = 2$. If $G = A_2(q)$, we call $h = h_{r_1}(\lambda)h_{r_2}(\lambda^{-1}) \in H$ and $u = x_{r_1+r_2}(t) \in U$, if $q \neq 4$, then $h \neq 1 \neq u$; if $G = B_2(q)$ we call $1 \neq h = h_{r_2}(\lambda) \in H$ and $1 \neq u = x_{r_1+r_2}(t) \in U$; if

$G = G_2(q)$ we call $1 \neq h = h_{r_1}(\lambda) \in H$ and $1 \neq u = x_{3r_1+2r_2}(t) \in U$. Then we have $[u, h] = 1$.

We suppose now that G is a twisted group. Let $f \colon K \longrightarrow K$ be the field automorphism, used to construct the twisted group (see [3], chapter 13), and we put $\bar{t} = f(t)$ for any $t \in K$.

If $G = {}^2A_l(q^2)$, $l \geq 5$ and l odd, let $r_1, r_2, \ldots, r_{2k-1}$ be a fundamental system for A_l. Then we have $1 \neq u = x_{r_1}(t)x_{r_{2k-1}}(\bar{t}) \in U$ and $1 \neq h = h_{r_k}(\lambda) \in H$, if $\overline{\lambda} = \lambda$.

If l is even, and $r_1, r_2, \ldots, r_{2k}$ are the simple roots of A_l, we have that $1 \neq u = x_{r_1}(t)x_{r_{2k}}(\bar{t}) \in U$ and $1 \neq h = h_{r_k}(\lambda)h_{r_{k+1}}(\overline{\lambda}) \in H$.

If $l = 2, 3, 4$, let r be the root that is the sum of all the simple roots of A_l. Then we have $1 \neq u = x_r(t) \in U$, with $t = \bar{t}$ for $l = 3, 4$ and $t + \bar{t} = 0$ for $l = 2$. We also have $1 \neq h = h_{r_1}(\lambda)h_{r_2}(\overline{\lambda}) \in H$ with λ a $(q+1)$-th root of unity for $l = 2$; $1 \neq h = h_{r_2}(\lambda) \in H$, with $\overline{\lambda} = \lambda$ for $l = 3$, and $1 \neq h = h_{r_2}(\lambda)h_{r_3}(\overline{\lambda}) \in H$ for $l = 4$.

Let G be ${}^2D_l(q^2)$ with $l \geq 4$. Let $r_1, r_2, \ldots, r_{l-2}, r_{l-1}, r_l$ be a fundamental system for D_l, then $1 \neq u = x_{r_{l-1}}(t)x_{r_l}(\bar{t}) \in U$ and $1 \neq h = h_{r_1}(\lambda) \in H$, where $\lambda = \overline{\lambda}$.

Let G be ${}^3D_4(q^3)$ and r be the root that is the sum of all the simple roots of D_4, then $1 \neq u = x_r(t) \in U$ with $t = \bar{t}$ and $1 \neq h = h_{r_1}(\lambda)h_{r_3}(\overline{\lambda})h_{r_4}(\overline{\overline{\lambda}}) \in H$.

Let G be ${}^2E_6(q^2)$ and let $r_1, r_2, \ldots, r_6$ be a fundamental system for E_6. Then $1 \neq u = x_{r_1}(t)x_{r_6}(\bar{t}) \in U$ and $1 \neq h = h_{r_2}(\lambda) \in H$, $\lambda = \overline{\lambda}$.

Let G be ${}^2F_4(q^2)$ and $r = r_1 + 2r_2 + 2r_3 + r_4$, then $1 \neq u = x_r(1) \in U$ and $1 \neq h = h_{r_1}(\lambda)h_{r_4}(\overline{\lambda}) \in H$, with r_4 a short root.

