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A blow-up criterion for the curve shortening flow by surface diffusion

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Abstract. A sharp criterion for finite time blow-up for the curve shortening flow by surface diffusion is given. It is also shown that a multiply folded circle attracts some nearby curves.

Key words: surface diffusion, curvature, blow-up.

1. Introduction

Let γ_0 be an immersed plane curve. We consider the Cauchy problem for the motion law

$$\frac{\partial\gamma}{\partial t}=-k_{ss}\boldsymbol{n},\quad \gamma(\cdot,0)=\gamma_0, \tag{1}$$

where s, $\boldsymbol{n}$ and k are respectively the arc-length, unit normal and curvature (with respect to $\boldsymbol{n}$) of the curve $\gamma(\cdot,t)$. This motion law was first proposed by Mullins [14] to model thermal grooving. More recent discussion on its physical significance can be found in Cahn-Taylor [6] and Cahn-Elliot-Novick-Cohen [5]. One may compare (1) with the well-studied motion law–the curve shortening flow

$$\frac{\partial\gamma}{\partial t}=k\boldsymbol{n}, \tag{2}$$

which was also proposed by Mullins. According to [6], one may view (1) and (2) respectively the negative H^{-1}-gradient and L^2-gradient flows of the surface energy (it is the length energy in the planar case). (1) can be obtained as the singular limit of the Cahn-Hilliard equation with a degenerate concentration-dependent mobility [5]. This is parallel to the well-known that fact that (2) arises as the singular limit of the Allen-Cahn equation. Further discussion on (1) and its higher dimensional counterpart can also be found in Bernoff-Bertozzi-Witelski [4].

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The curve shortening flow (2) has been studied in a rather detailed way and now a comprehensive understanding has been achieved, see, e.g., the book Chou-Zhu [7]. In contrast, few results are known for (1). The solvability of (1) for small time was established in Elliott-Garke [9] (see also Alvarez-Liu [1]) when γ_0 is a simple, closed C^4-curve. Also, it was shown that the flow exists globally and converges exponentially to a circle when the initial curve is very close to a circle. Their results were subsequently generalized to higher dimensions in Escher-Mayers-Simonett [11], where uniqueness of the flow is established and the regularity requirement on the initial data is reduced. Besides, various interesting behavior of the flow, including finite time blow-up, convergence to a multiply folded circle and almost self-similar shrinking, are illustrated by numerical experiments. In this paper sophisticated semi-group theory was used to prove existence. In Giga-Ito ([12] and [13]), local existence and uniqueness are proved based on a more elementary Lax-Milgram type theorem. More importantly, they show two basic properties, namely, embeddedness and convexity preserving, which have played an important role in the study of (2), are no longer valid for (1). The failure of the maximum principle makes things more complicated. For example, it is not clear at all how to introduce a good theory of weak solution such as the viscosity solution, which has been so sucessful in the study of (2). The difference may be explained by the structure of the flows. When the flow $\gamma(\cdot, t)$ is expressed as a family of local graphs or when we look at the evolution equation for its curvature, the curve shortening flow is a second-order parabolic equation but the curve shortening by surface diffusion is a fourth-order parabolic equation.

In this paper we shall use energy method to study (1). First of all, the unique solvability of (1) for small time enables us to define a unique maximal solution in a maximal interval $[0, \omega), \omega \leqslant \infty$. The number ω is called the life-span of the flow.

Proposition A *Let $\gamma(\cdot, t)$ be a solution of* (1) *where γ_0 is a closed, immersed, smooth curve with nonzero area. Then the solution exists as long as the L^2-norm of the curvature of $\gamma(\cdot, t)$ is finite. Furthermore, when ω is finite,*

$$\int_{\gamma(\cdot, t)} k^2(s, t)ds \geqslant C(\omega - t)^{-1/4}$$

for some constant C. When ω is infinity, the curvature of $\gamma(\cdot, t)$ converges smoothly to the curvature of a circle whose area is the same as the area of γ_0.

So, the situation is like the curve shortening flow. Although the equation is of higher order, the threshold for existence is again a bound on the curvature.

Next, we have a criterion for blow-up in finite time.

Proposition B *Let γ_0 be a closed, immersed curve with total curvature $2n\pi$. Suppose that $n \geqslant 2$ and the isoperimetric ratio of γ_0 satisfies*

$$\frac{L_0^2}{A_0} < 4n\pi,$$

Then ω is finite.

By reversing the orientation of the curve if necessary, we shall always assume the total curvature of the flow is non-negative.

