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Grassmann geometry on the 3-dimensional unimodular Lie groups II

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Abstract. We study the Grassmann geometry of surfaces in the special real linear group $SL(2, \mathbb{R})$.

Key words: Grassmann geometry, unimodular Lie group, special linear group, totally geodesic surface, flat surface, minimal surface, surface with constant mean curvature.

1. Introduction and the outline of Part I

This is a continuation of our previous paper [3] named Part I. Let G be a 3-dimensional unimodular Lie group with its Lie algebra $\mathfrak{g}$ and g a left invariant metric on G. Then the Riemannian metric g induces an inner product on $\mathfrak{g}$, where $\mathfrak{g}$ is identified with the tangent space T_eG at the unit element e of G. By J. Milnor [5], it is known that such a Lie group G is locally isomorphic to either of the unitary group $SU(2)$, the special linear group $SL(2, \mathbb{R})$, the group $E(2)$ of rigid motions of the Euclidean 2-plane, the group $E(1, 1)$ of rigid motions of the Minkowski 2-plane, the 3-dimensional Heisenberg group H_3, and the 3-dimensional real vector group $\mathbb{R}^3$, and moreover known that there exist an orthonormal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{g}$ and real numbers λ_1, λ_2, λ_3 such that

$$[E_2, E_3] = \lambda_1 E_1, \quad [E_3, E_1] = \lambda_2 E_2, \quad [E_1, E_2] = \lambda_3 E_3, \tag{1.1}$$

where the signatures of $\lambda_1, \lambda_2, \lambda_3$ determine the type of G locally. The constants $\lambda_1, \lambda_2, \lambda_3$ are in this paper called the *Milnor constants* of (G, g).

In this article we study the Grassmann geometry on such a Riemannian homogeneous manifold (G, g). Generally, the Grassmann geometry on a Riemannian homogeneous manifold (M, g) is defined as follows. Let $I_o(M, g)$ be the identity component of the isometry group of (M, g) and for an integer r, consider the Grassmann bundle $\mathrm{Gr}^r(TM)$ over M which consists

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of all r-dimensional linear subspaces of the tangent spaces of M. Then the Lie group $I_o(M,g)$ acts naturally on $\mathrm{Gr}^r(TM)$ by the differential of mappings. For an $I_o(M,g)$-orbit $\mathcal{O}$, a connected submanifold S of M is called an *$\mathcal{O}$-submanifold* if all tangent spaces of S belong to $\mathcal{O}$ and the geometry of $\mathcal{O}$-submanifolds is called the $\mathcal{O}$-geometry on (M,g). Grassmann geometry is a collective name for such a geometry, and its fundamental problem is to classify all the $\mathcal{O}$-geometries such that $\mathcal{O}$-submanifolds exist and moreover for each nonempty $\mathcal{O}$-geometry, to find typical $\mathcal{O}$-submanifolds such as minimal submanifolds, submanifolds with constant mean curvature, and so on. On this point of view, we consider the Grassmann geometry on a 3-dimensional unimodular Lie group (G,g) with a left invariant metric g, and in the previous Part I, we solve this problem for the almost cases of G where $r=2$, except one of the cases when $G=SL(2,\mathbb{R})$. (See Section 3 through Section 5 in Part I.) In this part II, we will study the exceptional case, for which the $\mathcal{O}$-geometry is nonempty, and will give an example of $\mathcal{O}$-surfaces with nonzero constant mean curvature.

We now retain the notations in Part I and recall the exceptional case. Let $G=SL(2,\mathbb{R})$ and take a left invariant metric g on G. Then it holds that $\dim I_o(G,g)=3$ or 4 and we can assume that the Milnor constants satisfy $\lambda_1<0<\lambda_2<\lambda_3$ or $\lambda_1<0<\lambda_2=\lambda_3$ according as $\dim I_o(G,g)=3$ or 4. Since the Grassmann geometry for the first case was solved in Part I (Section 5, 5.1), we consider the second case, which we called the Grassmann geometry of isotropy type $SO(2)$ in Part I. Put $\lambda_2=\lambda_3=\lambda$ and let $K_o=\{\varphi\in I_o(G,g);\ \varphi(e)=e\}$. Then if we identify $\mathfrak{g}$ $(=T_eG)$ with $\mathbb{R}^3$ by fixing the orthonomal basis $\{E_1,E_2,E_3\}$ of $\mathfrak{g}$, the K_o-action on $\mathfrak{g}$ is equal to the natural $SO(2)$-action on $\mathbb{R}^3$ which fixes the E_1-axis and acts on the (E_2E_3)-plane as rotations. Let $\mathrm{Gr}^2(\mathfrak{g})$, $\mathbb{R}P^2(\mathfrak{g})$, $S^2(\mathfrak{g})$ be the Grassmann manifold of 2-planes in $\mathfrak{g}$, the real projective 2-space over $\mathfrak{g}$, and the unit sphere in $\mathfrak{g}$ centered at the origin, respectively, and identify $\mathrm{Gr}^2(\mathfrak{g})$ with $\mathbb{R}P^2(\mathfrak{g})$ by the correspondence of a plane in $\mathfrak{g}$ to its orthogonal line in $\mathfrak{g}$. Moreover regard $\mathbb{R}P^2(\mathfrak{g})$ as the quotient space $S^2(\mathfrak{g})/_\sim$ where the equivalence relation $p\sim q$ implies that p and q are anti-podal each other. Then the space of $I_o(G,g)$-orbits in $\mathrm{Gr}^2(TG)$ is identified with the space of K_o-orbits in $\mathbb{R}P^2(\mathfrak{g})$, and moreover such a K_o-orbit is represented by a small circle in $S^2(\mathfrak{g})$ which is parallel to the (E_2E_3)-plane and which has the height h from the plane where $0\le h\le 1$. Hence the orbit space of $I_o(G,g)$-orbits in $\mathrm{Gr}^2(TG)$ is parametrized by the height h where $0\le h\le 1$. Denote by $\mathcal{O}(h)$ the

$I_o(G,g)$-orbit with height h.