Let G be ${}^2G_2(q^2)$ and χ the character defined on the root system Φ of $G = {}^2G_2$ by $\chi(a) = \chi(b) = -1$. By paragraph 6.4 of [15], we know that $1 \neq h_0 = h(\chi) \in H$ and by Proposition 13.6.4 [3], $1 \neq u = x_{a+b}(\bar{t})x_{3a+b}(t) \in U$.

So in any case that we have considered, B cannot be a Frobenius group because $[u, h] = 1$, for the $u \in U$ and $h \in H$ described in any singular case. □

Lemma 2 *Let G be a Chevalley group and $W(G)$ be its Weyl group. Then $Nl(W(G))$ is connected, except for $G = A_l$, with l prime or $l+1$ prime, and $G = D_l$, with l an odd prime.*

In any case, except $G = A_2$, the 2-subgroups of $W(G)$ lie in the same connected component of $Nl(W(G))$ and, except $G = A_2, A_3$, also the 3-

subgroups of $W(G)$ lie in this same connected component.

Proof. The Weyl groups of the Chevalley groups can be found in [3]. We recall them here.

$W(A_l) \cong S_{l+1}$ of order $(l+1)!$;
$W(B_l) \cong W(C_l) \cong Z_2^l S_l$ of order $2^l \cdot l!$;
$W(D_l) \cong Z_2^{l-1} S_l$ of order $2^{l-1} \cdot l!$;
$W(E_6) \cong O_6^-(2).2 \cong {}^2A_3(2).2 \cong C_2(3).2$ of order $2^7 \cdot 3^4 \cdot 5$;
$W(E_7) \cong Z_2 \times Sp_6(2)$ of order $2^{10} \cdot 3^4 \cdot 5 \cdot 7$;
$W(E_8) \cong 2.O_8^+(2).2$ of order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$;
$W(F_4) \cong W(D_4) : S_3$ of order $2^7 \cdot 3^2$;
$W(G_2) \cong D_{12}$ of order $2^2 \cdot 3$.

If l or $l-1$ are not prime numbers, then $Nl(S_l)$ is connected by Proposition 1, since $\Gamma(S_l)$ is connected (see [16]). In any case, except $l=3$ there is a connected component of $Nl(S_l)$, containing all the 2-subgroups of S_l, and if $l \geq 5$ also the 3-subgroups of S_l lie in this connected component. So the lemma is proved for $G = A_l$, $l \geq 4$.

$Nl(W(E_6))$ is connected since $\Gamma(W(E_6))$ is connecetd (see [5]). We conclude by Proposition 1.

$Nl(W(E_7))$ is connected by Lemma 1.

$Nl(W(F_4))$ is connected because $\Gamma(W(F_4))$ is connected, since there is an element of order 6 in $W(F_4)$. We conclude by Proposition 1.

$Nl(W(B_l))$, $Nl(W(E_8))$, $Nl(W(G_2))$ are contractible, and therefore connected, by Remark 2.2 of [10], because they have a non-trivial centre.

$Nl(W(D_l))$ is connected if l is not a prime. In fact $\Gamma(S_l)$ is connected if l, $l-1$ are not prime numbers (see [16]) and therefore also $\Gamma(W(D_l))$ because $\pi(W(D_l)) \subseteq \pi(S_l)$. If $l-1$ is a prime, let x be an element of order $l-1$ of $W = W(D_l)$. Then it can be easily checked that 2 divides $|C_W(x)|$ and therefore $\Gamma(W)$ is connected. We conclude again by Proposition 1.

If l is a prime, $l \geq 5$ there is a connected component of $Nl(S_l)$, containing all the 2-subgroups of S_l and also the 3-subgroups of S_l. The same is true for W, since $O = O_2(W) \neq 1$, and for any 2-Sylow subgroup P of W, we have $P \sim O$. □

Proposition 4 *If G is a finite group of Lie Type, over a finite field of characteristic p, then the p-groups of G are all in the same connected component of $Nl(G)$, except for $G = A_1(q)$ and ${}^2B_2(q)$.*