Figure 1 in [11] shows the evolution of the limaçon $r(\theta) = 1 + 1.7 \sin\theta$ under (1). The outer loop is almost circular, and the inner loop is very small compared to the outer one. A direct computation verifies the hypotheses of Proposition B and so the flow blows up in finite time. In fact, the numerical study in [11] shows that it blows up rapidly by contracting the inner loop. Notice that our criterion ensures blow-up even when the inner loop is just a little shorter than the outer one.

The isoperimetric ratio for an n-fold circle is equal to $4n\pi$. Since all n-fold circles are stationary solutions of (1), Proposition B is sharp in the sense that it no longer holds when $4n\pi$ is replaced by any larger number.

Proposition B implies that n-fold circles are not stable for $|n| \geqslant 2$. However, Figure 2 in [11] displays the evolution of the 4-leaf rose $r(\theta) = \sin 2\theta$. It converges to a three fold circle. We have the following result justifying behavior of this kind.

Proposition C *Let γ_0 be a locally convex, rotational symmetric curve with m leaves. Suppose $n/m < 1$ where $2n\pi$ is the total curvature of γ_0. There exists a small number ρ, which depends only on n, A_0 and L_0, such that (1) converges smoothly and exponentially to the n-fold circle centered at the origin and having the same enclosed area as γ_0, provided*

$$L_0 - 2n\pi < \rho,$$

and

$$\int_{\gamma_0} \left(\frac{dk_0}{ds}\right)^2 ds < \rho.$$

Propositions A–C will be proved in Section 3, after some energy estimates are derived in Section 2. In Section 4 we extend our results to a class of complete, non-compact curves.

2. Energy estimates

First, we write down the evolution of various geometric quantities along the flow (1). They can be obtained by direct computations, or may be deduced from the general formulas in Chou-Zhu [7]. We use the following notations:

$s(t)$	the arc-length of $\gamma(\cdot, t)$
$\boldsymbol{t}$ and $\boldsymbol{n}$	the unit tangent and normal of $\gamma(\cdot, t)$
θ	the normal angle of $\gamma(\cdot, t)$
$L(t)$	the length of $\gamma(\cdot, t)$
$A(t)$	the area of $\gamma(\cdot, t)$
$k(s, t)$	the curvature of $\gamma(s, t)$ with respect to $\boldsymbol{n}$. When γ is closed, $\boldsymbol{n}$ is the inner unit normal.

We have

$$\begin{aligned}
&\frac{ds}{dt} = kk_{ss},\\
&\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = -kk_{ss}\frac{\partial}{\partial s},\\
&\frac{d\boldsymbol{t}}{dt} = -\frac{d\boldsymbol{n}}{dt} = -k_{sss},\\
&\frac{d\theta}{dt} = -k_{sss},\\
&\frac{dk}{dt} = -k_{ssss} - k^2 k_{ss},\\
&\frac{dL}{dt} = \int_\gamma kk_{ss} ds,
\end{aligned} \tag{3}$$

and

$$\frac{dA}{dt} = \int_\gamma k_{ss} ds. \tag{4}$$

Recall that the (algebraic) area is defined only for closed curves and is given by

$$A = -\frac{1}{2}\int_\gamma \langle \gamma, \boldsymbol{n}\rangle ds.$$

It is equal to $\sum n_j A_j$ where A_j and n_j are respectively the area and winding number of the j-th component of the complement of γ in the plane. It follows from the last two formulas that the area is unchanged along the flow, but the length is strictly decreasing unless γ_0 is an n-fold circle:

$$-\int_0^t \int_\gamma k_s^2(\cdot, \tau) ds\, d\tau = L(t) - L(0). \tag{5}$$

Let

$$\begin{aligned} \bar{k}(t) &= \frac{1}{L(t)} \int_\gamma k(s,t) ds \\ &= \frac{2n\pi}{L(t)}, \end{aligned}$$

where $2n\pi$ is the total curvature of γ_0. By the Poincaré inequality, we have

$$\int_0^t \frac{\pi^2}{L^2(\tau)} \int_{\gamma(\cdot,\tau)} \left(k - \bar{k}\right)^2 (s,\tau) ds\, d\tau \leqslant L(0) - L(t).$$

Notice that π^2 is the minimum for the problem

$$\left\{ \frac{\int_0^1 u_x^2 dx}{\int_0^1 u^2 dx} : \int_0^1 u\, dx = 0 \right\}.$$

It follows that

$$\int_0^t \int_{\gamma(\cdot,\tau)} \left(k - \bar{k}\right)^2 (s,\tau) ds\, d\tau \leqslant \frac{L_0^2}{\pi^2} \left(L_0 - L(t)\right). \tag{6}$$