In Part I we solved the fundamental problem for the $\mathcal{O}(h)$-geometries except the case that $0 < h < \sqrt{\frac{\lambda}{\lambda-\lambda_1}}$. In fact, if $h = 1$, the $\mathcal{O}(1)$-geometry is empty, and if $0 \le h < 1$, the $\mathcal{O}(h)$-geometry is non-empty. Moreover, if $h = 0$, there exists an $\mathcal{O}(0)$-surface with any constant mean curvature; if $h = \sqrt{\frac{\lambda}{\lambda-\lambda_1}}$, any $\mathcal{O}(\sqrt{\frac{\lambda}{\lambda-\lambda_1}})$-surface with constant mean curvature is minimal and there exists such a minimal $\mathcal{O}(\sqrt{\frac{\lambda}{\lambda-\lambda_1}})$-surface; if $\sqrt{\frac{\lambda}{\lambda-\lambda_1}} < h < 1$, there exists no $\mathcal{O}(h)$-surface with constant mean curvature. (See Theorem 5.19 in Part I.)

From now on we consider the exceptional case of $\mathcal{O}(h)$-geometries where $0 < h < \sqrt{\frac{\lambda}{\lambda-\lambda_1}}$. This case was called the Grassmann geometry of Case (III) in Part I. We first analyse the constant mean curvature surface equation, shortly the CMC surface equation, of this case, by using the same way as the one for the other $\mathcal{O}(h)$-geometries which were solved in Part I. As a result we can see that the CMC surface equation has no constant solution for the case though it has any constant solution for the case that $h = \sqrt{\frac{\lambda}{\lambda-\lambda_1}}$. Moreover, by using a different method, we give an example of $\mathcal{O}(h)$-surfaces with nonzero constant mean curvature for all the cases that $0 < h < \sqrt{\frac{\lambda}{\lambda-\lambda_1}}$.

2. CMC surface equation for Grassmann geometry of Case (III)

We first recall the CMC surface equation of general case. The solvability of this equation implies the existence of $\mathcal{O}(h)$-surface with constant mean curvature, under the condition that among solutions of this equation there exists one which satisfies a certain regularity condition. This condition is weak and it may be satisfied for almost cases. The general CMC surface equation is now given in the following form:

$$(z\cos\theta + x\sin\theta)\left(\theta_t\frac{\partial(z,w)}{\partial(a,b)} + \theta_a\frac{\partial(z,w)}{\partial(b,t)} + \theta_b\frac{\partial(z,w)}{\partial(t,a)}\right)$$
$$+ (y\cos\theta - w\sin\theta)\left(\theta_t\frac{\partial(w,y)}{\partial(a,b)} + \theta_a\frac{\partial(w,y)}{\partial(b,t)} + \theta_b\frac{\partial(w,y)}{\partial(t,a)}\right)$$
$$+ (x\cos\theta - z\sin\theta)\left(\theta_t\frac{\partial(y,z)}{\partial(a,b)} + \theta_a\frac{\partial(y,z)}{\partial(b,t)} + \theta_b\frac{\partial(y,z)}{\partial(t,a)}\right)$$

$$+ K\left(y_t \frac{\partial(z,w)}{\partial(a,b)} + z_t \frac{\partial(w,y)}{\partial(a,b)} + w_t \frac{\partial(y,z)}{\partial(a,b)} \right) = 0 \tag{2.2}$$

where $\theta = \gamma t + \varphi$, $\gamma = -2\{(1-h^2) + \frac{\lambda_1}{\lambda} h^2\}$, K is an arbitrary constant, and t, a, b and $\varphi = \varphi(a,b)$ are respectively the variables and the unknown function of this equation. Moreover for our Case (III), functions x, y, z, w are given in the following form:

$$\begin{aligned} x(t) &= A^1(t,a,b)e^{\sigma t} + A^2(t,a,b)e^{-\sigma t}, \\ y(t) &= B^1(t,a,b)e^{\sigma t} + B^2(t,a,b)e^{-\sigma t}, \\ z(t) &= C^1(t,a,b)e^{\sigma t} + C^2(t,a,b)e^{-\sigma t}, \\ w(t) &= D^1(t,a,b)e^{\sigma t} + D^2(t,a,b)e^{-\sigma t} \end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
A^1(t,a,b) &= \frac{\mu}{2\sigma}\left\{ \cos\left(\frac{\gamma}{2}t - \tau\right)\sqrt{1-a^2+b^2} + \sin\left(\frac{\gamma}{2}t - \tau\right)a + \sin\left(\frac{\gamma}{2}t + \varphi\right)b \right\}, \\
A^2(t,a,b) &= \frac{\mu}{2\sigma}\left\{ \cos\left(\frac{\gamma}{2}t + \tau\right)\sqrt{1-a^2+b^2} + \sin\left(\frac{\gamma}{2}t + \tau\right)a - \sin\left(\frac{\gamma}{2}t + \varphi\right)b \right\}, \\
B^1(t,a,b) &= \frac{\mu}{2\sigma}\left\{ -\cos\left(\frac{\gamma}{2}t + \varphi\right)\sqrt{1-a^2+b^2} + \sin\left(\frac{\gamma}{2}t + \varphi\right)a + \sin\left(\frac{\gamma}{2}t - \tau\right)b \right\}, \\
B^2(t,a,b) &= \frac{\mu}{2\sigma}\left\{ \cos\left(\frac{\gamma}{2}t + \varphi\right)\sqrt{1-a^2+b^2} - \sin\left(\frac{\gamma}{2}t + \varphi\right)a + \sin\left(\frac{\gamma}{2}t + \tau\right)b \right\}, \\
C^1(t,a,b) &= \frac{\mu}{2\sigma}\left\{ -\sin\left(\frac{\gamma}{2}t - \tau\right)\sqrt{1-a^2+b^2} + \cos\left(\frac{\gamma}{2}t - \tau\right)a + \cos\left(\frac{\gamma}{2}t + \varphi\right)b \right\}, \\
C^2(t,a,b) &= \frac{\mu}{2\sigma}\left\{ -\sin\left(\frac{\gamma}{2}t + \tau\right)\sqrt{1-a^2+b^2} + \cos\left(\frac{\gamma}{2}t + \tau\right)a - \cos\left(\frac{\gamma}{2}t + \varphi\right)b \right\}, \\
D^1(t,a,b) &= \frac{\mu}{2\sigma}\left\{ \sin\left(\frac{\gamma}{2}t + \varphi\right)\sqrt{1-a^2+b^2} + \cos\left(\frac{\gamma}{2}t + \varphi\right)a + \cos\left(\frac{\gamma}{2}t - \tau\right)b \right\}, \\
D^2(t,a,b) &= \frac{\mu}{2\sigma}\left\{ -\sin\left(\frac{\gamma}{2}t + \varphi\right)\sqrt{1-a^2+b^2} - \cos\left(\frac{\gamma}{2}t + \varphi\right)a + \cos\left(\frac{\gamma}{2}t + \tau\right)b \right\}.
\end{aligned} \tag{2.4}$$

