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On construction of continuous functions with cusp singularities

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Abstract. In this paper, we study various constructions of continuous functions on $\mathbf{R}$ which have the prescribed cusp singularities at each point. As applications, we get some generalizations of the results given in our previous paper [7], which discuss the cusp singularities of the classical Weierstrass functions and Takagi function.

Key words: wavelets, scaling exponents, singularities, Weierstrass functions, spline functions, Takagi function.

1. Introduction

Let s be a positive number, which is not an integer and let x_0 be a point in $\mathbf{R}^n$. Then a function f on $\mathbf{R}^n$ belongs to the pointwise Hölder space $C^s(x_0)$, if there exists a polynomial P of degree less than s such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^s$$

in a neighborhood of x_0. The pointwise Hölder exponent of a function f at a point x_0 in $\mathbf{R}^n$ is defined as

$$H(f, x_0) = \sup\{s > 0; f \in C^s(x_0)\}.$$

If a continuous function f does not belong to $C^s(x_0)$ for every $s > 0$, then $H(f, x_0) = 0$.

However the pointwise Hölder exponent of a function f at a point x_0 in $\mathbf{R}^n$ is not stable under the pseudo-differential operators. Similarly it does not fully characterize the oscillatory behavior on a neighborhood of x_0. This implies that $f \in C^s(x_0)$ cannot be characterized by size estimates on the wavelet coefficients of f.

Here let us recall the definition of the weak scaling exponent characterizing the local oscillatory behavior.

$\mathcal{S}_0(\mathbf{R}^n)$ denotes the closed subspace of the Schwartz class $\mathcal{S}(\mathbf{R}^n)$ such

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that

$$\int_{\mathbf{R}^n} x^\alpha \psi(x)\, dx = 0$$

for every multi-index α in $\mathbf{Z}_+^n$. Then a tempered distribution f belongs to $\Gamma^s(x_0)$, if for every ψ in $\mathcal{S}_0(\mathbf{R}^n)$, there exists a constant $C(\psi)$ such that

$$\left| \int_{\mathbf{R}^n} f(x) \frac{1}{a^n} \psi\left(\frac{x - x_0}{a}\right) dx \right| \le C(\psi) a^s, \quad 0 < a \le 1.$$

The weak scaling exponent of a function f at a point x_0 in $\mathbf{R}^n$ is defined as

$$\beta(f, x_0) = \sup \{ s \in \mathbf{R};\, f \text{ locally belongs to } \Gamma^s(x_0)\}.$$

Since it is known that the pointwise Hölder space $C^s(x_0)$ is contained in local $\Gamma^s(x_0)$, it is obvious that

$$H(f, x_0) \le \beta(f, x_0).$$

Now we recall the definition of the two-microlocal spaces $C_{x_0}^{s,s'}$, which characterize this weak scaling exponent.

Let φ be a function in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$ such that

$$\hat{\varphi}(\xi) = \begin{cases} 1 & \text{on } |\xi| \le \dfrac{1}{2} \\ 0 & \text{on } |\xi| \ge 1 \end{cases},$$

where $\hat{\varphi}$ is the Fourier transform of φ. For every non-negative integer j, we define the convolution operator $S_j(f) = f * \varphi_{\frac{1}{2^j}}$ where $\varphi_a(x) = \frac{1}{a^n}\varphi\left(\frac{x}{a}\right)$, and the difference operator $\Delta_j = S_{j+1} - S_j$. Then

$$I = S_0 + \sum_{j=0}^{\infty} \Delta_j.$$

Let $\psi = \varphi_{\frac{1}{2}} - \varphi$. Then $\psi \in \mathcal{S}_0(\mathbf{R}^n)$ and

$$\Delta_j(f) = f * \psi_{\frac{1}{2^j}}.$$

Let s and s' be two real numbers and x_0 a point in $\mathbf{R}^n$. Then a tempered distribution f belongs to the two-microlocal spaces $C_{x_0}^{s,s'}$, if there exists a constant C such that

$$|S_0(f)(x)| \le C(1 + |x - x_0|)^{-s'}$$

and

$$|\Delta_j(f)(x)| \le C2^{-js}(1+2^j|x-x_0|)^{-s'}$$

for every $j \in \mathbf{Z}_+$ and $x \in \mathbf{R}^n$.

The following remarkable theorems with respect to the two-microlocal spaces $C_{x_0}^{s,s'}$ and $\Gamma^s(x_0)$ were given in [5].

Theorem A [5, Theorem 1.8] *Let s and s' be two real numbers and x_0 a point in $\mathbf{R}^n$ and let us assume two positive integers r and N satisfying*

$$r + s + \inf(s', n) > 0$$

and

$$N > \sup(s, s+s').$$

Let ψ be a function such that

$$|\partial^\alpha \psi(x)| \le \frac{C(q)}{(1+|x|)^q}, \quad |\alpha| \le r, \quad q \ge 1$$

and

$$\int_{\mathbf{R}^n} x^\beta \psi(x)\, dx = 0, \quad |\beta| \le N-1.$$

If a function or a distribution f belongs to the two-microlocal spaces $C_{x_0}^{s,s'}$, then we have

$$\left| \int_{\mathbf{R}^n} f(x) \frac{1}{a^n} \overline{\psi\left(\frac{x-b}{a}\right)}\, dx \right| \le Ca^s \left(1 + \frac{|b-x_0|}{a}\right)^{-s'},$$
$$0 < a \le 1, \quad |b-x_0| \le 1.$$

Theorem B [5, Theorem 1.2] *Let s be a real number and let f be a function or a distribution defined on a neighborhood V of x_0.*

Then f locally belongs to $\Gamma^s(x_0)$ if and only if f locally belongs to the two-microlocal spaces $C_{x_0}^{s,s'}$ for some s'.

Several scientists have been interested in constructing irregular functions. The well-known example is the Weierstrass function [8]. It is an example of a nowhere differentiable continuous function. Hardy gave better estimates of the regularities for the Weierstrass function

$$\mathcal{W}_c(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x) \tag{1}$$

and its sine series

$$\mathcal{W}_s(x) = \sum_{n=0}^{\infty} a^n \sin(b^n \pi x), \tag{2}$$

where $0 < a < 1$, $b > 1$ and $ab \geq 1$ [3]. He proved that these functions do not possess finite derivatives at each point x and showed more precisely that if $ab > 1$ and $\xi = \frac{\log(\frac{1}{a})}{\log b}$, then these functions satisfy

$$\mathcal{W}_c(x+h) - \mathcal{W}_c(x) = O(|h|^{\xi}) \quad \text{and} \quad \mathcal{W}_s(x+h) - \mathcal{W}_s(x) = O(|h|^{\xi})$$

for each x, but satisfy neither

$$\mathcal{W}_c(x+h) - \mathcal{W}_c(x) = o(|h|^{\xi}) \quad \text{nor} \quad \mathcal{W}_s(x+h) - \mathcal{W}_s(x) = o(|h|^{\xi})$$

for any x.