Now, we proceed to derive some energy inequalities for (1). First, we have, by (3) and (4),

$$\frac{d}{dt}\int_\gamma k^2(s,t) ds = -2\int_\gamma k_{ss}^2 ds + 3\int_\gamma k^2 k_s^2 ds,$$

and

$$\frac{d}{dt}\int_\gamma (k-\bar{k})^2 ds = -2\int_\gamma k_{ss}^2 ds + 3\int_\gamma (k-\bar{k})^2 k_s^2 ds$$
$$+ 6\bar{k}\int_\gamma (k-\bar{k})k_s^2 ds + 2\bar{k}^2\int_\gamma k_s^2 ds. \tag{7}$$

We shall use the interpolation inequalities: For periodic functions with zero mean,

$$\left\|u^{(j)}\right\|_{L^r} \leqslant C\left\|u\right\|_{L^p}^{1-\theta}\left\|u^{(k)}\right\|_{L^q}^{\theta}, \quad \theta\in(0,1),$$

where r, q, p, j and k satisfy $p,q,r>1$, $j\geqslant 0$,

$$\frac{1}{r} = j + \theta\left(\frac{1}{q} - k\right) + (1-\theta)\frac{1}{p},$$

and

$$\frac{j}{k} \leqslant \theta \leqslant 1.$$

Here the constant C depends on r, p, q, j and k only. Using this interpolation inequality, we have

$$\left(\int_\gamma (k-\bar{k})^4 ds\right)^{\frac{1}{2}} \leqslant C\left(\int_\gamma (k-\bar{k})^2 ds\right)^{\frac{7}{8}}\left(\int_\gamma k_{ss}^2 ds\right)^{\frac{1}{8}},$$

and

$$\left(\int_\gamma k_s^4 ds\right)^{\frac{1}{2}} \leqslant C\left(\int_\gamma (k-\bar{k})^2 ds\right)^{\frac{3}{8}}\left(\int_\gamma k_{ss}^2 ds\right)^{\frac{5}{8}}.$$

Hence

$$\begin{aligned}\int_\gamma (k-\bar{k})^2 k_s^2 ds &\leqslant \left(\int_\gamma (k-\bar{k})^4 ds\right)^{\frac{1}{2}}\left(\int_\gamma k_s^4 ds\right)^{\frac{1}{2}}\\ &\leqslant C_1\left(\int_\gamma (k-\bar{k})^2 ds\right)^{\frac{5}{4}}\left(\int_\gamma k_{ss}^2 ds\right)^{\frac{3}{4}}\\ &\leqslant \varepsilon_1\int_\gamma k_{ss}^2 ds + \frac{27}{256}C_1^4\varepsilon_1^{-3}\left(\int_\gamma (k-\bar{k})^2 ds\right)^5.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\int_\gamma (k-\bar{k})k_s^2 ds\right| &\leqslant \left(\int_\gamma (k-\bar{k})^2 ds\right)^{\frac{1}{2}} \left(\int_\gamma k_s^4 ds\right)^{\frac{1}{2}} \\ &\leqslant C_2 \left(\int_\gamma (k-\bar{k})^2 ds\right)^{\frac{7}{8}} \left(\int_\gamma k_{ss}^2 ds\right)^{\frac{5}{8}} \\ &\leqslant \varepsilon_2 \int_\gamma k_{ss}^2 ds + \frac{3}{8}\left(\frac{5}{8}\right)^{\frac{5}{2}} C_2^{\frac{7}{2}} \varepsilon_2^{-\frac{5}{2}} \left(\int_\gamma (k-\bar{k})^2 ds\right)^{\frac{7}{3}},\end{aligned}$$

and

$$\begin{aligned}\int_\gamma k_s^2 ds &\leqslant \left(\int_\gamma (k-\bar{k})^2 ds\right)^{\frac{1}{2}} \left(\int_\gamma k_{ss}^2 ds\right)^{\frac{1}{2}} \\ &\leqslant \varepsilon_3 \int_\gamma k_{ss}^2 ds + \frac{1}{4\varepsilon_3}\int_\gamma (k-\bar{k})^2 ds.\end{aligned}$$

On the other hand, an isoperimetric inequality of Rado [15] asserts that for any closed, immersed curve,

$$L^2 \geqslant 4\pi \Sigma n_j A_j.$$

By the area preserving property of the flow, we have

$$L^2(t) \geqslant 4\pi A_0,$$

that's, $|\bar{k}| \leqslant n\pi^{1/2}(A_0)^{-1/2}$. (We may assume A_0 is positive by reversing the orientation of the flow if necessary. Note that (1) is independent of the orientation of the curves.) Putting these estimates into (7), we have

$$\frac{d}{dt}\int_\gamma (k-\bar{k})^2 ds \leqslant (3\varepsilon_1 + 6\varepsilon_2 + 2\varepsilon_3 - 2)\int_\gamma k_{ss}^2 ds + p(E),$$