Moreover μ, σ, τ are the constants given in the following way:

$$\mu = \sqrt{\frac{|\lambda_1|}{\lambda}} h\sqrt{1-h^2}, \quad \sigma = \sqrt{\mu^2 - \upsilon^2}, \quad \upsilon = \nu + \gamma/2, \quad \nu = 1 - h^2,$$
$$\cos\tau = \frac{\sigma}{\sqrt{\sigma^2+\upsilon^2}} = \frac{\sigma}{\mu}, \quad \sin\tau = \frac{\upsilon}{\sqrt{\sigma^2+\upsilon^2}} = \frac{\upsilon}{\mu}. \tag{2.5}$$

See Part I for the details. If φ is constant, the equation (2.2) is reduced to the following equation:

$$\gamma\Big[(z\cos\theta + x\sin\theta)\frac{\partial(z,w)}{\partial(a,b)} + (y\cos\theta - w\sin\theta)\frac{\partial(w,y)}{\partial(a,b)} + (x\cos\theta - z\sin\theta)\frac{\partial(y,z)}{\partial(a,b)}\Big] + K\Big(y_t\frac{\partial(z,w)}{\partial(a,b)} + z_t\frac{\partial(w,y)}{\partial(a,b)} + w_t\frac{\partial(y,z)}{\partial(a,b)}\Big) = 0. \tag{2.6}$$

Now we consider the equation (2.6). If we take notice of the variable t, the CMC surface equation (2.2) of Case (III) is formed by using trigonometric functions and exponential functions. Hence we can rearrange the equation (2.6) to the following form:

$$P_+(t,a,b)e^{3\sigma t} + P_-(t,a,b)e^{-3\sigma t} + Q_+(t,a,b)e^{\sigma t} + Q_-(t,a,b)e^{-\sigma t} = 0 \tag{2.7}$$

where $P_\pm$, $Q_\pm$ are the trigonometric functions with respect to the parameter t given in the following way.

$$P_+ = \gamma\Big[T^1\frac{\partial(C^1,D^1)}{\partial(a,b)} + T^2\frac{\partial(D^1,B^1)}{\partial(a,b)} + T^3\frac{\partial(B^1,C^1)}{\partial(a,b)}\Big] + K\Big[\big(B^1_t + \sigma B^1\big)\frac{\partial(C^1,D^1)}{\partial(a,b)} + \big(C^1_t + \sigma C^1\big)\frac{\partial(D^1,B^1)}{\partial(a,b)} + \big(D^1_t + \sigma D^1\big)\frac{\partial(B^1,C^1)}{\partial(a,b)}\Big], \tag{2.8}$$

$$P_- = \gamma\Big[S^1\frac{\partial(C^2,D^2)}{\partial(a,b)} + S^2\frac{\partial(D^2,B^2)}{\partial(a,b)} + S^3\frac{\partial(B^2,C^2)}{\partial(a,b)}\Big] + K\Big[\big(B^2_t - \sigma B^2\big)\frac{\partial(C^2,D^2)}{\partial(a,b)} + \big(C^2_t - \sigma C^2\big)\frac{\partial(D^2,B^2)}{\partial(a,b)} + \big(D^2_t - \sigma D^2\big)\frac{\partial(B^2,C^2)}{\partial(a,b)}\Big], \tag{2.9}$$