Next let us recall the definition of the Takagi function [6]. Let θ^* be the 1-periodic function such that

$$\theta^*(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ 1-x & \text{if } \frac{1}{2} \leq x < 1 \end{cases}.$$

Then the Takagi function is defined by

$$\mathcal{T}(x) = \sum_{n=0}^{\infty} \frac{\theta^*(2^n x)}{2^n}. \tag{3}$$

It is another example of a nowhere differentiable continuous function.

Using the scaling exponents, Meyer defined two types of singularities of functions as follows [5]: a point x_0 in $\mathbf{R}^n$ is called a cusp singularity of a function f, when

$$H(f, x_0) = \beta(f, x_0) < \infty,$$

while a point x_0 in $\mathbf{R}^n$ is called an oscillating singularity of a function f, when

$$H(f, x_0) < \beta(f, x_0).$$

When a point x_0 is a cusp singularity of a function f, the pointwise Hölder exponent can be found by computing the size estimates on the wavelet coefficients of f inside the influence cone. Using this fact, we construct continuous functions which have a prescribed cusp singularity at each point x_0 in $\mathbf{R}$.

Daoudi and his team [2] studied the following problem which was raised by Lévy Véhel:

Let s be a function from $[0,1]$ to $[0,1]$. Under what conditions on s does there exist a continuous function f from $[0,1]$ to $\mathbf{R}$ such that $H(f,x) = s(x)$ for all x in $[0,1]$?

They solved the problem as follows: "For a function s from $[0,1]$ to $[0,1]$, there exist a continuous function f on $[0,1]$ such that $H(f,x) = s(x)$ for all x in $[0,1]$ if and only if s is a function which can be represented as a limit inferior of a sequence of continuous functions on $[0,1]$." Further, they constructed such f by various methods, – as the Weierstrass type function, using Schauder bases and using Iterated Function System.

On the other hand, Andersson [1] proved a similar characterization for a function s from $\mathbf{R}$ to $[0,\infty]$ and constructed f satisfying $H(f,x) = s(x)$ for all x in $\mathbf{R}$ by a method using orthogonal wavelets.

In the rest of the paper we study, for a given function on $\mathbf{R}$, various constructions of a function f satisfying

$$H(f,x) = \beta(f,x) = s(x), \quad x \in \mathbf{R},$$

using orthonormal wavelets in Section 2, as the Weierstrass type function in Section 3 and using spline functions in Section 4.

2. Construction using orthonormal wavelets

In this section, using orthonormal wavelets, we construct a continuous function which has a prescribed cusp singularity at each point in $\mathbf{R}$.

The following Lemma 1 is used in the proof of Theorems 1 and 2.

Lemma 1 *Let s be a function from $\mathbf{R}$ to $[0,\infty]$, which is the lower limit of a sequence of real continuous functions $\{t_l\}_{l\in\mathbf{N}}$. Then there exists a sequence $\{s_l\}_{l\in\mathbf{Z}_+}$ of infinitely differentiable non-negative functions with* *compact supports such that*

(i) $s(x) = \liminf_{l\to\infty} s_l(x), \quad x \in \mathbf{R}$,

(ii) *For each* x_0 *in* $\mathbf{R}$, *there exists a positive integer* l_0 *such that*

$$s_l(x) \geq \frac{1}{\sqrt{l+1}}, \quad l \geq l_0, \quad |x - x_0| \leq 1.$$

(iii) *There exists a sequence* $\{C_k\}_{k\in\mathbf{Z}_+} \subset (0,\infty)$ *such that*

$$\sup_{x\in\mathbf{R}} |s_l^{(k)}(x)| \leq C_k l^{k+1}, \quad l \in \mathbf{Z}_+,$$

where $s_l^{(k)}$ *is the k-th derivative of* s_l.

Proof. Let η be a non-negative infinitely differentiable function supported on $[-1,1]$ satisfying $\eta(x) = 1$ if $|x| \leq \frac{1}{4}$, $\sup_{x\in\mathbf{R}} \eta(x) = 1$ and $\int_{\mathbf{R}} \eta(x)\,dx = 1$. If we put

$$\tilde{t}_l(x) = \eta\left(\frac{x}{l}\right) \min\left(\max\left(t_l(x), \frac{1}{\sqrt{l+1}}\right), l\right), \quad l \in \mathbf{N},$$

it is easy to see that $\{\tilde{t}_l\}_{l\in\mathbf{N}}$ satisfies

$$\liminf_{l\to\infty} \tilde{t}_l(x) = s(x), \quad x \in \mathbf{R},$$

$$\tilde{t}_l(x) \geq \frac{1}{\sqrt{l+1}}, \quad |x| \leq \frac{l}{4},$$

$$\tilde{t}_l(x) = 0, \quad |x| \geq l$$

and

$$\sup_{x\in\mathbf{R}} \tilde{t}_l(x) \leq l.$$

Since each $\tilde{t}_l$ is uniformly continuous, we can choose a strictly increasing sequence of positive integers $\{p_l\}_{l\in\mathbf{N}}$ such that

$$\sup_{|x-y|\leq\frac{1}{p_l}} |\tilde{t}_l(x) - \tilde{t}_l(y)| \leq \frac{1}{l}, \quad l \in \mathbf{N}.$$

Under these circumstances, we define $s_l(x)$ for $l \in \mathbf{Z}_+$ and $x \in \mathbf{R}$ by

$$s_l(x) = \begin{cases} 0 & \text{if } 0 \le l < p_1 \\ \displaystyle\int_{\mathbf{R}} p_m \eta(p_m(x-y)) \tilde{t}_m(y)\, dy & \text{if } p_m \le l < p_{m+1}, \ m \in \mathbf{N}. \end{cases}$$

If we put $C_k = \int_{\mathbf{R}} |\eta^{(k)}(x)|\, dx$ for $k \in \mathbf{Z}_+$, then $\{s_l\}_{l \in \mathbf{Z}_+}$ satisfies the required properties (i), (ii) and (iii). To prove (i) we have

$$\begin{aligned} |s_l(x) - \tilde{t}_m(x)| &= \left| \int_{\mathbf{R}} p_m \eta(p_m(x-y)) \left(\tilde{t}_m(y) - \tilde{t}_m(x)\right) dy \right| \\ &\le \sup_{|x-y| \le \frac{1}{p_m}} |\tilde{t}_m(y) - \tilde{t}_m(x)| \int_{\mathbf{R}} \eta(y)\, dy \\ &\le \frac{1}{m}, \quad p_m \le l < p_{m+1}. \end{aligned}$$