where

$$p(E) = \frac{81}{256} C_1^4 \varepsilon_1^{-3} E^5 + \frac{9}{4}\left(\frac{5}{8}\right)^{\frac{5}{2}} C_2^{\frac{7}{2}} \varepsilon_2^{-\frac{5}{2}} E^{\frac{7}{3}} + \frac{1}{2}\varepsilon_3^{-1} E,$$

and

$$E \equiv \int_\gamma (k-\bar{k})^2 ds.$$

By choosing ε_1, $i = 1, 2, 3$ so that $3\varepsilon_1 + 6\varepsilon_2 + 2\varepsilon_3 = 2$, we have

$$\frac{dE}{dt} \leqslant C_3(E + E^5), \tag{8}$$

where C_3 depends only on the initial area, total curvature, and the best constants in these interpolation inequalities. When n is equal to zero, we have $\bar{k} = 0$ and

$$\frac{dE}{dt} \leqslant C_3 E^5. \tag{8$'$}$$

Next, we compute

$$\begin{aligned}
\frac{d}{dt}\int_\gamma k_s^2 ds &= -2\int_\gamma k_{sss}^2 + 2\int_\gamma k^2 k_{ss}^2 + \frac{1}{3}\int_\gamma k_s^4 \\
&= -2\int_\gamma k_{sss}^2 + 2\int_\gamma (k - \bar{k})^2 k_{ss}^2 + 4\bar{k}\int_\gamma (k - \bar{k}) k_{ss}^2 \\
&\quad + 2\bar{k}^2 \int_\gamma k_{ss}^2 + \frac{1}{3}\int_\gamma k_s^4.
\end{aligned}$$

As before,

$$\begin{aligned}
\int_\gamma (k - \bar{k})^2 k_{ss}^2 ds &\leqslant \left(\int_\gamma (k - \bar{k})^4 ds\right)^{\frac{1}{2}} \left(\int_\gamma k_{ss}^4 ds\right)^{\frac{1}{2}} \\
&\leqslant C\left(\int_\gamma (k - \bar{k})^2 ds\right)^{\frac{7}{6}} \left(\int_\gamma k_{sss}^2 ds\right)^{\frac{5}{6}}, \\
\int_\gamma (k - \bar{k}) k_{ss}^2 ds &\leqslant \left(\int_\gamma (k - \bar{k})^2 ds\right)^{\frac{1}{2}} \left(\int_\gamma k_{ss}^4 ds\right)^{\frac{1}{2}} \\
&\leqslant C\left(\int_\gamma (k - \bar{k})^2 ds\right)^{\frac{3}{4}} \left(\int_\gamma k_{sss}^2 ds\right)^{\frac{3}{4}},
\end{aligned}$$

and

$$\int_\gamma k_s^4 ds \leqslant C\left(\int_\gamma (k - \bar{k})^2 ds\right)^{\frac{7}{6}} \left(\int_\gamma k_{sss}^2 ds\right)^{\frac{5}{6}}.$$

As a result of these estimates, we arrive at

$$\frac{d}{dt}\int_\gamma k_s^2 ds \leqslant C_4\left(E^3 + E^7\right), \tag{9}$$

where C_4 depends on various quantities like C_3. Finally,

$$\frac{d}{dt}\int_\gamma k_{ss}^2 ds = -2\int_\gamma k_{ssss}^2 ds - 2\int_\gamma k^2 k_{ss} k_{ssss} ds - 2\int_\gamma k k_s k_{ss}^2 ds.$$

By the interpolation inequality, we have

$$\int_\gamma k^8 ds \leqslant C\left(\int_\gamma k^2 ds\right)^{\frac{29}{8}}\left(\int_\gamma k_{ssss}^2 ds\right)^{\frac{3}{8}},$$

and

$$\int_\gamma k_{ss}^4 ds \leqslant C\left(\int_\gamma k^2 ds\right)^{\frac{7}{8}}\left(\int_\gamma k_{ssss}^2 ds\right)^{\frac{9}{8}}.$$

Therefore,

$$\int_\gamma k^4 k_{ss}^2 ds \leqslant C\left(\int_\gamma k^2 ds\right)^{\frac{9}{4}}\left(\int_\gamma k_{ssss}^2 ds\right)^{\frac{3}{4}}.$$

On the other hand,

$$\begin{aligned}\left|\int_\gamma k k_s k_{ss}^2 ds\right| &\leqslant \left(\int_\gamma k^4 ds\right)^{\frac{1}{4}}\left(\int_\gamma k_s^4 ds\right)^{\frac{1}{4}}\left(\int_\gamma k_{ss}^4 ds\right)^{\frac{1}{2}} \\ &\leqslant C\left(\int_\gamma k^2 ds\right)^{\frac{5}{4}}\left(\int_\gamma k_{ssss}^2 ds\right)^{\frac{3}{4}}.\end{aligned}$$

Putting these together, we get

$$\frac{d}{dt}\int_\gamma k_{ss}^2 ds \leqslant C_5\left(E^9 + E^5\right). \tag{10}$$

By the same procedure, one can obtain a general inequality

$$\frac{d}{dt}\int_\gamma \left(\frac{d^n k}{ds^n}\right)^2 ds \leqslant C E^{2n+5},$$

for $E \geqslant 1$. However, (8), (9) and (10) are sufficient for our purpose.