$$
\begin{aligned}
Q_+ = \gamma\Big[& T^1\left(\frac{\partial(C^1,D^2)}{\partial(a,b)} + \frac{\partial(C^2,D^1)}{\partial(a,b)}\right) + S^1\frac{\partial(C^1,D^1)}{\partial(a,b)} \\
& + T^2\left(\frac{\partial(D^1,B^2)}{\partial(a,b)} + \frac{\partial(D^2,B^1)}{\partial(a,b)}\right) + S^2\frac{\partial(D^1,B^1)}{\partial(a,b)} \\
& + T^3\left(\frac{\partial(B^1,C^2)}{\partial(a,b)} + \frac{\partial(B^2,C^1)}{\partial(a,b)}\right) + S^3\frac{\partial(B^1,C^1)}{\partial(a,b)}\Big] \\
+ K\Big[& \big(B^1_t + \sigma B^1\big)\left(\frac{\partial(C^1,D^2)}{\partial(a,b)} + \frac{\partial(C^2,D^1)}{\partial(a,b)}\right) + \big(B^2_t - \sigma B^2\big)\frac{\partial(C^1,D^1)}{\partial(a,b)} \\
& + \big(C^1_t + \sigma C^1\big)\left(\frac{\partial(D^1,B^2)}{\partial(a,b)} + \frac{\partial(D^2,B^1)}{\partial(a,b)}\right) + \big(C^2_t - \sigma C^2\big)\frac{\partial(D^1,B^1)}{\partial(a,b)} \\
& + \big(D^1_t + \sigma D^1\big)\left(\frac{\partial(B^1,C^2)}{\partial(a,b)} + \frac{\partial(B^2,C^1)}{\partial(a,b)}\right) + \big(D^2_t - \sigma D^2\big)\frac{\partial(B^1,C^1)}{\partial(a,b)}\Big],
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
Q_- = \gamma\Big[& S^1\left(\frac{\partial(C^1,D^2)}{\partial(a,b)} + \frac{\partial(C^2,D^1)}{\partial(a,b)}\right) + T^1\frac{\partial(C^2,D^2)}{\partial(a,b)} \\
& + S^2\left(\frac{\partial(D^1,B^2)}{\partial(a,b)} + \frac{\partial(D^2,B^1)}{\partial(a,b)}\right) + T^2\frac{\partial(D^2,B^2)}{\partial(a,b)} \\
& + S^3\left(\frac{\partial(B^1,C^2)}{\partial(a,b)} + \frac{\partial(B^2,C^1)}{\partial(a,b)}\right) + T^3\frac{\partial(B^2,C^2)}{\partial(a,b)}\Big] \\
+ K\Big[& \big(B^2_t - \sigma B^2\big)\left(\frac{\partial(C^1,D^2)}{\partial(a,b)} + \frac{\partial(C^2,D^1)}{\partial(a,b)}\right) + \big(B^1_t + \sigma B^1\big)\frac{\partial(C^2,D^2)}{\partial(a,b)} \\
& + \big(C^2_t - \sigma C^2\big)\left(\frac{\partial(D^1,B^2)}{\partial(a,b)} + \frac{\partial(D^2,B^1)}{\partial(a,b)}\right) + \big(C^1_t + \sigma C^1\big)\frac{\partial(D^2,B^2)}{\partial(a,b)} \\
& + \big(D^2_t - \sigma D^2\big)\left(\frac{\partial(B^1,C^2)}{\partial(a,b)} + \frac{\partial(B^2,C^1)}{\partial(a,b)}\right) + \big(D^1_t + \sigma D^1\big)\frac{\partial(B^2,C^2)}{\partial(a,b)}\Big]
\end{aligned} \tag{2.11}
$$

where we put

$$
\begin{aligned}
T^1 &= C^1\cos\theta + A^1\sin\theta, \quad & S^1 &= C^2\cos\theta + A^2\sin\theta, \\
T^2 &= B^1\cos\theta - D^1\sin\theta, \quad & S^2 &= B^2\cos\theta - D^2\sin\theta, \\
T^3 &= A^1\cos\theta - C^1\sin\theta, \quad & S^3 &= A^2\cos\theta - C^2\sin\theta,
\end{aligned}
$$

and moreover put

$$T^4 = D^1 \cos\theta + B^1 \sin\theta, \quad S^4 = D^2 \cos\theta + B^2 \sin\theta.$$

The functions T^i, S^i, $1 \le i \le 4$, are trigonometric with respect to the variable t, and particularly when $i = 1, 2, 3$, they are defined as the following relations hold:

$$z\cos\theta + x\sin\theta = T^1 e^{\sigma t} + S^1 e^{-\sigma t},$$
$$y\cos\theta - w\sin\theta = T^2 e^{\sigma t} + S^2 e^{-\sigma t},$$
$$x\cos\theta - z\sin\theta = T^3 e^{\sigma t} + S^3 e^{-\sigma t}$$

where the terms $(z\cos\theta + x\sin\theta)$, $(y\cos\theta - w\sin\theta)$, $(x\cos\theta - z\sin\theta)$ appear in the equation (2.6).

Note that $\sigma > 0$, $\gamma < 0$, and $\upsilon > 0$ since $0 < h < \sqrt{\frac{\lambda}{\lambda - \lambda_1}}$ and $\lambda_1 < 0$. Moreover note that, with respect to the variable t, functions T^i, S^i, $1 \le i \le 4$, and A_j, B_j, C_j, D_j, $j = 1, 2$, have the period $\gamma/2$. In fact, it holds that

$$\begin{aligned} &T^1_t = \frac{\gamma}{2} T^3, \quad T^3_t = -\frac{\gamma}{2} T^1, \qquad T^2_t = -\frac{\gamma}{2} T^4, \quad T^4_t = \frac{\gamma}{2} T^2, \\ &S^1_t = \frac{\gamma}{2} S^3, \quad S^3_t = -\frac{\gamma}{2} S^1, \qquad S^2_t = -\frac{\gamma}{2} S^4, \quad S^4_t = \frac{\gamma}{2} S^2, \\ &A^j_t = \frac{\gamma}{2} C^j, \quad C^j_t = -\frac{\gamma}{2} A^j, \qquad B^j_t = \frac{\gamma}{2} D^j, \quad D^j_t = -\frac{\gamma}{2} B^j, \quad (j = 1, 2). \end{aligned} \tag{2.12}$$

Then the functions $P_\pm$, $Q_\pm$ also have the period $\gamma/2$ with respect to the variable t, and since the terms $P_+ e^{3\sigma t}$, $P_- e^{-3\sigma t}$, $Q_+ e^{\sigma t}$, $Q_- e^{-\sigma t}$ in the equlity (2.7) are independent as real functions with respect to the variable t, it follows that $P_\pm = 0$ and $Q_\pm = 0$. Moreover since $P_\pm$ and $Q_\pm$ are trigonometric functions with respect to t, these are determined by the values of the functions and their differentials at any fixed real number t_0. Hence $P_\pm = 0$ (resp. $Q_\pm = 0$) if and only if $P_\pm(t_0, a, b) = (P_\pm)_t(t_0, a, b) = 0$ $(Q_\pm(t_0, a, b) = (Q_\pm)_t(t_0, a, b) = 0)$.