This proves the desired result. To prove (ii) we choose $m_0 \in \mathbf{N}$ such that $\frac{m_0}{4} - \frac{1}{m_0} \ge |x_0| + 1$ and put $l_0 = p_{m_0}$. For a positive integer $l \ge l_0$, choose $m \in \mathbf{N}$ such that $p_m \le l < p_{m+1}$. Then if $|x - x_0| \le 1$, we have

$$\begin{aligned} s_l(x) &= \int_{\mathbf{R}} p_m \eta(p_m(x-y)) \tilde{t}_m(y)\, dy \\ &\ge \inf_{|x-y| \le \frac{1}{p_m}} \tilde{t}_m(y) \int_{\mathbf{R}} \eta(y)\, dy \\ &\ge \inf_{|y| \le |x_0| + 1 + \frac{1}{m}} \tilde{t}_m(y) \\ &\ge \inf_{|y| \le \frac{m}{4}} \tilde{t}_m(y) \\ &\ge \frac{1}{\sqrt{m+1}} \ge \frac{1}{\sqrt{l+1}}. \end{aligned}$$

To prove (iii) we choose $m \in \mathbf{N}$, for a given $l \in \mathbf{N}$, such that $p_m \le l < p_{m+1}$. Then we have

$$\begin{aligned} |s_l^{(k)}(x)| &= \left| \int_{\mathbf{R}} p_m^{k+1} \eta^{(k)}(p_m(x-y)) \tilde{t}_m(y)\, dy \right| \\ &\le p_m^k \sup_{|x-y| \le \frac{1}{p_m}} \tilde{t}_m(y) \int_{\mathbf{R}} |\eta^{(k)}(y)|\, dy \\ &\le C_k m p_m^k \le C_k l^{k+1}. \end{aligned}$$

□

Theorem 1 *Let s be a function from $\mathbf{R}$ to $[0,\infty]$, which is the lower limit of a sequence of continuous functions. Then there exists a sequence $\{s_l\}_{l\in\mathbf{Z}_+}$ of differentiable functions such that*

$$s(x) = \liminf_{l\to\infty} s_l(x), \quad x \in \mathbf{R} \tag{4}$$

and

$$\sup_{x\in\mathbf{R}} |s_l'(x)| \le C_1 l^2, \quad l \in \mathbf{Z}_+. \tag{5}$$

Let ψ be an orthonormal wavelet in the Schwartz class $\mathcal{S}(\mathbf{R})$. If we define a continuous function f by

$$f(x) = \sum_{l=2}^{\infty}\sum_{m=0}^{\infty} c(l,m)\psi(2^l x - m),$$

where

$$c(l,m) = \min\left(2^{-ls_l\left(\frac{m}{2^l}\right)}, 2^{-\frac{l}{\log l}}\right),$$

then we have

$$H(f,x_0) = \beta(f,x_0) = s(x_0)$$

at each point x_0 in $\mathbf{R}$.

Proof. The existence of $\{s_l\}_{l\in\mathbf{Z}_+}$ satisfying (4) and (5) follows from Lemma 1. Since

$$\begin{aligned}
&\lim_{j\to\infty} \sup_{|x-y|\le 2^{-\frac{j}{(\log j)^2}}} |s_j(x) - s_j(y)| \\
&\le \lim_{j\to\infty} \sup_{x\in\mathbf{R}} |s_j'(x)| \sup_{|x-y|\le 2^{-\frac{j}{(\log j)^2}}} |x-y| \\
&\le C_1 \lim_{j\to\infty} j^2 2^{-\frac{j}{(\log j)^2}} \\
&= 0,
\end{aligned}$$

$H(f,x_0) = s(x_0)$ at each point $x_0 \in \mathbf{R}$ (cf. [1] p. 441, proof of Theorem 1). We only need to compute the value of $\beta(f,x_0)$.

Let us assume f locally belongs to $\Gamma^s(x_0)$. Then by Theorem B, f locally belongs to $C_{x_0}^{s,s'}$ for some $s' < 0$. On the other hand, $\psi \in \mathcal{S}_0(\mathbf{R})$

(cf. [4, 2. Corollary 3.7]). By Theorem A, there exist two constants $C \in (0,\infty)$ and $\delta \in \left(0,\frac{1}{2}\right)$ such that

$$\left|\int f(x)\frac{1}{a}\overline{\psi\left(\frac{x-b}{a}\right)}\,dx\right| \le Ca^s\left(1+\frac{|b-x_0|}{a}\right)^{-s'},$$
$$0<a\le\delta,\quad |b-x_0|\le\delta. \tag{6}$$

Let j_0 be a positive integer such that $\frac{1}{2^{j_0}} \le \delta$. For every $j \ge j_0$, there exists $k_j \in \mathbf{Z}$ such that $\frac{k_j}{2^j} \le x_0 < \frac{k_j+1}{2^j}$ and we define a_j and b_j by $a_j = \frac{1}{2^j}$ and $b_j = \frac{k_j}{2^j}$. Then $|b_j - x_0| \le a_j$ and by (6), we have

$$\left|\int f(x)2^j\overline{\psi(2^jx-k_j)}\,dx\right| \le \frac{C2^{-s'}}{2^{js}},\quad j\ge j_0. \tag{7}$$

We estimate the left hand side of (7) as follows:

$$\begin{aligned}&\left|\int f(x)2^j\overline{\psi(2^jx-k_j)}\,dx\right|\\ &=\left|\sum_{l=2}^{\infty}\sum_{m=-\infty}^{\infty}c(l,m)\int\psi(2^lx-m)2^j\overline{\psi(2^jx-k_j)}\,dx\right|\\ &=c(j,k_j).\end{aligned} \tag{8}$$

By (7) and (8), $f \in \Gamma^s(x_0)$ implies

$$c(j,k_j)=\min\left(2^{-js_j\left(\frac{k_j}{2^j}\right)},2^{-\frac{j}{\log j}}\right)\le\frac{C2^{-s'}}{2^{js}},\quad j\ge j_0. \tag{9}$$

Observe that

$$\begin{aligned}\lim_{j\to\infty}\left|s_j\left(\frac{k_j}{2^j}\right)-s_j(x_0)\right| &\le \lim_{j\to\infty}\sup_{x\in\mathbf{R}}|s_j'(x)|\left(x_0-\frac{k_j}{2^j}\right)\\ &\le C_1\lim_{j\to\infty}\frac{j^2}{2^j}\\ &=0.\end{aligned}$$

By (9), we have

$$\begin{aligned}s&\le\liminf_{j\to\infty}\max\left(s_j\left(\frac{k_j}{2^j}\right),\frac{1}{\log j}\right)\\ &=\liminf_{j\to\infty}s_j\left(\frac{k_j}{2^j}\right)\end{aligned}$$

$$
\begin{aligned}
&= \liminf_{j\to\infty} s_j(x_0) + \lim_{j\to\infty}\left(s_j\left(\frac{k_j}{2^j}\right) - s_j(x_0)\right) \\
&= s(x_0).
\end{aligned}
$$

Therefore $\beta(f, x_0) \leq s(x_0) = H(f, x_0)$. Since $H(f, x_0) \leq \beta(f, x_0)$ is trivial, we have $H(f, x_0) = \beta(f, x_0) = s(x_0)$. □

3. Use of Weierstrass type functions

In this section, we construct the Weierstrass type continuous function which has a prescribed cusp singularity at each point in $\mathbf{R}$.