3. Proofs of the Propositions

Proof of Proposition A. From [12] we know that for any closed, immersed H^4-curve, there exists a unique solution of (1) in some $[0, t_1), t_1 > 0$ where t_1 depends on the H^4-norm of the initial curve. Now, suppose that $E(t)$ is uniformly bounded in $[0, T)$ for some T. By integrating (9) and (10), we see that it implies a uniform bound on the L^2-norms of the first and the second derivatives of the curvature, which in term implies a H^4-bound on the curve itself. (In fact, using the differential inequality next to (9), one can obtain a uniform H^k-bound on the curve with a prescribed k.) We may then use the local existence result in [12] again to extend the flow beyond T. Taking T to be ω, we conclude that E must become unbounded as a finite ω is approached.

When ω is finite, we can integrate (8) from t to ω to obtain

$$E^4(t) \geqslant \frac{1}{8C}(\omega - t)^{-1},$$

when t is close to ω. We have obtained the desired lower bound for the blow-up rate.

Finally, when ω is infinity, it follows from (5) and the isoperimetric inequality that

$$\int_0^\infty \int_\gamma k_s^2(s, \tau) ds\, d\tau \leqslant L_0,$$

for some constant L_0. Hence for any $\varepsilon > 0$, there exists j_0 such that we can find, by the mean-value theorem, $t_j \in [j, j+1]$ satisfying $E(t_j) \leqslant \varepsilon$ for all $j \geqslant j_0$. From (8) it is clear that we can find a sufficiently small ε such that $E(t)$ is less than 1 for all t in $[t_j, t_j + 2]$. It means that $E(t)$ is uniformly bounded in $[j_0 + 1, \infty)$. It follows from (9) and parabolic regularity that all spatial and time derivatives of k are uniformly bounded. In view of (5), any sequence $\{k(\cdot, t_j)\}$ contains a subsequence $\{k(\cdot, t_{j_i})\}$ converging smoothly to a constant as $t_{j_i} \to \infty$. Since the flow preserves both area and total curvature, the constant must be the same for any converging subsequence. We have shown that $k(\cdot, t)$ tends to a constant as $t \to \infty$. The proof of Proposition A is completed. □

Before proving Propositions B and C, we put down some comments. First, ω is always finite when $n = 0$. For, by (5), we have

$$\begin{aligned} \frac{dL}{dt} &\leqslant -\frac{\pi^2}{L^2(t)} \int_\gamma k^2 ds \\ &\leqslant -\frac{\pi^2}{L^3(t)} \left(\int_\gamma |k| ds \right)^2 \end{aligned}$$

$$\leqslant -\frac{4\pi^4}{L^3(t)}$$

Hence

$$\omega \leqslant L_0^4/8\pi^3.$$

Second, we may call the flow develops a **type I singularity** if

$$\int_{\gamma(\cdot,t)} k^2 ds \leqslant C(\omega - t)^{-1/4}.$$

In view of Proposition A, the blow-up rate is the lowest for type I singularities. Recall that for the curve shortening flow (2), a type I singularity satisfies the lowest blow-up rate

$$|k|_{\max}(t) \leqslant C(\omega - t)^{-1/2}.$$

A theorem of Altschuler [2] states that if a flow develops only type I singularities, then in fact it must shrink to a point and, after rescaling it so that its area is always the same, the normalized flow converges to a contracting self-similar solution of (2). All contracting self-similar solutions of (2) were completely classified by Abresch-Langer [7]. We believe the present situation is similar. We rescale the flow by setting

$$\gamma(s,t) = (\omega - t)^{\frac{1}{4}} \widetilde{\gamma}(s,t).$$

Then $\widetilde{\gamma}$ satisfies the flow

$$\frac{\partial \widetilde{\gamma}}{\partial \tau} = -\widetilde{k}_{\widetilde{s}\widetilde{s}} \boldsymbol{n} + \frac{1}{4}\widetilde{\gamma}.$$

where $\tau = -\log(\omega - t)$, and its stationary solution is a contracting self-similar solution satisfying

$$\gamma = 4k_{ss}\boldsymbol{n}.$$

It would be very interesting to find and classify all closed, contracting self-similar solutions of (1).