We first consider the equalities $P_\pm = 0$. Set

$$P_\pm = [\gamma(P_\pm)]\gamma + [K(P_\pm)]K = 0$$

where

$$[\gamma(P_+)] = T^1\frac{\partial(C^1,D^1)}{\partial(a,b)} + T^2\frac{\partial(D^1,B^1)}{\partial(a,b)} + T^3\frac{\partial(B^1,C^1)}{\partial(a,b)},$$

$$[K(P_+)] = \big(B^1_t + \sigma B^1\big)\frac{\partial(C^1,D^1)}{\partial(a,b)} + \big(C^1_t + \sigma C^1\big)\frac{\partial(D^1,B^1)}{\partial(a,b)} + \big(D^1_t + \sigma D^1\big)\frac{\partial(B^1,C^1)}{\partial(a,b)}$$

and

$$[\gamma(P_-)] = S^1\frac{\partial(C^2,D^2)}{\partial(a,b)} + S^2\frac{\partial(D^2,B^2)}{\partial(a,b)} + S^3\frac{\partial(B^2,C^2)}{\partial(a,b)},$$

$$[K(P_-)] = \big(B^2_t - \sigma B^2\big)\frac{\partial(C^2,D^2)}{\partial(a,b)} + \big(C^2_t - \sigma C^2\big)\frac{\partial(D^2,B^2)}{\partial(a,b)} + \big(D^2_t - \sigma D^2\big)\frac{\partial(B^2,C^2)}{\partial(a,b)}.$$

Now, taking t_0 such that $(\gamma/2)t_0+\varphi = 0$, thus, $\theta = 0$, we calculate the terms $[\gamma(P_\pm)](t_0,a,b)$, $[\gamma(P_\pm)]_t(t_0,a,b)$ and $[K(P_\pm)](t_0,a,b)$, $[K(P_\pm)]_t(t_0,a,b)$ by using the explicit data (2.4) and the properties (2.12), and consequently we have the following lemma.

Lemma 2.1 (1) *It holds that*

$$[\gamma(P_+)](t_0,a,b) = \left(\frac{\mu}{2\sigma}\right)^3\left[\cos(\varphi+\tau)\frac{a}{\sqrt{1-a^2+b^2}} + \frac{b}{\sqrt{1-a^2+b^2}} + \sin(\varphi+\tau)\right] \times [2\cos(\varphi+\tau)]\big[\cos(\varphi+\tau)\sqrt{1-a^2+b^2} - \sin(\varphi+\tau)a\big]$$

and

$$[K(P_+)](t_0,a,b) = \left(\frac{\mu}{2\sigma}\right)^3\gamma\left[\cos(\varphi+\tau)\frac{a}{\sqrt{1-a^2+b^2}} + \frac{b}{\sqrt{1-a^2+b^2}} + \sin(\varphi+\tau)\right] \times \big[\cos(\varphi+\tau)\sqrt{1-a^2+b^2} - \sin(\varphi+\tau)a\big],$$

and thus it holds that

$$2\cos(\varphi+\tau)+K=0. \tag{2.13}$$

Also, the equality $(P_+)_t(t_0,a,b)=0$ *induces the same equality* (2.13).

(2) *Moreover it holds that*

$$\begin{aligned}&[\gamma(P_-)](t_0,a,b)\\&\quad=\left(\frac{\mu}{2\sigma}\right)^3\left[\cos(\varphi-\tau)\frac{a}{\sqrt{1-a^2+b^2}}-\frac{b}{\sqrt{1-a^2+b^2}}+\sin(\varphi-\tau)\right]\\&\qquad\times[2\cos(\varphi-\tau)]\big[\cos(\varphi-\tau)\sqrt{1-a^2+b^2}-\sin(\varphi-\tau)a\big]\end{aligned}$$

and

$$\begin{aligned}&[K(P_-)](t_0,a,b)\\&\quad=\left(\frac{\mu}{2\sigma}\right)^3\gamma\left[\cos(\varphi-\tau)\frac{a}{\sqrt{1-a^2+b^2}}-\frac{b}{\sqrt{1-a^2+b^2}}+\sin(\varphi-\tau)\right]\\&\qquad\times\big[-\cos(\varphi-\tau)\sqrt{1-a^2+b^2}+\sin(\varphi-\tau)a\big],\end{aligned}$$

and thus it holds that

$$2\cos(\varphi-\tau)-K=0. \tag{2.14}$$

Also, the equality $(P_-)_t(t_0,a,b)=0$ *induces the same equality* (2.14).

The equalities (2.13) and (2.14) induce that $\cos\varphi=0$ and $\sin\varphi=\pm1$.

We next consider the equality $Q_+=0$. In the same way as the case of $P_\pm$ we put

$$Q_+=[\gamma(Q_+)]\gamma+[K(Q_+)]K=0$$

and calculate $[\gamma(Q_+)](t_0,a,b)$ and $[K(Q_+)](t_0,a,b)$ by using the explicit data (2.4), the properties (2.12), and the result that $\cos\varphi=0$ and $\sin\varphi=\pm1$.

We first assume that $\cos\varphi=0$ and $\sin\varphi=-1$. Then it follows that $K=-2\sin\tau$ and we have the following lemma.

Lemma 2.2 *It holds that*

$$[\gamma(Q_+)](t_0,a,b) = \left(\frac{\mu}{2\sigma}\right)^3 \left(\frac{2}{\sqrt{1-a^2+b^2}}\right)\Big[(\cos\tau)(\cos^2\tau+1)+(\cos^3\tau)b^2-(\cos\tau)b - (\sin\tau\cos\tau)ab+(\sin\tau)a\sqrt{1-a^2+b^2}-(\sin^2\tau)b\sqrt{1-a^2+b^2}\Big]$$

and

$$[K(Q_+)](t_0,a,b) = \left(\frac{\mu}{2\sigma}\right)^3 \Bigg\{\left(\frac{\gamma}{2}\right)\left(\frac{2}{\sqrt{1-a^2+b^2}}\right)\times\Big[(\sin 2\tau)-2(\sin 2\tau)a^2+(\sin 2\tau)b^2 - 2(\cos\tau\cos 2\tau)ab-(1+2\sin^2\tau)a\sqrt{1-a^2+b^2} - 3(\sin\tau)b\sqrt{1-a^2+b^2}\Big] - (\sigma)\left(\frac{2}{\sqrt{1-a^2+b^2}}\right)\times(2\cos\tau)\left[\frac{1}{2}(\sin 2\tau)ab+(\cos\tau)a^2 + (\sin\tau)a\sqrt{1-a^2+b^2}+(\sin^2\tau)b\sqrt{1-a^2+b^2}\right]\Bigg\},$$

and then it does not hold that $Q_+(t_0,a,b)=0$ *for any* a, b.