We begin with the following lemma.

Lemma 2 *Let $s \in [0, \infty]$, $l_0 \in \mathbf{Z}_+$ and $\{s_l\}_{l\in\mathbf{Z}_+} \subset \mathbf{R}$ be such that*

(a) $\liminf_{l\to\infty} s_l = s$,

(b) $s_l \geq \dfrac{1}{\sqrt{l+1}}, \quad l \geq l_0.$

Suppose $\lambda > 1$ and $\{\theta_l\}_{l\in\mathbf{Z}_+} \subset \mathbf{R}$ are chosen arbitrary.

(i) *If $m \in \mathbf{Z}_+$ and $\{\alpha_l\}_{l\in\mathbf{Z}_+}$ is a bounded sequence in $\mathbf{R}$ and if we define a continuous function f by*

$$f(x) = \sum_{l=0}^{\infty} \frac{\alpha_l l^m}{\lambda^{l s_l}} \sin(\lambda^l x + \theta_l), \quad x \in \mathbf{R},$$

then we have

$$H(f, x_0) \geq s$$

at each point x_0 in $\mathbf{R}$.

(ii) *If we define a continuous function g by*

$$g(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{l s_l}} \sin(\lambda^l x + \theta_l), \quad x \in \mathbf{R},$$

then we have

$$H(g, x_0) = \beta(g, x_0) = s$$

at each point x_0 in $\mathbf{R}$.

Proof. (i) By (b), f is a continuous function on $\mathbf{R}$ and hence we have only to show (i) when $s > 0$.

Let $x_0 \in \mathbf{R}$ be fixed arbitrary.

First, we consider the case $0 < s \leq 1$. Let $\varepsilon \in (0, s)$ be arbitrary. By (a), we can choose $l_0 \in \mathbf{Z}_+$ such that $s_l > s - \frac{\varepsilon}{2}$ for $l \geq l_0$ and we put $f_1(x) = \sum_{l=l_0}^{\infty} \frac{\alpha_l l^m}{\lambda^{l s_l}} \sin(\lambda^l x + \theta_l)$. To show $H(f, x_0) \geq s - \varepsilon$, it suffices to show $f_1 \in C^{s-\varepsilon}(x_0)$ since $H(f - f_1, x_0) = \infty$ is obvious. Let x be a real number such that $|x - x_0| < \frac{1}{\lambda^{l_0}}$ and choose $N \in \mathbf{Z}_+$ such that $\frac{1}{\lambda^{N+1}} \leq |x - x_0| < \frac{1}{\lambda^N}$. Then we have

$$\begin{aligned} |f_1(x) - f_1(x_0)| &= \left| \sum_{l=l_0}^{\infty} \frac{\alpha_l l^m}{\lambda^{l s_l}} (\sin(\lambda^l x + \theta_l) - \sin(\lambda^l x_0 + \theta_l)) \right| \\ &\leq \left| \sum_{l=l_0}^{N-1} \frac{\alpha_l l^m}{\lambda^{l s_l}} (\sin(\lambda^l x + \theta_l) - \sin(\lambda^l x_0 + \theta_l)) \right| \\ &\quad + \left| \sum_{l=N}^{\infty} \frac{\alpha_l l^m}{\lambda^{l s_l}} (\sin(\lambda^l x + \theta_l) - \sin(\lambda^l x_0 + \theta_l)) \right| \\ &= \mathrm{A}_1 + \mathrm{A}_2. \end{aligned} \tag{10}$$

Observe first that there exists a constant $M_1 \in (0, \infty)$ such that

$$|\alpha_l| l^m \leq M_1 \lambda^{\frac{l\varepsilon}{2}}, \quad l \geq l_0. \tag{11}$$

To estimate A_1 and A_2 we use (11) to obtain

$$\begin{aligned} \mathrm{A}_1 &\leq 2 \sum_{l=l_0}^{N-1} \frac{|\alpha_l| l^m}{\lambda^{l s_l}} \left| \cos\left(\frac{\lambda^l (x + x_0)}{2} + \theta_l \right) \sin\left(\frac{\lambda^l (x - x_0)}{2} \right) \right| \\ &\leq \sum_{l=l_0}^{N-1} |\alpha_l| l^m \lambda^{l(1-s_l)} |x - x_0| \\ &\leq M_1 \sum_{l=l_0}^{N-1} \lambda^{l(1-s+\varepsilon)} |x - x_0| \\ &= \frac{M_1 \lambda^{l_0(1-s+\varepsilon)} (\lambda^{(N-l_0)(1-s+\varepsilon)} - 1)}{\lambda^{1-s+\varepsilon} - 1} |x - x_0| \\ &\leq \frac{M_1 \lambda^{N(1-s+\varepsilon)}}{\lambda^{1-s+\varepsilon} - 1} |x - x_0| \\ &\leq \frac{M_1}{\lambda^{1-s+\varepsilon} - 1} |x - x_0|^{s-\varepsilon}, \end{aligned}$$

$$\begin{aligned}
\mathrm{A}_2 &\le 2\sum_{l=N}^{\infty} \frac{|\alpha_l| l^m}{\lambda^{l s_l}} \left| \cos\left(\frac{\lambda^l(x+x_0)}{2} + \theta_l\right) \sin\left(\frac{\lambda^l(x-x_0)}{2}\right) \right| \\
&\le 2\sum_{l=N}^{\infty} \frac{|\alpha_l| l^m}{\lambda^{l s_l}} \\
&\le 2M_1 \sum_{l=N}^{\infty} \frac{1}{\lambda^{l(s-\varepsilon)}} \\
&= \frac{\frac{2M_1}{\lambda^{N(s-\varepsilon)}}}{1-\frac{1}{\lambda^{s-\varepsilon}}} \\
&\le \frac{2M_1\lambda^{2(s-\varepsilon)}}{\lambda^{s-\varepsilon}-1}|x-x_0|^{s-\varepsilon}.
\end{aligned}$$

The estimates for A_1 and A_2 with (10) show that there exists a constant $M_2 \in (0,\infty)$ such that

$$|f_1(x) - f_1(x_0)| \le M_2|x-x_0|^{s-\varepsilon}, \quad |x-x_0| < \frac{1}{\lambda^{l_0}}.$$

Thus $H(f_1, x_0) \ge s-\varepsilon$ and hence $H(f, x_0) \ge s-\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $H(f, x_0) \ge s$.