Third, the inequalities (6) and (8) enable us to solve (1) for γ_0' s satisfying only the regularity requirement

$$\int_{\gamma_0} k_0^2 ds < \infty.$$

To see this, let's apply the mean-value theorem to (5) to find some $t^* \in (0,t)$ such that

$$\int_\gamma k_s^2(s,t^*)ds \leqslant L_0/t.$$

Therefore,

$$\begin{aligned}\int_\gamma k_s^2(s,t)ds &\leqslant \int_\gamma k_s^2(s,t^*)ds + \int_{t^*}^t C(E^3+E^7)ds \\ &\leqslant L_0/t + C'. \end{aligned} \tag{11}$$

This is because by (8) the second term in the right hand side of (11) is uniformly bounded on $(0,T]$ for some T depending only on the L^2-curvature of the initial curve. For any initial curve with finite L^2-curvature, we may approximate it by a sequence of smooth curves $\{\gamma_j^0\}$ whose L^2-curvature are uniformly bounded. By (8) and (11) we know that there is a uniform $T>0$ such that the flow, γ_j, starting at γ_j^0 exists in $[0,T]$. By passing to a converging subsequence we see that the flows $\{\gamma_j\}$ approach to a flow of (1) as $j\to\infty$. To see that it takes γ_0 as its initial curve we may first represent the flow locally as graphs and then adapt, for instance, the argument in Section 2 of Bernis-Friedman [3] to establish a uniform Hölder bound on these graphs. By letting t go to 0 we see that it takes γ_0 as its initial curve.

We point out that all results in this paper apply to flows whose initial curves have L^2-curvature.

Proof of Proposition B. Suppose on the contrary that the flow exists for all time. By Proposition A, it subconverges smoothly to an n-fold circle. Since area is constant along the flow, the radius of the limit circle is equal to $(A_0/n\pi)^{1/2}$. By the curve shortening property of the flow,

$$\frac{L_0^2}{A_0} \geqslant \frac{(2n\pi)^2 A_0}{n\pi} \times \frac{1}{A_0} = 4n\pi.$$

The contradiction shows that Proposition B must hold. □

The reader should not be left with the impression that long time existence of (1) always holds for curves with total curvature 2π. Some of these curves do blow up in finite time, although we don't have a result as general as Proposition B. As an illustration, consider a circle centered at the origin O and oriented in the counterclockwise direction. Place two small circles on the top and bottom of this circle respectively. These three circles together form a smooth, immersed curve γ symmetric with respect to the x- and the y-axes. The orientation of the small circles are clockwise, and so the total curvature of this curve is -2π. Suppose that the isoperimetric ratio of γ is not greater than 8π. We claim that the flow (1) starting at this curve blows up in finite time. For, if it exists for all time, by Proposition A it converges to a circle centered at O with total curvature -2π. Since the winding number of the initial curve around O is 1 and it changes to -1 eventually, there is a time t^* at which the flow touches O. By symmetry, the flow at t^* splits into a closed curve C touching O and its image under reflection with respect to the x-axis. Denoting the perimeter and area of C by l and a respectively, we have,

$$8\pi \geqslant \frac{L_0^2}{A_0} > \frac{(2l)^2}{2a} \geqslant 8\pi.$$

Contradiction holds. Hence the flow cannot exist for long.

Now, we proceed to prove Proposition C. First, we need to recall some basic properties of the support function of a locally convex curve with positive curvature. For such a curve γ, we may use the normal angle θ, i.e, the unit outer normal is given by $(\cos\theta, \sin\theta)$, to parametrise γ. Then the support function is given by

$$h(\theta) = \langle \gamma(s(\theta)), (\cos\theta, \sin\theta) \rangle .$$

For a flow of (1) consisting of closed, locally convex curves $\gamma(\cdot, t)$, the support function $h(\theta, t)$ satisfies

$$\begin{aligned}\frac{\partial h}{\partial t} &= \left\langle \boldsymbol{t}\frac{\partial s}{\partial t} + \frac{\partial \gamma}{\partial t}, (\cos\theta, \sin\theta) \right\rangle \\ &= k_{ss}.\end{aligned}$$

By the formulas $d\theta/ds = k$ and $k = (h_{\theta\theta} + h)^{-1}$, h satisfies the parabolic equation

$$h_t = k^2 \left(\frac{1}{h_{\theta\theta} + h} \right)_{\theta\theta} + k \left[\left(\frac{1}{h_{\theta\theta} + h} \right)_{\theta} \right]^2 .$$

In fact, it can be shown that this equation is equivalent to (1) ([7]).