Proof. We check the last statement. To do so, we assume that $Q_+(t_0,a,b)=0$ for any a and b and take notice of the coefficients of the term $(2a^2/\sqrt{1-a^2+b^2})$ in the equality. Since the term does not appear in $[\gamma(Q_+)](t_0,a,b)$, we may take out it from $[K(Q_+)](t_0,a,b)$. Then it holds that

$$\frac{\gamma}{2}\times 2\sin 2\tau+2\sigma\cos^2\tau=0, \text{ thus, } \gamma\sin\tau\cos\tau+\sigma\cos^2\tau=0.$$

Since $\cos\tau\neq 0$, it follows that $\gamma\sin\tau+\sigma\cos\tau=0$, and moreover by (2.5), it follows that $\upsilon\gamma+\sigma^2=0$. Also, again by (2.5) it follows that

$$\sigma^2=-\left(\frac{\gamma}{2}\right)^2-\left(\frac{\gamma}{2}\right)(1-h^2), \quad \upsilon=-\frac{\lambda_1}{\lambda}h^2, \quad \gamma=-2\left\{(1-h^2)+\frac{\lambda_1}{\lambda}h^2\right\}.$$

Substituting these into the equality $\upsilon\gamma + \sigma^2 = 0$, we can see that $h = 0$. This is not the case. Hence we have the last claim of the lemma. □

We next assume that $\cos\varphi = 0$ and $\sin\varphi = 1$. Then it follows that $K = 2\sin\tau$ and we have the following lemma.

Lemma 2.3 *It holds that*

$$
\begin{aligned}
&[\gamma(Q_+)](t_0, a, b)\\
&= \left(\frac{\mu}{2\sigma}\right)^3 \Bigg[-(\sin\tau\sin 2\tau)\sqrt{1-a^2+b^2} + 6(\sin\tau)a - 6(\sin^2\tau)b\\
&\qquad + 2\cos\tau(1+\cos^2\tau)\frac{a^2}{\sqrt{1-a^2+b^2}} - (\sin 2\tau)\frac{ab}{\sqrt{1-a^2+b^2}}\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
&[K(Q_+)](t_0, a, b)\\
&= \left(\frac{\mu}{2\sigma}\right)^3 \Bigg\{ \left(\frac{\gamma}{2}\right) \Bigg[-2(1+2\sin^2\tau)a + 2\sin\tau(1+2\sin^2\tau)b\\
&\qquad + 2\cos\tau(2\sin^2\tau - 1)\frac{ab}{\sqrt{1-a^2+b^2}}\\
&\qquad - (\sin 2\tau)\frac{a^2}{\sqrt{1-a^2+b^2}} + (\sin 2\tau)\sqrt{1-a^2+b^2}\Bigg]\\
&\quad + (\sigma)\Bigg[2(\sin\tau\sin 2\tau)b - 2(\sin 2\tau)a\\
&\qquad - 4(\cos^2\tau)\frac{a^2}{\sqrt{1-a^2+b^2}} + 2(\cos\tau\sin 2\tau)\frac{ab}{\sqrt{1-a^2+b^2}}\Bigg]\Bigg\},
\end{aligned}
$$

and then it does not hold that $Q_+(t_0, a, b) = 0$ for any a, b.

Proof. We check the last statement. To do so, we assume that $Q_+(t_0, a, b) = 0$ for any a and b. Noting that $K = 2\sin\tau$ and substituting the above results for the terms $[\gamma(Q_+)](t_0, a, b)$ and $[K(Q_+)](t_0, a, b)$ into the equality

$$Q_+(t_0, a, b) = [\gamma(Q_+)](t_0, a, b)\gamma + [K(Q_+)](t_0, a, b)K = 0,$$

we have the following equality:

$$\begin{aligned}&\gamma(4\cos^2\tau)\Bigg[(\sin\tau)a-(\sin^2\tau)b+(\cos\tau)\frac{a^2}{\sqrt{1-a^2+b^2}}\\&\qquad\qquad\qquad\qquad-\left(\frac{1}{2}\sin 2\tau\right)\frac{ab}{\sqrt{1-a^2+b^2}}\Bigg]\\&\quad-\sigma(4\sin 2\tau)\Bigg[(\sin\tau)a-(\sin^2\tau)b+(\cos\tau)\frac{a^2}{\sqrt{1-a^2+b^2}}\\&\qquad\qquad\qquad\qquad-\left(\frac{1}{2}\sin 2\tau\right)\frac{ab}{\sqrt{1-a^2+b^2}}\Bigg]=0,\end{aligned}$$

thus, $\gamma\cos\tau-2\sigma\sin\tau=0$. By (2.5) and the fact that $\sigma\neq 0$, it follows that $\gamma-2\upsilon=0$. Again by (2.5) we can see that $h=1$. This is not the case. Hence we have the last claim of the lemma. □

Summing up Lemma 2.1, Lemma 2.2 and Lemma 2.3, we have the following result.

Proposition 2.4 *The CMC surface equation* (2.2) *for Grassmann geometry of Case* (III) *has no constant solution.*

Remark 2.5 Here a constant solution for the CMC surface equation (2.2) means that the unknown function $\varphi(a,b)$ is locally constant. For the Grassmann geometry when $h=\sqrt{\frac{\lambda}{\lambda-\lambda_1}}$, called that of Case (II) in Part I, its corresponding CMC surface equation has constant solutions together with the minimality condition of $\mathcal{O}\big(\sqrt{\frac{\lambda}{\lambda-\lambda_1}}\big)$-surfaces, while Proposition 2.4 implies that in our Case (III) that $0<h<\sqrt{\frac{\lambda}{\lambda-\lambda_1}}$ the CMC surface equation has no constant solution. This indicates that the state of Grassmann geometry may change at the value $h=\sqrt{\frac{\lambda}{\lambda-\lambda_1}}$. In fact, in Grassmann geometry of Case (II) there are no $\mathcal{O}\big(\sqrt{\frac{\lambda}{\lambda-\lambda_1}}\big)$-surfaces with nonzero constant mean curvature (Theorem 5.19 in Part I), and on the other hand in Grassmann geometry of Case (III) there exists an $\mathcal{O}(h)$-surface with nonzero constant mean curvature for any h. The latter fact will be proved in the following sections.