Proof. We first suppose that $H \neq \{1\}$ and $G \neq A_2(4)$, $A_1(q)$ or ${}^2B_2(q)$. As U is a p-Sylow subgroup of G, it is enough to prove that all its conjugates lie in the same connected component of $Nl(G)$. Any $g \in G$ is uniquely expressible in the form $g = bnu$, with $b \in B$, $n \in N_G(H)$ and $u \in U$. We observe that U is a normal subgroup of B and that $H \sim U$ in $Nl(G)$, by Proposition 3. Then

$$U = U^u \sim H^u = H^{nu} \sim U^{nu} = U^g.$$

A direct computation shows that the 2-subgroups of $A_2(4)$ are all in the same connected component of $Nl(G)$.

If $H = \{1\}$, then G is a Chevalley group and $q = 2$. Then $N = N_{W(G)}$ is isomorphic to $W(G)$, the Weyl group of G. If $r_1, \ldots, r_l$ are the simple roots of G, we put $\overline{n} = n_{w_{r_1}}$ the element of N, corresponding to the involution w_{r_1}, and $u = x_{r_3}(t)$, if $l \geq 3$ or $u = x_{r_1+r_2}(t)$ if $G = B_2(q)$ or $u = x_{3r_1+2r_2}(t)$ if $G = G_2(q)$, where t is the only non-zero element of the field with two elements. Then $1 \neq u \in U$ and $\overline{n}^2 = 1$ and $u^{\overline{n}} = u$. By Lemma 2, we know that all the 2-subgroups of $N \cong W(G)$ are in the same connected component of $Nl(G)$. So for any $n \in N$ we have $\langle \overline{n} \rangle^n \sim \langle \overline{n} \rangle \sim \langle \overline{n}, u \rangle \sim U$.

We can therefore conclude because any $g \in G$ can be written as $g = u_1 n u_2$ with $u_i \in U$, $i = 1, 2$ and $n \in N$ and then

$$U = U^{u_2} \sim \langle \overline{n} \rangle^{u_2} \sim (\langle \overline{n} \rangle^n)^{u_2} \sim U^{nu_2} = U^{u_1 n u_2} = U^g.$$ □

Corollary 1 *If G is a Chevalley group of Lie rank $l > 2$ over the field with two elements, then a maximal torus $\overline{H}$ of order $q + 1 = 3$ lies in the connected component of the 2-subgroups in $Nl(G)$.*

Proof. By Lemma 2, the 2-subgroups and the 3-subgroups of $W(G)$ lie in the same connected component of $Nl(W(G))$ and $N = N_{W(G)} \cong W(G)$, except $G = A_3(2)$, $A_2(2)$.

Therefore there must exist an element g in G, an element $x \in N$ of order 3 and a 3-Sylow subgroup Q of G such that

$$\overline{H} \sim Q \sim \langle x \rangle^g \sim \langle \overline{n} \rangle^g \sim U^g \sim U.$$

Since $A_2(2) \cong A_1(7)$, we only have to consider the group $G = A_3(2) \cong A_8$. The statement is again true, as it can be easily checked. □

Remark The groups $A_1(q) \cong PSL(2, q)$, ${}^2B_2(q) \cong Sz(q)$ and $A_2(4) \cong PSL(3, 4)$ are groups with a partition. Therefore the connected components are exactly $Nl(R)$, where R is one of the subgroups of G, forming the partition (see for example Proposition 2.9 of [10]). We observe that if $G \cong PSL(3,4)$, then the subgroups forming the partition are exactly the Sylow subgroups of G.

In all the remaining cases, we have that $\widetilde{H}$ lie in $[U]$, where $\widetilde{H} = \overline{H}$ if $q = 2$ and $\widetilde{H} = H$ otherwise.

