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A new generalization of Besov-type and Triebel-Lizorkin-type spaces and wavelets

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Abstract. In this paper we introduce a new function space which unifies and generalizes the Besov-type and the Triebel-Lizorkin-type function spaces introduced by S. Jaffard and D. Yang- W. Yuan. This new function space covers the Besov spaces and the Triebel-Lizorkin spaces in the homogeneous case, and further the Morrey spaces. We define the new function space through wavelet expansions. We establish characterizations of the new function space such as the φ-transform characterization in the sense of Frazier-Jawerth, the atomic and molecular decomposition characterization. Moreover, in the inhomogeneous case, we give a characterization by local polynomial approximation. As application, we investigate the boundedness of the Calderòn-Zygmund operator and the trace theorem on the new function space.

Key words: wavelet, Besov space, Triebel-Lizorkin space, trace theorem, Calderòn-Zygmund operator, atomic and molecular decomposition.

1. Introduction

It is well known that function spaces are now of increasing applications in many areas of modern analysis, in particular, harmonic analysis and partial differential equations. The most general scales, probably, are the scales of Besov spaces and Triebel-Lizorkin spaces which cover many well known classical concrete function spaces such as Lebesgue spaces, Lipschitz spaces, Sobolev spaces, Hardy spaces and the space BMO.

In recent years D. Yang and W. Yuan in [10], [11] introduced a new class of Besov-type and Triebel-Lizorkin-type spaces which includes the Q spaces. S. Jaffard in [4] introduced the oscillation spaces in order to quantify the degree of correlations between positions of large wavelet coefficients through the scales. In this paper new function spaces are introduced which unify and generalize the function spaces in [4] and [10], [11]. This new function space covers the Besov and Triebel-Lizorkin spaces in the homogeneous case, and further the Morrey spaces.

The plan of next sections in the paper is as follows:

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In Section 2 we define corresponding sequence spaces of our new function spaces and we give some embedding properties for the sequence spaces.

Furthermore, we discuss the almost diagonality and we give conditions under which the almost diagonal operators are bounded on the corresponding sequence spaces.

In Section 3 we will define our new function spaces by wavelet expansions through the sequence spaces. Those new function spaces cover the Besov spaces and Triebel-Lizorkin spaces in the homogeneous case and the Morrey spaces. Furthermore, the function spaces introduced by Jaffard [4], Yang-Yuan [10], [11] and Sawano-Tanaka [7], are special cases of the new function spaces.

In Section 4 we investigate equivalent characterizations of the new function spaces. We establish the φ-transform characterization in the sense of Frazier-Jawerth [1] and further the smooth atomic and molecular decomposition characterization of the new function spaces.

In Section 5, as applications, we investigate the boundedness of the Calderòn-Zygmund operators and the trace theorem on the new function space.

In Section 6 we describe the corresponding results of previous sections for the inhomogeneous cases. Moreover, we give a characterization by local polynomial approximation treated as in [5] for inhomogeneous cases.

We use C to denote a positive constant different in each occasion. But it will depend on the parameters appearing in each assertion. The same notations C are not necessarily the same on any two occurrences. We set $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. Sequence spaces

We consider dyadic cubes in $\mathbb{R}^n$ of the form $Q = [0\ 2^{-l})^n + 2^{-l}k$ for $k \in \mathbb{Z}^n$ and $l \in \mathbb{Z}$ and use notations $l(Q) = 2^{-l}$ for the side length and $x_Q = 2^{-l}k$ for the corner point. Throughout the paper, when dyadic cubes Q appear as indices, it is understood that Q runs over all dyadic cubes of the above form $Q = [0\ 2^{-l})^n + 2^{-l}k$ in $\mathbb{R}^n$. We denote by χ_E the characteristic function of a set E in $\mathbb{R}^n$.