3. Bianchi-Cartan-Vranceanu metrics on $SL(2,\mathbb{R})$

In the previous section we have studied the existence problem of $\mathcal{O}(h)$-surfaces with constant mean curvature by analysing the CMC surface equation (2.2). In this and next sections we will consider the problem by using a Lie-theoretic method called *invariant surface*, and give an example of $\mathcal{O}(h)$-surfaces with nonzero constant mean curvature for the Grassmann geometry of Case (III) on $SL(2,\mathbb{R})$.

We first consider details of the space $(SL(2,\mathbb{R}), g)$ whose Milnor constants satisfy that $\lambda_1 < 0 < \lambda_2 = \lambda_3\ (= \lambda)$, on which we have studied the Grassmann geometry of Case (III). Such a metric g is called a *Bianchi-Cartan-Vranceanu metric* [1] on $SL(2,\mathbb{R})$.

Let us denote by $H^2(-4)$ the hyperbolic 2-space of constant curvature -4 realized as the upper half plane equipped with the Poincaré metric:

$$H^2(-4) = \left(\{(x,y) \in \mathbb{R}^2 \mid y > 0\}, \frac{dx^2 + dy^2}{4y^2} \right).$$

Then $SL(2,\mathbb{R})$ acts isometrically and transtively on $H^2(-4)$ as linear fractional transformation group. The isotropy subgroup of $SL(2,\mathbb{R})$ at $(0,1)$ is the special orthogonal group $SO(2)$. The natural projection $\pi : SL(2,\mathbb{R}) \to H^2(-4)$ is given explicitly by

$$\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \frac{1}{c^2 + d^2}(ac + bd, 1).$$

Here we recall the *Iwasawa decomposition* of $SL(2,\mathbb{R})$. It is given as follows: $SL(2,\mathbb{R}) = NAK$ where

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \cong (\mathbb{R}, +),$$

$$A = \left\{ \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \,\middle|\, y > 0 \right\} \cong SO^+(1,1),$$

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \,\middle|\, \theta \in \mathbb{R} \right\} = SO(2).$$

We refer (x, y, θ) as a coordinate system of $SL(2,\mathbb{R})$ and denote by ψ the mapping of coordinate system, *i.e.*,

$$\psi(x,y,\theta) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where the coordinate point $(0,1,0)$ corresponds to the identity I of $SL(2,\mathbb{R})$. Then, by a direct computaion we have the following formulas:

$$\begin{aligned} \psi^{-1}\frac{\partial\psi}{\partial x} &= \frac{1}{y}(\cos^2\theta E - \sin^2\theta F - \sin\theta\cos\theta H), \\ \psi^{-1}\frac{\partial\psi}{\partial y} &= \frac{1}{2y}(\sin 2\theta E + \sin 2\theta F + \cos 2\theta H), \\ \psi^{-1}\frac{\partial\psi}{\partial\theta} &= E - F, \end{aligned} \tag{3.15}$$

where

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Evaluating these at the identity $(0,1,0)$, we get

$$\psi^{-1}\frac{\partial\psi}{\partial x}\Big|_{(0,1,0)} = E, \quad \psi^{-1}\frac{\partial\psi}{\partial y}\Big|_{(0,1,0)} = \frac{1}{2}H, \quad \psi^{-1}\frac{\partial\psi}{\partial\theta}\Big|_{(0,1,0)} = E - F.$$

These vectors E, $(1/2)H$ and $E-F$ form the basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$ which corresponds to the basis $\{(\partial/\partial x)_I, (\partial/\partial y)_I, (\partial/\partial z)_I\}$ of coordinate vectors at I.

On the other hand, in Part I, Section 5.2.1, we equiped a basis of $\mathfrak{sl}(2,\mathbb{R})$

$$e_1 = \frac{\lambda}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \frac{\sqrt{\lambda|\lambda_1|}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \frac{\sqrt{\lambda|\lambda_1|}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that they satisfy the relation (1.1) and the triple $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathfrak{sl}(2,\mathbb{R})$ with respect to the considered left invariant metric g. Hereafter, for simplicity of computations we choose the Milnor constants as $\lambda = 2$ and $\lambda_1 = -2$. Namely,

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In fact, we can choose such a basis by changing each original vector e_i where $i = 1, 2, 3$ with a suitable scalar multiple. Though this choice also change the left invariant metric g, it has no effect on the following argument. The inner product $\langle\, ,\, \rangle$ on $\mathfrak{sl}(2, \mathbb{R})$ induced by the changed metric g is given in the following:

$$\langle X, Y \rangle = \frac{1}{2}({}^t XY), \quad X, Y \in \mathfrak{sl}(2, \mathbb{R}).$$

The metric on $SL(2, \mathbb{R})/SO(2)$ induced from g coincides with the Poincaré metric $(dx^2 + dy^2)/(4y^2)$ and the natural projection π is a Riemannian submersion with totally geodesic fibers. We here note that the left invariant vector fields E_1, E_2, E_3 satisfying (1.1) correspond to e_1, e_2 and e_3, respectively, *i.e.*, $(E_i)_I = e_i$ where $i = 1, 2, 3$.