Proposition 5 *Let G be a finite simple group of Lie type. Then*

a) *If R is a nilpotent subgroup of G, then either $R \sim U$ or there exists T, a maximal torus of G, such that $R \sim T$;*

b) *if T is a maximal torus of G that is isolated, then $[T] = Nl(T)$; moreover, if G is not of the type $A_1(q)$, $A_2(4)$ or ${}^2B_2(q)$,*

c) *any maximal torus T which is not isolated lie in $[U]$;*

d) *the connected components of $Nl(G)$ are $[U]$ and $Nl(T)$, for T isolated maximal tori.*

Proof. a) Let g be an element of R; if p divides the order of g, then there exists $n \in \mathbb{Z}$ such that g^n is a p-element and so $g^n \in U^x$ for some $x \in G$. Then by Proposition 4 we have that $\langle g \rangle \sim \langle g^n \rangle \leq U^x \sim U$.

If p does not divide the order of g, then g is a semisimple element and so it is contained in a maximal torus T. So $R \sim \langle g \rangle \sim T$.

b) We suppose that $R \sim T$, for some nilpotent subgroup R of G with $R \not\leq T$. Then there exists a nilpotent subgroup $\widetilde{R}$ such that $\widetilde{R} \cap T > 1$, but $\widetilde{R} \not\leq T$. Therefore, if the center $Z(\widetilde{R})$ of $\widetilde{R}$ is contained in $\widetilde{R} \cap T$, we take $1 \neq x \in Z(\widetilde{R})$ and $k \in \widetilde{R} \setminus T$, while if $Z(\widetilde{R}) \not\leq \widetilde{R} \cap T$ we take $k \in Z(\widetilde{R}) \setminus T$ and $1 \neq x \in \widetilde{R} \cap T$. Then $1 \neq x \in T$ and $k \notin T$, but $[x,k] = 1$, against our hypothesis that $T = C_G(x)$.

c) If T is not isolated, there exists an element $y \in T$ such that $C_G(y) \not\leq T$, with $|y| = r$, r a prime in $\pi_1(G)$ (see Lemma 5 of [16]). This means that $\langle y \rangle \sim \langle x \rangle$ for some involution x. Therefore $T \sim \langle y \rangle \sim \langle x \rangle \sim U$, because in our hypothesis, all the 2-subgroups of G lie in $[U]$, by the Remark preceding this proposition.

d) If R is a nilpotent subgroup of G, then by a), we have that either $R \sim U$ or $R \sim T$ for some maximal torus T. If $T \sim U$, then $R \sim U$, if $T \not\sim U$, by c), T is isolated and therefore applying b), we obtain that $R \leq T$. □

Remark If we consider two maximal tori T_1 and T_2 that are isolated and have the same order, then they are conjugate. In fact, if we take a prime r dividing $|T_1|$ and an r-Sylow subgroup R of G contained in T_1, there exists an element $g \in G$ such that $R^g \leq T_2$. Then $T_1 \cap T_2^{g^{-1}} = R > 1$, and therefore we have that $T_1 \sim T_2^{g^{-1}}$. By Proposition 5 d) this is possible if and only if $T_1 = T_2^{g^{-1}}$. We conclude that T_1 and T_2 are conjugate.

Then we need to know the number of the conjugates of each of these maximal tori which are isolated; this is exactly $|G : N_G(T)|$. From Proposition 3.3.6 of [4] we obtain that

$$N_G(T)/T \cong C_{W,F}(w) = \{x \in W : x^{-1}wF(x) = w\},$$

where F is the Frobenius map and W is the Weyl group of $\widetilde{G}$ such that $\widetilde{G}^F = G$. Moreover in [7], Chapter 5 and 7, these centralizers $C_{W,F}(w)$ are calculated for any finite simple group of Lie type.

We shall denote by $n(T)$ the order of the group $N_G(T)/T$, so that for any maximal torus T that is isolated, we have that the number of its conjugates is exactly $|G|/n(T)|T|$.

The groups G such that $Nl(G)$ is not connected are exactly the groups G such that $\Gamma(G)$ is not connected, by Proposition 1. All the simple groups G such that $\Gamma(G)$ is not connected have been described in [8], [9] and [16]. Therefore it is now enough to calculate the connected components of $Nl(G)$, in the case in which $Nl(G)$ is not connected.