Let $s \in \mathbb{R}$ and $0 < q \leq \infty$. For a sequence $c = (c_Q)$ indexed by dyadic cubes Q, we define

$$\|c\|_{\dot{b}^s_{pq}} = \left(\sum_{l\in\mathbb{Z}} \left\| \sum_{l(Q)=2^{-l}} l(Q)^{-s} |c_Q| \chi_Q \right\|^q_{L^p(\mathbb{R}^n)} \right)^{1/q} \qquad \text{when } 0 < p \le \infty,$$

$$\|c\|_{\dot{f}^s_{pq}} = \left\| \left\{ \sum_{l\in\mathbb{Z}} \left(\sum_{l(Q)=2^{-l}} l(Q)^{-s} |c_Q| \chi_Q \right)^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \qquad \text{when } 0 < p < \infty,$$

and

$$\|c\|_{\dot{f}^s_{\infty q}} = \sup_Q l(Q)^{-\frac{n}{q}} \left\| \left\{ \sum_{j\ge -\log_2 l(Q)} \left(\sum_{l(P)=2^{-j}} l(P)^{-s} |c_P| \chi_P \right)^q \right\}^{1/q} \right\|_{L^q(Q)},$$

with the usual modification for $q = \infty$.

For a sequence $c = (c_Q)$ we define some sequences indexed by dyadic cubes Q:

$$c_{b^s_{pq}}(Q) = \left(\sum_{j\ge -\log_2 l(Q)} \left\| \sum_{l(P)=2^{-j}} l(P)^{-s} |c_P| \chi_P \right\|^q_{L^p(Q)} \right)^{1/q}, \qquad 0 < p \le \infty,$$

$$c_{f^s_{pq}}(Q) = \left\| \left\{ \sum_{j\ge -\log_2 l(Q)} \left(\sum_{l(P)=2^{-j}} l(P)^{-s} |c_P| \chi_P \right)^q \right\}^{1/q} \right\|_{L^p(Q)}, \quad 0 < p < \infty,$$

and

$$c_{f^s_{\infty q}}(Q) = l(Q)^{-\frac{n}{q}} \left\| \left\{ \sum_{j\ge -\log_2 l(Q)} \left(\sum_{l(P)=2^{-j}} l(P)^{-s} |c_P| \chi_P \right)^q \right\}^{1/q} \right\|_{L^q(Q)},$$

with the usual modification for $q = \infty$.

We use notations $\dot{a}^s_{pq}$, $\dot{e}^s_{pq}$ to denote either $\dot{b}^s_{pq}$ or $\dot{f}^s_{pq}$, and a^s_{pq}, e^s_{pq} to denote either b^s_{pq} or f^s_{pq}.

We define the sequence spaces

$$\dot{a}^s_{pq} = \left\{ c = (c_Q) : \|c\|_{\dot{a}^s_{pq}} < \infty \right\},$$

and

$$\dot{a}^s_{pq}\left(e^{s'}_{\zeta\eta}\right) = \left\{ c = (c_Q) : \|c\|_{\dot{a}^s_{pq}\left(e^{s'}_{\zeta\eta}\right)} < \infty \right\}$$

where $\|c\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} = \big\|\{c_{e^{s'}_{\zeta\eta}(Q)}\}\big\|_{\dot{a}^s_{pq}}$.

We have following properties for the sequence spaces $\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$. In what follows, the symbol $\subset$ stands for continuous embeddings.

Theorem 1 (The embedding theorem) *Suppose that s, $s' \in \mathbb{R}$ and $0 < p$, q, ζ, $\eta \leq \infty$.*

(1) $\dot{a}^{s-\epsilon_1}_{pq}\big(e^{s'+\epsilon_1}_{\zeta\eta}\big) \subset \dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) \subset \dot{a}^{s+\epsilon_2}_{pq}\big(e^{s'-\epsilon_2}_{\zeta\eta}\big)$ *if* $0 < \epsilon_1, \epsilon_2$.

(2) $\dot{a}^{s+\frac{n}{\zeta_1}}_{pq}\big(e^{s'}_{\zeta_1\eta}\big) \subset \dot{a}^{s+\frac{n}{\zeta_2}}_{pq}\big(e^{s'}_{\zeta_2\eta}\big)$ *if* $0 < \zeta_2 \leq \zeta_1 < \infty$.

(3) $\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) \subset \dot{a}^{s+s'-\frac{n}{\zeta}}_{pq}$.

(4) $\dot{a}^0_{\infty\infty}\big(e^s_{pq}\big) = \dot{e}^s_{pq}$.

(5) $\dot{f}^s_{\infty q} = \dot{f}^{\frac{n}{p}}_{\infty\infty}\big(f^s_{pq}\big)$.

(6) *When* $s < \dfrac{n}{p}$, $\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) = \{0\}$.