4. Invariant surfaces

In this section we consider a surface S in $SL(2, \mathbb{R})$ invariant under the action of nilpotent group N. Such a surface is called an *N-invariant surface.*

The position vector field φ of S is parametrized as $\varphi(u, v) = (v, y(u), u)$. In matrix form,

$$\varphi(u, v) = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y(u)} & 0 \\ 0 & 1/\sqrt{y(u)} \end{pmatrix} \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix}.$$

The surface S is an orbit of a curve $(y(u), u)$ in $AK \subset SL(2, \mathbb{R})$ under the action of N. Then the partial derivatives of φ are given by

$$\varphi_* \frac{\partial}{\partial u} = \frac{y'}{2y} \epsilon_2 + \epsilon_3, \quad \varphi_* \frac{\partial}{\partial v} = \frac{1}{2y}(\epsilon_1 + \epsilon_3),$$

where

$$\epsilon_1 = 2y \frac{\partial}{\partial x} - \frac{\partial}{\partial \theta}, \quad \epsilon_2 = 2y \frac{\partial}{\partial y}, \quad \epsilon_3 = \frac{\partial}{\partial \theta}.$$

Hence, by (3.15) the induced metric ds^2 on S is:

$$\left\{1+\left(\frac{y'}{2y}\right)^2\right\}du^2+\frac{1}{y}dudv+\frac{1}{2y^2}dv^2.$$

Take a unit normal vector field $\vec{n}$ as

$$\vec{n}=\frac{1}{\alpha}\left(\frac{y'}{2y}\epsilon_1+\epsilon_2-\frac{y'}{2y}\epsilon_3\right)$$

where α is a function given by

$$\alpha=\sqrt{1+2\left(\frac{y'}{2y}\right)^2}.$$

Then the second fundamental form II of S is given by

$$\text{II}(\partial_u,\partial_u)=\frac{-2(y')^2+y''y}{2\alpha y^2},\quad \text{II}(\partial_u,\partial_v)=\frac{-(y')^2+2y^2}{4\alpha y^3},$$

$$\text{II}(\partial_v,\partial_v)=\frac{1}{\alpha y^2}>0.$$

These formulas imply that S has no geodesic points. The mean curvature function H is given by

$$H=\frac{1}{2\alpha^3y^2}(y''y+2y^2).$$

For more informations on N-invariant surfaces, we refer to [2].

Let us consider the height function $h=g(\vec{n},E_1)$. Direct computations show that

$$h(u,v)=\frac{1}{\sqrt{1+2\left(\frac{y'}{2y}\right)^2}}\frac{y'}{2y}. \tag{4.16}$$

We now assume that h is a *constant*. Since E_1 and $\vec{n}$ have unit length, we have $0\leq h\leq 1$. Put $\tau=(\log y)/2$. Then we have

$$(\tau')^2=h^2(1+2(\tau')^2),\quad \text{equivalently,}\quad (1-2h^2)(\tau')^2=h^2.$$

This equation implies that $h \neq 1/\sqrt{2}$. Now we solve the equation

$$\left(\frac{d\tau}{du}\right)^2 = \frac{h^2}{1-2h^2}.$$

Since the left hand side is non-negative, we have $0 \leq h < \frac{1}{\sqrt{2}}$.

Remark 4.1 In Part I, we distingushed the cases $h = \sqrt{\frac{\lambda}{\lambda-\lambda_1}}$ or $h \neq \sqrt{\frac{\lambda}{\lambda-\lambda_1}}$. In our choice, these two cases correspond to the cases $h = \frac{1}{\sqrt{2}}$ or $h \neq \frac{1}{\sqrt{2}}$, respectively. Moreover the condition $0 < h < 1/\sqrt{2}$ corresponds to our case (III) $0 < h < \sqrt{\frac{\lambda}{\lambda-\lambda_1}}$.

Under the condition that the height function $h(u,v)$ is a constant h, the ODE (4.16) has the general solution

$$y(u) = a \exp\left(\frac{2hu}{\sqrt{1-2h^2}}\right), \ a > 0.$$

Since the choice of the constant a has no effect on the following argument, we assume that $a = 1$. Then we get an N-invarinat surface

$$\varphi(u,v) = \left(v, \exp\left(\frac{2hu}{\sqrt{1-2h^2}}\right), u\right),$$

and it satisfies the following conditions:

$$\varphi(0,0) = (0,1,0) = I, \quad \alpha(u,v) = \frac{1}{\sqrt{1-2h^2}}, \quad H(u,v) = \sqrt{1-2h^2},$$

and

$$\vec{n}(0,0) = \left\{h(\epsilon_1 - \epsilon_3) + \sqrt{1-2h^2}\epsilon_2\right\}\big|_{(0,0,1)} = h(-e_1 + e_2) + \sqrt{1-2h^2}e_3.$$

The last condition can be moreover rewritten as follows:

$$\vec{n}(0,0) = \sqrt{1-h^2}(\sin\vartheta e_2 - \cos\vartheta e_3) - he_1,$$

$$\text{where} \quad \cos\vartheta = -\frac{\sqrt{1-2h^2}}{\sqrt{1-h^2}}, \quad \sin\vartheta = \frac{h}{\sqrt{1-h^2}}.$$

Compare this with the expression (5.55) in Part I. By these consideration we can see that this N-invariant surface is an $\mathcal{O}(h)$-surface with constant mean curvature $\sqrt{1-2h^2} < 1$.

Remark 4.2 When $h = 0$, we have $\varphi(u, v) = (v, 1, u)$. Thus the resulting surface is the Hopf cylinder over the curve $\gamma = \{(x, 1) \in H^2(-4)\}$. This Hopf cylinder has non-zero constant mean curvature 1. Note that, a Hopf cylinder is of constant mean curvature 1 if and only if its base curve is a horocycle or a line $y =$ constant. (See [2, Proposition 2.3], [4, Proposition 4.3]).

Summing up the argument in this section, we have the following existence theorem for the Grassmann geometry of type (III).

Theorem 4.3 *Let h be a constant such that $0 < h < \sqrt{\frac{\lambda}{\lambda-\lambda_1}}$. Then there exists an $\mathcal{O}(h)$-surface with nonzero constant mean curvature in the space $(SL(2,\mathbb{R}), g)$ with Bianchi-Cartan-Vranceanu metric.*

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