By Proposition 5 and last Remark, these are exactly $[U]$ and $[T_i] = Nl(T_i)$, for $i > 1$, with all its conjugates, where T_i are the maximal tori of G such that $\pi_i(G) = \pi(|T_i|)$ for $i > 1$, as it is described in [16].

So now we can describe $Nl(G)$ for any finite simple group of Lie type such that $Nl(G)$ is not connected. In Tables 1a) and 1b), we describe G, $|T_i|$ and $n(T_i)$.

As observed in the introduction, the group G acts by conjugacy on the poset $Nl(G)$ of nilpotent subgroups of G. We would like to describe this action, as it is done in Corollary 2.8 of [10] for the soluble groups.

Corollary 2 *Let G be a finite simple group of Lie type with $t(G) \geq 2$. Then the number of G-orbits of $Nl(G)$ is exactly $t(G)$, the number of connected components of $\Gamma(G)$. Moreover, if G has not a partition, then there is only one G-orbit, $[U]$, fixed by G.*

Table 1a. Connected components of $Nl(G)$, except $[U]$, with G finite simple group of Lie type, not twisted, $G \not\cong A_1(q), A_2(4)$

$G=A_l(q)$ with $l>2$ and $l+1$ a prime	$\|T_2\|=(q^l+q^{l-1}+\cdots+1)/(q-1,l+1)$	$n(T_2)=l+1$
$G=A_l(q)$ with l odd prime and $(q-1)\|(l+1)$	$\|T_2\|=q^{l-1}+q^{l-2}+\cdots+1$	$n(T_2)=l$
$G=A_2(q)$ with $q\neq 2,4$	$\|T_2\|=(q^2+q+1)/(q-1,3)$	$n(T_2)=3$
$G=B_l(q)$ with $l=2^n\geq 4$, q odd	$\|T_2\|=(q^l+1)/(q-1,2)$	$n(T_2)=2l$
$G=B_l(q)$ with l prime and $q=3$	$\|T_2\|=q^{l-1}+q^{l-2}+\cdots+1$	$n(T_2)=2l$
$G=C_l(q)$ with $l=2^n$	$\|T_2\|=(q^l+1)/(q-1,2)$	$n(T_2)=2l$
$G=C_l(q)$ with l prime and $q=2,3$	$\|T_2\|=q^{l-1}+q^{l-2}+\cdots+1$	$n(T_2)=2l$
$G=D_l(q)$ with l prime and $q=2,3,5$	$\|T_2\|=q^{l-1}+q^{l-2}+\cdots+1$	$n(T_2)=l$
$G=D_l(q)$ with $l-1$ prime and $q=2,3$	$\|T_2\|=q^{l-2}+q^{l-3}+\cdots+1$	$n(T_2)=2(l-1)$
$G=E_6(q)$	$\|T_2\|=(q^6+q^3+1)/(3,q-1)$	$n(T_2)=9$
$G=E_7(q)$ with $q=2,3$	$\|T_2\|=q^6+q^3+1$	$n(T_2)=18$
	$\|T_3\|=(q^7-1)/(q-1)$	$n(T_3)=14$
$G=E_8(q)$	$\|T_2\|=q^8+q^7-q^5-q^4-q^3+q+1$	$n(T_2)=30$
	$\|T_3\|=q^8-q^7+q^5-q^4+q^3-q+1$	$n(T_3)=30$
	$\|T_4\|=q^8-q^4+1$	$n(T_4)=24$
moreover, if $q\equiv 0,1,4(5)$	$\|T_5\|=q^8-q^6+q^4-q^2+1$	$n(T_5)=20$
$G=F_4(q)$	$\|T_2\|=q^4-q^2+1$	$n(T_2)=12$
moreover if $q=2^f$	$\|T_3\|=q^4+1$	$n(T_3)=8$
$G=G_2(q)$ with $q\equiv 1(3)$	$\|T_2\|=q^2-q+1$	$n(T_2)=6$
$G=G_2(q)$ with $q\equiv -1(3)$	$\|T_2\|=q^2+q+1$	$n(T_2)=6$
$G=G_2(q)$ with $q\equiv 0(3)$	$\|T_2\|=q^2-q+1$	$n(T_2)=6$
	$\|T_3\|=q^2+q+1$	$n(T_3)=6$