Next, we consider the case $n < s \le n+1$ for some $n \in \mathbf{N}$. In this case, f is n-times continuously differentiable on $\mathbf{R}$ and we have

$$f^{(n)}(x) = \sum_{l=0}^{\infty} \frac{\alpha_l l^m}{\lambda^{l(s_l-n)}} \sin\left(\lambda^l x + \theta_l + \frac{n\pi}{2}\right).$$

Thus $H(f^{(n)}, x_0) \ge s-n$ by an argument similar to the case where $0 < s \le 1$ and hence $H(f, x_0) \ge s$ holds even for $1 < s < \infty$.

Finally, we consider the case $s = \infty$. In this case, f is obviously infinitely differentiable at x_0 and hence $H(f, x_0) = \infty$.

(ii) $H(g, x_0) \ge s$ follows from (i), if we put $\alpha_l = 1$ for $l \in \mathbf{Z}_+$ and $m = 0$ in (i).

For $\beta(g, x_0)$, let us assume g locally belongs to $\Gamma^\rho(x_0)$. Let ψ be a function in $\mathcal{S}_0(\mathbf{R})$ such that $\hat{\psi}(\xi) = 0$ if $|\xi - 1| \ge \frac{\lambda-1}{\lambda}$ and $\hat{\psi}(1) = 2$. Then there exist two constants $M_3 \in (0,\infty)$ and $\eta \in (0,1]$ such that

$$\left| \int g(x)\frac{1}{a}\psi\left(\frac{x-x_0}{a}\right) dx \right| \le M_3 a^\rho, \quad 0 < a \le \eta. \tag{12}$$

Let j_0 be a non-negative integer such that $\frac{1}{\lambda^{j_0}} \le \eta$. For every $j \ge j_0$, we put $a_j = \frac{1}{\lambda^j}$. By (12), we have

$$\left|\int g(x)\lambda^j\psi(\lambda^j(x-x_0))\,dx\right| \le \frac{M_3}{\lambda^{j\rho}}, \quad j \ge j_0. \tag{13}$$

We estimate the left hand side of (13) as follows:

$$\begin{aligned}
&\left|\int g(x)\lambda^j\psi(\lambda^j(x-x_0))\,dx\right| \\
&= \left|\int \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l}} \sin(\lambda^{l-j}x + \lambda^l x_0 + \theta_l)\psi(x)\,dx\right| \\
&= \left|\sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l}} \int \frac{e^{i(\lambda^{l-j}x+\lambda^l x_0+\theta_l)} - e^{-i(\lambda^{l-j}x+\lambda^l x_0+\theta_l)}}{2i}\psi(x)\,dx\right| \\
&= \left|\sum_{l=0}^{\infty} \frac{e^{i(\lambda^l x_0+\theta_l)}\hat{\psi}(-\lambda^{l-j}) - e^{-i(\lambda^l x_0+\theta_l)}\hat{\psi}(\lambda^{l-j})}{2i\lambda^{ls_l}}\right| \\
&= \frac{|\hat{\psi}(1)|}{2\lambda^{js_j}} \\
&= \frac{1}{\lambda^{js_j}}.
\end{aligned} \tag{14}$$

By (13) and (14), $g \in \Gamma^\rho(x_0)$ implies $\frac{1}{\lambda^{js_j}} \le \frac{M_3}{\lambda^{j\rho}}$ for every $j \ge j_0$ and hence $\rho \le \liminf_{j\to\infty} s_j = s \le H(g,x_0)$. Therefore $\beta(g,x_0) \le s \le H(g,x_0)$. Since $H(g,x_0) \le \beta(g,x_0)$ is trivial, we have $H(g,x_0) = \beta(g,x_0) = s$. □

Theorem 2 *Let s be a function from $\mathbf{R}$ to $[0,\infty]$, which is the lower limit of a sequence of continuous functions and let $\{s_l\}_{l\in\mathbf{Z}_+}$ be a sequence of continuous functions satisfying part* (i), (ii) *and* (iii) *of Lemma* 1.

Suppose $\lambda > 1$ and $\{\theta_l\}_{l\in\mathbf{Z}_+} \subset \mathbf{R}$ are chosen arbitrary. If we define a continuous function f by

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l(x)}} \sin(\lambda^l x + \theta_l),$$

then we have

$$H(f,x_0) = \beta(f,x_0) = s(x_0)$$

at each point x_0 in $\mathbf{R}$.

Proof. First, we consider the case $n \le s(x_0) < n+1$ for some $n \in \mathbf{Z}_+$. Using the Taylor expansion we have

$$\frac{1}{\lambda^{ls_l(x)}} = \frac{1}{\lambda^{ls_l(x_0)}} + \sum_{j=1}^{n} \frac{1}{j!} \frac{d^j}{dx^j} \frac{1}{\lambda^{ls_l(x)}}\bigg|_{x=x_0} (x-x_0)^j + \frac{1}{(n+1)!} \frac{d^{n+1}}{dx^{n+1}} \frac{1}{\lambda^{ls_l(x)}}\bigg|_{x=\xi_l} (x-x_0)^{n+1}, \tag{15}$$

where $\xi_l \in (\min(x, x_0), \max(x, x_0))$. It goes without saying that if $n = 0$ the second term in the right hand side of (15) does not appear. By (15), we can write

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l(x)}} \sin(\lambda^l x + \theta_l) = f_1(x) + f_2(x) + f_3(x), \tag{16}$$

where

$$f_1(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l(x_0)}} \sin(\lambda^l x + \theta_l), \tag{17}$$

$$f_2(x) = \sum_{l=0}^{\infty} \sum_{j=1}^{n} \frac{1}{j!} \frac{d^j}{dx^j} \frac{1}{\lambda^{ls_l(x)}}\bigg|_{x=x_0} \sin(\lambda^l x + \theta_l)(x-x_0)^j \tag{18}$$

and

$$f_3(x) = \frac{1}{(n+1)!} \sum_{l=0}^{\infty} \frac{d^{n+1}}{dx^{n+1}} \frac{1}{\lambda^{ls_l(x)}}\bigg|_{x=\xi_l} \sin(\lambda^l x + \theta_l)(x-x_0)^{n+1}, \tag{19}$$

where $\xi_l \in (\min(x, x_0), \max(x, x_0))$.