Now, a locally convex, closed curve is called a rotational symmetric curve with m leaves if its support function is T-period where $T = 2n\pi/m$

and $(n, m) = 1$. Let's denote the class of all these curves by $\mathcal{K}(n, m)$. The perimeter and area of $\gamma \in \mathcal{K}(n, m)$ are given by

$$\begin{aligned} L &= \int_0^{2n\pi} (h_{\theta\theta} + h)\, d\theta \\ &= \int_0^{2n\pi} h\, d\theta \\ &= m \int_0^T h\, d\theta, \end{aligned}$$

and

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2n\pi} (h_{\theta\theta} + h) h\, d\theta \\ &= \frac{m}{2} \int_0^T \left(h^2 - h_\theta^2\right) d\theta, \end{aligned}$$

respectively. We have the following isoperimetric inequality.

Lemma 3.1 *For any $\gamma \in \mathcal{K}(n, m)$, $n/m < 1$,*

$$\frac{L^2}{A} \geqslant 4n\pi,$$

with equality holds if and only if γ is an n-fold circle.

Proof. Let the Fourier expansion of the support function of γ be

$$h = \sum_{k=0}^{\infty} a_k \cos \frac{2k\pi\theta}{T} + \sum_{k=1}^{\infty} b_k \sin \frac{2k\pi\theta}{T}.$$

We have

$$\begin{aligned} \frac{L^2}{A} &= 2m \frac{\left(\int_0^T h\, d\theta\right)^2}{\int_0^T \left(h^2 - h_\theta^2\right) d\theta} \\ &= \frac{2mT^2 a_0^2}{T a_0^2 + \frac{T}{2} \sum_{k=1}^{\infty} \left[1 - \left(\frac{2k\pi}{T}\right)^2\right] \left(a_k^2 + b_k^2\right)} \end{aligned}$$

$$\begin{aligned}&\geqslant 2mT\\&= 4n\pi,\end{aligned}$$

provided $n/m < 1$. By a routine application of the direct method, one concludes that,

$$\mu = \inf\left\{\frac{L^2}{A} : h \text{ is an } H^1\text{-function of period } T.\right\}$$

has a minimizer whose support function is denoted by h^*. Taking first variation, we have

$$2\int_0^T \phi\, h\theta - \mu \int_0^T (h^*\varphi - h^*_\theta \phi_\theta)\, d\theta = 0,$$

for all smooth ϕ. By elliptic regularity, h^* is smooth and satisfies

$$h^*_{\theta\theta} + h^* = \frac{2}{\mu}.$$

Hence $h^* = 2/\mu + a\cos\theta + b\sin\theta$, i.e., the minimizer is a circle. □

Proof of Proposition C. From (8) and (9) we know that

$$E(t) \leqslant E(0)e^{ct},$$

and

$$\int_\gamma k_s^2(s,t)ds \leqslant \rho + C\left(E^3(t) + E^7(t)\right).$$

When ρ is small, $|k_0 - \bar{k}_0|$ and $E(0)$ are small. We fix ρ so that the flow exists in $[0,1]$ and $k > 0$. We claim that $\omega = \infty$. Suppose on the contrary that ω is finite. By the mean-value theorem there exists some $t^* \in (\omega - 1, \omega)$ such that

$$\begin{aligned}\int_\gamma k_s^2(s,t^*)ds &= \int_{\omega-1}^{\omega}\int k_s^2(s,\tau)ds\, d\tau\\&= L(\omega-1) - L(\omega)\\&\leqslant L_0 - L(\omega)\\&\leqslant L_0 - 2(n\pi A_0)^{1/2} \qquad \text{(by Lemma 3.1)}\\&\leqslant \rho.\end{aligned}$$

Using t^* as the initial time, we can extend the flow beyond ω, contradiction holds. Hence the flow exists for all time. By Proposition A the flow subconverges to an n fold circle with area A_0. By rotational symmetry, all limit circles must be centered at the origin and hence they are the same.

Hence the entire flow converges to an n-fold circle smoothly. Exponential decay can be proved by looking at the eigenvalues of the linearised equation at this circle. A similar situation can be found in [11], and we shall not repeat it here. $\square$

4. Periodic curves

The results proved in the previous sections not only apply to closed but also to complete curves. Recall that when dealing with non-closed curves, one usually assumes either the curve is complete or it satisfies some boundary conditions. Moreover, as for the curve shortening flow, people have studied the special case where the initial curve is a graph [8]. As a consequence of the maximum principle, the flow remains as graphs over the same axis as long as its slope is bounded at the boundary or near infinity. Here the situation changes drastically for (1). Nice behavior along the boundary or near infinity does not guarantee interior regularity. As an illustration, we have

Proposition 4.1 *There exists a solution of* (1) *of the form* $(x, u(x,t))$, $(x,t) \in \mathbb{R}\times[0,T), T > 0$, *such that* (1) u_0 *is smooth,* (2) u *and its derivatives are uniformly bounded near* $\pm\infty$ *for* $t \in [0,T)$, (3) u *is unformly bounded in* $\mathbb{R} \times [0,T)$, *and yet* (4) $|u_x(0,t)|$ *becomes unbounded as* $t \uparrow T$.