Table 1b. Connected components of $Nl(G)$, except $[U]$, with G finite simple group of Lie type, twisted, $G \not\cong {}^2B_2(q)$

$G={}^2A_l(q^2)$ with $l+1$ prime	$\lvert T_2\rvert=((-q)^l+(-q)^{l-1}+\cdots+(-q)+1)/(q+1,l+1)$	$n(T_2)=l+1$
$G={}^2A_l(q^2)$ with l odd prime and $(q+1)\lvert(l+1)$	$\lvert T_2\rvert=(-q)^{l-1}+(-q)^{l-2}+\cdots+(-q)+1$	$n(T_2)=l$
moreover if $q=2$ and $l=5$ or $q=3$ and $l=3$	$\lvert T_3\rvert=(q^{l+1}-1)/(q+1)(l+1,q+1)$	$n(T_3)=l+1$
$G={}^2A_3(2^2)$	$\lvert T_2\rvert=(q^4-1)/(q+1)$	$n(T_2)=4$
$G={}^2D_l(q^2)$ with $l=2^n$	$\lvert T_2\rvert=(q^l+1)/(2,q+1)$	$n(T_2)=l$
$G={}^2D_l(2^2)$ with $l=2^n+1$	$\lvert T_2\rvert=2^{l-1}+1$	$n(T_2)=2(l-1)$
$G={}^2D_l(3^2)$ with l prime and $l\neq 2^n+1$	$\lvert T_2\rvert=(3^l+1)/4$	$n(T_2)=l$
$G={}^2D_l(3^2)$ with l not a prime and $l=2^n+1$	$\lvert T_2\rvert=(3^{l-1}+1)/2$	$n(T_2)=2(l-1)$
$G={}^2D_l(3^2)$ with l a prime and $l=2^n+1$	$\lvert T_2\rvert=(3^l+1)/4$	$n(T_2)=l$
	$\lvert T_3\rvert=(3^{l-1}+1)/2$	$n(T_3)=2(l-1)$
$G={}^2E_6(q^2)$	$\lvert T_2\rvert=(q^6\rvert+q^3+1)/(3,q+1)$	$n(T_2)=9$
moreover if $q=2$	$\lvert T_3\rvert=(q^4-q^2+1)(q^2-q+1)/(3,q+1)=\{13\}$	$n(T_3)=12$
	$\lvert T_4\rvert=(q^4+1)(q+1)(q-1)/(3,q+1)=\{17\}$	$n(T_4)=8$
$G={}^3D_4(q^2)$	$\lvert T_2\rvert=q^4-q^2+1$	$n(T_2)=9$
$G={}^2F_4(q^2)$	$\lvert T_2\rvert=q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1$	$n(T_2)=12$
	$\lvert T_3\rvert=q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1$	$n(T_3)=12$
$G={}^2F_4(2)'$	$\lvert T_2\rvert=q^4-q^2+1$	$n(T_2)=6$
$G={}^2G_2(q^2)$	$\lvert T_2\rvert=q^2-\sqrt{3}q+1$	$n(T_2)=6$
	$\lvert T_3\rvert=q^2+\sqrt{3}q+1$	$n(T_3)=6$

Proof. It is enough to consider the connected component of $Nl(G)$. If G has a partition, then the behaviour of G on $Nl(G)$ is as described in Proposition 2.9 of [10]. Otherwise, we observe that by Proposition 5 d), and the following Remark, G acts fixing the component $[U]$ and transitively permuting the components $Nl(T_i^g)$, where T_i is a maximal torus and a $\pi_i(G)$-Hall subgroup of G, for $i \geq 2$. □

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