By part (ii) of Lemma 2, $H(f_1, x_0) = \beta(f_1, x_0) = s(x_0)$ follows at once. f_2 does not appear if $n = 0$, and if $n \ge 1$ we have

$$f_2(x) = \sum_{l=0}^{\infty} \sum_{j=1}^{n} \sum_{k=1}^{j} \sum_{(*)_j} \frac{1}{j!} \frac{(-\log \lambda)^k l^k \alpha_{j,i_1,\dots,i_k} s_l^{(i_1)}(x_0) \dots s_l^{(i_k)}(x_0)}{\lambda^{ls_l(x_0)}} \cdot \sin(\lambda^l x + \theta_l)(x-x_0)^j, \tag{20}$$

where $\sum_{(*)_j}$ mean the summation under the condition $i_1 + \cdots + i_k = j$ with $i_1 \le \cdots \le i_k$ and $\{\alpha_{j,i_1,\dots,i_k}\}$ are positive integers satisfying $\sum_{(*)_j} \alpha_{j,i_1,\dots,i_k} \le$

$(k+1)^j$. By (20), part (iii) of Lemma 1 and part (i) of Lemma 2, we can deduce that $H(f_2, x_0) \geq s(x_0) + 1$. For f_3, we have

$$f_3(x) = \frac{1}{(n+1)!} \sum_{l=0}^{\infty} \sum_{k=1}^{n+1} \sum_{(*)_{n+1}} \frac{(-\log\lambda)^k l^k \alpha_{n+1,i_1,\ldots,i_k} s_l^{(i_1)}(\xi_l) \ldots s_l^{(i_k)}(\xi_l)}{\lambda^{l s_l(\xi_l)}} \cdot \sin(\lambda^l x + \theta_l)(x - x_0)^{n+1}, \quad (21)$$

where $\sum_{(*)_{n+1}}$ mean the summation under the condition $i_1 + \cdots + i_k = n+1$ with $i_1 \leq \cdots \leq i_k$ and $\{\alpha_{n+1,i_1,\ldots,i_k}\}$ are positive integers satisfying $\sum_{(*)_{n+1}} \alpha_{n+1,i_1,\ldots,i_k} \leq (k+1)^{n+1}$. By (21) and part (iii) of Lemma 1, we can deduce that $H(f_3, x_0) \geq n+1$. By the estimates for f_1, f_2 and f_3, and (16), we can conclude that $H(f, x_0) = \beta(f, x_0) = s(x_0)$.

Next, we consider the case $s(x_0) = \infty$. Let n be a positive integer and let $f = f_1 + f_2 + f_3$, where f_1, f_2 and f_3 are defined by (17), (18) and (19), respectively. But in this case, we have $H(f_1, x_0) = H(f_2, x_0) = \infty$ and $H(f_3, x_0) \geq n+1$ by part (iii) of Lemma 1 and part (i) of Lemma 2, since $\liminf_{l\to\infty} s_l(x_0) = \infty$. By the estimates for f_1, f_2 and f_3, and (16), we have $H(f, x_0) \geq n+1$. Since n is arbitrary, we can conclude that $H(f, x_0) = \beta(f, x_0) = s(x_0)$ even for $s(x_0) = \infty$. □

In the case where s is a continuous function, we have the following result.

Theorem 3 *Let s be a continuous function from $\mathbf{R}$ to $(0, \infty)$ such that*

$$s(x_0) < H(s, x_0)$$

at each point x_0 in $\mathbf{R}$. Suppose $\lambda > 1$ and $\{\theta_l\}_{l\in\mathbf{Z}_+} \subset \mathbf{R}$ are chosen arbitrary. If we define a continuous function f by

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{l s(x)}} \sin(\lambda^l x + \theta_l),$$

then we have

$$H(f, x_0) = \beta(f, x_0) = s(x_0)$$

at each point x_0 in $\mathbf{R}$.

Proof. Let $x_0 \in \mathbf{R}$ be fixed arbitrary and let x be a real number such that

$|x - x_0| < 1$. Then we have

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls(x_0)}} \sin(\lambda^l x + \theta_l) + \sum_{l=0}^{\infty} \left(\frac{1}{\lambda^{ls(x)}} - \frac{1}{\lambda^{ls(x_0)}} \right) \sin(\lambda^l x + \theta_l)$$
$$= f_1(x) + f_2(x). \tag{22}$$

By part (ii) of Lemma 2, $H(f_1, x_0) = \beta(f_1, x_0) = s(x_0)$ follows at once. Let ε be a positive number such that $s(x_0)+\varepsilon < H(s, x_0)$ and $s(x_0)+\varepsilon \notin \mathbf{N}$. Then $s \in C^{s(x_0)+\varepsilon}(x_0)$ and there exist a polynomial P of degree at most $[s(x_0) + \varepsilon]$, two constants $C \in (0, \infty)$ and $\delta \in (0, 1)$ such that

$$s(x) = s(x_0) + P(x - x_0) + Q(x - x_0)$$

and

$$|Q(x - x_0)| \leq C|x - x_0|^{s(x_0)+\varepsilon}, \quad |x - x_0| \leq \delta.$$

To estimate f_2, using the mean value theorem, we write

$$\frac{1}{\lambda^{ls(x)}} - \frac{1}{\lambda^{ls(x_0)}} = \frac{(-\log\lambda) l (s(x) - s(x_0))}{\lambda^{l\tau_l}},$$

where $\tau_l \in [\min(s(x), s(x_0)), \max(s(x), s(x_0))]$. Then we have

$$\left| f_2(x) - \left((-\log\lambda) \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\tau_l}} \sin(\lambda^l x + \theta_l) \right) P(x - x_0) \right|$$
$$= (\log\lambda) \left| \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\tau_l}} \sin(\lambda^l x + \theta_l) \right| |Q(x - x_0)|$$
$$\leq C(\log\lambda) \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\tau_l}} |x - x_0|^{s(x_0)+\varepsilon}.$$

Hence $H(f_2, x_0) \geq s(x_0) + \varepsilon$. By the estimates for f_1 and f_2, and (22), we can conclude that $H(f, x_0) = \beta(f, x_0) = s(x_0)$. □

Corollary 1 *Each point in* $\mathbf{R}$ *is a cusp singularity of the Weierstrass functions.*

Proof. Let $\mathcal{W}_c$ and $\mathcal{W}_s$ be the Weierstrass functions (for the definitions of $\mathcal{W}_c$ and $\mathcal{W}_s$, see (1) and (2)). If we put $\lambda = b$, $s(x) = \frac{\log(\frac{1}{a})}{\log b}$ and $\theta_l = \frac{\pi}{2}$ for $l \in \mathbf{Z}_+$ or $\theta_l = 0$ for $l \in \mathbf{Z}_+$, then we have $H(\mathcal{W}_c, x) = \beta(\mathcal{W}_c, x) = \frac{\log(\frac{1}{a})}{\log b} = H(\mathcal{W}_s, x) = \beta(\mathcal{W}_s, x)$ at each point x in $\mathbf{R}$ from Theorem 3. □

4. Construction using spline functions

In this section, using spline functions [9], we construct a continuous function which has a prescribed cusp singularity at each point in $\mathbf{R}$.