Proof. Fix an odd function $\phi(y), y \in (-1,1)$, satisfying (a) $d\phi/dy < 0$ $(y \neq 0)$ and $(y, \phi(y))$ is asymptotic to the vertical line $y = 1$ as $y \uparrow 1$, and (b) $\phi^{(j)} = 0, j = 1,2,3,4$ and $\phi^{(5)}(0) = -1$. Let $\gamma(\cdot,t)$ be the flow (1) whose initial curve is given by $(y, \phi(y))$. By modifying the arguments in [11] or [13], one can show that the flow exists in $[0, t_1)$ be some $t_1 > 0$. By further restricting t_1, we may assume that each $\gamma(\cdot,t)$ is also a graph $(y, v(y,t))$ over $(-1,1)$.

Now, the normal velocity of the flow is given by

$$\gamma_t \cdot \boldsymbol{n} = \frac{v_t}{\sqrt{1+v_y^2}}$$

Therefore, v satisfies the equation

$$-v_t = \left(\frac{1}{\sqrt{1+v_y^2}} \left(\frac{v_{yy}}{(1+v_y^2)^{3/2}} \right)_y \right)_y .$$

By (a), (b) and Taylor expansion

$$\begin{aligned} v(y,t) &= v(y,0) + v_t(y,0)t + O(t^2) \\ &= \frac{\phi^{(5)}(0)y^5}{120} - \phi^{(5)}(0)yt + O(t^2 + |y|^7) \\ &= -y\left(\frac{y^4}{120} - t\right) + O(t^2 + |y|^7) \\ &> 0, \end{aligned}$$

for $y > 0$, $y^4 \ll t \ll 1$.

Let ψ be another odd function close to ϕ such that (a) and (b) hold for ψ except now $d\psi/dy < 0$ in $(-1,1)$. When ψ is very close to ϕ, the flow starting at $(y,\psi(y))$ exists in $[0,t_1/2)$ and assumes the form $(y,w(y,t))$. Moreover, $w(y,t) > 0$ for $x > 0$ and $x^4 \ll t \ll 1$. Since $\partial w/\partial y < 0$, we may represent this flow as a family of evolving graphs $(x,u(x,t))$ over the x-axis. By continuity, there must be some $T < t_1/2$ such that $\partial u/\partial x\ (0,T)$ blows up. □

Remark A fuller discussion on the graph-losing property of (1) can be found in [10]. We thank the referee for providing us with this reference.

In view of this proposition, we do not consider evolving graphs. Instead we consider periodic curves. A complete curve is called a **periodic curve** (with period L) if there exists a vector ξ such that

$$\gamma(s+L) = \gamma(s) + \xi, \quad \text{for all } s \in \mathbb{R}.$$

Proposition 4.2 *Let γ_0 be a periodic curve with period L_0. Then* (1) *has a unique maximal solution in* $[0,\omega)$ *such that*

$$\gamma(s + L(t), t) = \gamma(s) + \xi, \quad \textit{for all } s \in \mathbb{R},$$

and

$$\int_0^{L(t)} k(s,t)ds = 2n\pi, \quad \textit{for all } t \in [0,\omega).$$

Moreover, ω is always finite when $n \neq 0$. When $n = 0$ and k_0 is small, ω is infinity and the flow converges to a straight line.

Proof. We outline the proof of the proposition as follows. By representing the flow as graphs over γ_0, one can follow the arguments in [11] or [13] to show that a unique maximal solution exists. It follows from the uniqueness of the solution that each $\gamma(\cdot, t)$ is periodic with period $L(t)$ along the same direction ξ. When ω is infinity, the proof of Proposition A shows that the curvature $k(\cdot, t)$ converges smoothly to some constant k_∞. Since $L(t) \geqslant |\xi|$, the constant is non-zero when $n \neq 0$. So, $\gamma(\cdot, t)$ converges to a circle, but this is impossible. Hence ω must be finite when $n \neq 0$. On the other hand, k_∞ is zero when $n = 0$. In other words, the flow converges to a straight line. Finally, when k_0 is small, one can argue as in the proof of Proposition C that ω is infinity. The proof of Proposition 4.2 is completed. □

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