Let a be a positive real number and for a positive integer n, $C^n(\mathbf{R})$ be the set of all functions f defined on $\mathbf{R}$ such that all the derivatives of f up to order n exist and $f^{(n)}$ is continuous on $\mathbf{R}$. For $n = 0$, we mean the set of all continuous functions on $\mathbf{R}$. A spline of order n with nodes in $a\mathbf{Z}$ is a function f defined on $\mathbf{R}$ which is of class $C^{n-1}(\mathbf{R})$ and is a polynomial of degree at most n when restricted to each interval of the form $[ka, (k+1)a]$ for an integer k.

Lemma 3 *For a positive integer n, suppose θ is the 1-periodic spline of order n with nodes in $\frac{1}{\lambda}\mathbf{Z}$, which is not a constant function, where λ is a positive integer greater than 1. If we define a continuous function f by*

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls}} \theta(\lambda^l x),$$

where $0 < s \leq n$, then we have

$$H(f, x_0) = \beta(f, x_0) = s$$

at each point x_0 in $\mathbf{R}$.

Proof. Let $x_0 \in \mathbf{R}$ be fixed arbitrary. For $H(f, x_0)$, we divide the proof into the following two cases.

First, we consider the case $0 < s \leq 1$. We first prove that $H(f, x_0) \geq s$ in the case $s < 1$. Let x be a real number such that $|x - x_0| < 1$ and choose $N \in \mathbf{Z}_+$ such that $\frac{1}{\lambda^{N+1}} \leq |x - x_0| < \frac{1}{\lambda^N}$. Then we have

$$\begin{aligned} |f(x) - f(x_0)| &= \left| \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls}} (\theta(\lambda^l x) - \theta(\lambda^l x_0)) \right| \\ &\leq \left| \sum_{l=0}^{N-1} \frac{1}{\lambda^{ls}} (\theta(\lambda^l x) - \theta(\lambda^l x_0)) \right| \\ &\qquad + \left| \sum_{l=N}^{\infty} \frac{1}{\lambda^{ls}} (\theta(\lambda^l x) - \theta(\lambda^l x_0)) \right| \\ &= \mathrm{A}_1 + \mathrm{A}_2. \end{aligned} \tag{23}$$

To estimate A_2 we have

$$\begin{aligned}
A_2 &\leq \sum_{l=N}^{\infty} \frac{1}{\lambda^{ls}} |\theta(\lambda^l x) - \theta(\lambda^l x_0)| \\
&\leq 2 \sup_{x\in\mathbf{R}} |\theta(x)| \sum_{l=N}^{\infty} \frac{1}{\lambda^{ls}} \\
&= \frac{\frac{2\sup_{x\in\mathbf{R}}|\theta(x)|}{\lambda^{Ns}}}{1-\frac{1}{\lambda^s}} \\
&\leq \frac{2\lambda^{2s}\sup_{x\in\mathbf{R}}|\theta(x)|}{\lambda^s - 1}|x - x_0|^s.
\end{aligned}$$

Observe that the estimate for A_2 holds even for $s = 1$. To estimate A_1 we use the relation

$$|\theta(x) - \theta(y)| \leq C_1|x - y|,$$

where $C_1 = \sup_{x\in\mathbf{R}\setminus\frac{\mathbf{Z}}{\lambda}} |\theta'(x)| < \infty$. Then we have

$$\begin{aligned}
A_1 &\leq \sum_{l=0}^{N-1} \frac{1}{\lambda^{ls}} |\theta(\lambda^l x) - \theta(\lambda^l x_0)| \\
&\leq C_1 \sum_{l=0}^{N-1} \lambda^{l(1-s)} |x - x_0| \\
&= \frac{C_1(\lambda^{N(1-s)} - 1)}{\lambda^{1-s} - 1}|x - x_0| \\
&\leq \frac{C_1}{\lambda^{1-s} - 1}|x - x_0|^s.
\end{aligned}$$

$H(f, x_0) \geq s$ now follows from the estimates for A_1 and A_2, and (23).

To prove that $H(f, x_0) \geq s$ when $s = 1$ we recall that (23) and the estimate for A_2 are still valid in this case. Thus we need to find an upper bound for A_1. Let $\varepsilon > 0$ be fixed arbitrary. To estimate A_1 we write

$$\begin{aligned}
A_1 &\leq \sum_{l=0}^{N-1} \frac{1}{\lambda^{l}} |\theta(\lambda^l x) - \theta(\lambda^l x_0)| \\
&\leq C_1 N |x - x_0|
\end{aligned}$$

$$\leq \frac{C_1}{\log\lambda}|x-x_0|\log\frac{1}{|x-x_0|}$$
$$\leq C_2|x-x_0|^{1-\varepsilon}.$$

for some constant $C_2 \in (0,\infty)$. Hence there exists a constant $C_3 \in (0,\infty)$ such that

$$|f(x)-f(x_0)| \leq C_3|x-x_0|^{1-\varepsilon}.$$

Therefore $H(f,x_0) \geq 1-\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $H(f,x_0) \geq s$ holds even for $s=1$.

Next, we consider the case $m < s \leq m+1$ for some positive integer $m < n$. Since $f^{(m)}(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{l(s-m)}}\theta^{(m)}(\lambda^l x)$, $H(f^{(m)},x_0) \geq s-m$ by an argument similar to the case where $0 < s \leq 1$. Therefore $H(f,x_0) \geq s$ holds even for $1 < s \leq n$.

For $\beta(f,x_0)$, let us assume f locally belongs to $\Gamma^{\rho}(x_0)$. Then by Theorem B, f locally belongs to $C^{\rho,\rho'}_{x_0}$ for some $\rho' < 0$. Let M be an integer greater than ρ. Let ψ be a function supported on $[0,1]$, has $M-1$ vanishing moments. By Theorem A, there exist two constants $C_4 \in (0,\infty)$ and $\delta \in (0,1]$ such that

$$\left|\int f(x)\frac{1}{a}\psi\left(\frac{x-b}{a}\right)dx\right| \leq C_4 a^{\rho}\left(1+\frac{|b-x_0|}{a}\right)^{-\rho'},$$
$$0 < a \leq \delta, \quad |b-x_0| \leq \delta. \tag{24}$$

Let j_0 be a non-negative integer such that $\frac{1}{\lambda^{j_0}} \leq \delta$. For every $j \geq j_0$, there exists $k_j \in \mathbf{Z}$ such that $\frac{k_j}{\lambda^j} \leq x_0 < \frac{k_j+1}{\lambda^j}$ and we define a_j and b_j by $a_j = \frac{1}{\lambda^j}$ and $b_j = \frac{k_j}{\lambda^j}$. Then $|b_j - x_0| \leq a_j$ and by (24), we have

$$\left|\int f(x)\lambda^j\psi(\lambda^j x - k_j)\,dx\right| \leq \frac{C_4 2^{-\rho'}}{\lambda^{j\rho}}, \quad j \geq j_0. \tag{25}$$

We estimate the left hand side of (25) as follows:

$$\left|\int f(x)\lambda^j\psi(\lambda^j x - k_j)\,dx\right| = \left|\int_0^1 \sum_{l=0}^{\infty}\frac{1}{\lambda^{ls}}\theta(\lambda^{l-j}(x+k_j))\psi(x)\,dx\right|.$$

Then we have

$$\begin{aligned}
&\sum_{l=0}^{\infty} \frac{1}{\lambda^{ls}} \theta(\lambda^{l-j}(x+k_j)) \\
&= \sum_{l=0}^{j-1} \frac{1}{\lambda^{ls}} \theta(\lambda^{l-j}(x+k_j)) + \sum_{l=j}^{\infty} \frac{1}{\lambda^{ls}} \theta(\lambda^{l-j}(x+k_j)) \\
&= \frac{1}{\lambda^{js}} \sum_{l=1}^{j} \lambda^{ls} \theta\left(\frac{x+k_j}{\lambda^l}\right) + \frac{1}{\lambda^{js}} \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls}} \theta(\lambda^l x).
\end{aligned}$$

Since θ is a spline of order n with nodes in $\frac{1}{\lambda}\mathbf{Z}$, $\sum_{l=1}^{j} \lambda^{ls}\theta\left(\frac{x+k_j}{\lambda^l}\right)$ is a polynomial of degree at most n on the support of ψ. Thus $\frac{1}{\lambda^{js}} \int_0^1 \sum_{l=1}^{j} \lambda^{ls}\theta\left(\frac{x+k_j}{\lambda^l}\right)\psi(x)\,dx = 0$. Hence

$$\left|\int f(x)\lambda^j\psi(\lambda^j x - k_j)\,dx\right| = \frac{1}{\lambda^{js}}\left|\int_0^1 \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls}}\theta(\lambda^l x)\psi(x)\,dx\right|. \tag{26}$$

Since $\sum_{l=0}^{\infty} \frac{1}{\lambda^{ls}}\theta(\lambda^l x)$ is not a polynomial, we can select a wavelet ψ such that

$$\int_0^1 \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls}}\theta(\lambda^l x)\psi(x)\,dx = 1. \tag{27}$$

By (25), (26) and (27), $f \in \Gamma^\rho(x_0)$ implies $\frac{1}{\lambda^{js}} \le \frac{C_4 2^{-\rho'}}{\lambda^{j\rho}}$ for every $j \ge j_0$ and hence $\rho \le s \le H(f, x_0)$. Therefore $\beta(f, x_0) \le s \le H(f, x_0)$. Since $H(f, x_0) \le \beta(f, x_0)$ is trivial, we have $H(f, x_0) = \beta(f, x_0) = s$. □

In the case where s is a continuous function, we have the following result.

Theorem 4 *For a positive integer n, suppose θ is the* 1*-periodic spline of order n with nodes in $\frac{1}{\lambda}\mathbf{Z}$, which is not a constant function, where λ is a positive integer greater than* 1*. Let s be a continuous function from $\mathbf{R}$ to $(0, n]$ such that*

$$s(x_0) < H(s, x_0)$$

at each point x_0 in $\mathbf{R}$. If we define a continuous function f by

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls(x)}}\theta(\lambda^l x),$$

then we have

$$H(f, x_0) = \beta(f, x_0) = s(x_0)$$

at each point x_0 in $\mathbf{R}$.

Proof. Let $x_0 \in \mathbf{R}$ be fixed arbitrary and let x be a real number such that $|x - x_0| < 1$. Then we have

$$\begin{aligned} f(x) &= \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls(x_0)}} \theta(\lambda^l x) + \sum_{l=0}^{\infty} \left(\frac{1}{\lambda^{ls(x)}} - \frac{1}{\lambda^{ls(x_0)}} \right) \theta(\lambda^l x) \\ &= f_1(x) + f_2(x). \end{aligned} \tag{28}$$

By Lemma 3, $H(f_1, x_0) = \beta(f_1, x_0) = s(x_0)$ follows at once. Let ε be a positive number such that $s(x_0) + \varepsilon < H(s, x_0)$ and $s(x_0) + \varepsilon \notin \mathbf{N}$. Then $s \in C^{s(x_0)+\varepsilon}(x_0)$ and there exist a polynomial P of degree at most $[s(x_0) + \varepsilon]$, two constants $C \in (0, \infty)$ and $\delta \in (0, 1)$ such that

$$s(x) = s(x_0) + P(x - x_0) + Q(x - x_0)$$

and

$$|Q(x - x_0)| \le C|x - x_0|^{s(x_0)+\varepsilon}, \quad |x - x_0| \le \delta.$$

To estimate f_2, using the mean value theorem, we write

$$\frac{1}{\lambda^{ls(x)}} - \frac{1}{\lambda^{ls(x_0)}} = \frac{(-\log \lambda) l (s(x) - s(x_0))}{\lambda^{l\tau_l}},$$

where $\tau_l \in [\min(s(x), s(x_0)), \max(s(x), s(x_0))]$. Then we have

$$\begin{aligned} &\left| f_2(x) - \left((-\log \lambda) \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\tau_l}} \theta(\lambda^l x) \right) P(x - x_0) \right| \\ &\quad = (\log \lambda) \left| \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\tau_l}} \theta(\lambda^l x) \right| |Q(x - x_0)| \\ &\quad \le C(\log \lambda) \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\tau_l}} \sup_{x \in \mathbf{R}} |\theta(x)| |x - x_0|^{s(x_0)+\varepsilon}. \end{aligned}$$

Hence $H(f_2, x_0) \ge s(x_0) + \varepsilon$. By the estimates for f_1 and f_2, and (28), we can conclude that $H(f, x_0) = \beta(f, x_0) = s(x_0)$. □

Corollary 2 *Each point in $\mathbf{R}$ is a cusp singularity of the Takagi function.*

Proof. Let $\mathcal{T}$ be the Takagi function (for the definition of $\mathcal{T}$, see (3)). If we put $\lambda = 2$, $s(x) = 1$ and $\theta = \theta^*$, then we have $H(\mathcal{T}, x) = \beta(\mathcal{T}, x) = 1$ at each point x in $\mathbf{R}$ from Theorem 4. $\square$

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