(7) *When* $s > \dfrac{n}{p}$,

$$\dot{b}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) = \dot{b}^{s+s'-\frac{n}{\zeta}}_{pq} \quad \textit{if} \;\; 0 < p \leq \zeta \leq \eta \leq \infty,$$

$$\dot{b}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) = \dot{b}^{s+s'-\frac{n}{p}}_{pq} \quad \textit{if} \;\; 0 < \zeta \leq p \leq \eta \leq \infty.$$

(8) *When* $s = \dfrac{n}{p}$,

$$\dot{b}^{s+s'-\frac{n}{\zeta}}_{pp} \subset \dot{b}^s_{p\infty}\big(e^{s'}_{\zeta\eta}\big) \subset \dot{b}^{s+s'-\frac{n}{\zeta}}_{p\infty} \quad \textit{if} \;\; 0 < p \leq \zeta \leq \eta \leq \infty,$$

$$\dot{b}^{s'}_{\zeta\zeta} \subset \dot{b}^s_{p\infty}\big(e^{s'}_{\zeta\zeta}\big) \subset \dot{b}^{s+s'-\frac{n}{\zeta}}_{pp} \quad \textit{if} \;\; 0 < \zeta \leq p \leq \infty.$$

Proof. The properties (1) through (6) are simple consequences of both the monotonicity of the l^q-norm and Hölder's inequality. (See [9, Proposition 2 in 2.3.2]). We will prove the first statement of (7). By (3) we have the embedding $\dot{b}^{s+s'-n/\zeta}_{pq} \supset \dot{b}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$. Therefore it suffices to prove the converse inclusion. Applying the monotonicity of the l^p-norm, we have for $0 < p \leq \zeta \leq \eta \leq \infty$,

$$
\begin{aligned}
&\|c\|_{\dot b^s_{pq}(e^{s'}_{\zeta\eta})} \le \|c\|_{\dot b^s_{pq}(e^{s'}_{\zeta\zeta})} \\
&\le \left\{ \sum_{l\in\mathbb{Z}} \left(\sum_{l(Q)=2^{-l}} 2^{\left(s-\frac{n}{p}\right)lp} \sum_{j\ge l} 2^{j\left(s'-\frac{n}{\zeta}\right)p} \sum_{l(P)=2^{-j},\, P\subset Q} |c_P|^p \right)^{q/p} \right\}^{1/q} \\
&= \left\{ \sum_{l\in\mathbb{Z}} \left(2^{\left(s-\frac{n}{p}\right)lp} \sum_{j\ge l} 2^{j\left(s'-\frac{n}{\zeta}\right)p} \sum_{l(Q)=2^{-l}} \sum_{l(P)=2^{-j},\, P\subset Q} |c_P|^p \right)^{q/p} \right\}^{1/q} \\
&= \left\{ \sum_{l\in\mathbb{Z}} \left(2^{\left(s-\frac{n}{p}\right)lp} \sum_{j\ge l} 2^{j\left(s'-\frac{n}{\zeta}\right)p} \sum_{l(P)=2^{-j}} |c_P|^p \right)^{q/p} \right\}^{1/q} \\
&\le C\left\{ \sum_{l\in\mathbb{Z}} \left(2^{\left(s+s'-\frac{n}{\zeta}\right)lp} 2^{-nl} \sum_{l(P)=2^{-l}} |c_P|^p \right)^{q/p} \right\}^{1/q} = C\|c\|_{\dot b^{s+s'-\frac{n}{\zeta}}_{pq}}
\end{aligned}
$$

where the last inequality follows from Hardy's inequality if $sp-n>0$. This yields the first statement of (7). In order to prove the second statement of (7), we observe from (2), (3) and the first statement of (7),

$$
\dot b^{s+s'-\frac{n}{p}}_{pq} = \dot b^s_{pq}\left(e^{s'}_{pp}\right) \subset \dot b^{s+\frac{n}{\zeta}-\frac{n}{p}}_{pq}\left(e^{s'}_{\zeta p}\right) \subset \dot b^{s+\frac{n}{\zeta}-\frac{n}{p}}_{pq}\left(e^{s'}_{\zeta\eta}\right) \subset \dot b^{s+s'-\frac{n}{p}}_{pq}
$$

if $0<\zeta\le p\le\infty$. Hence we have the second statement of (7). Similarly to (7) by the monotonicity of l^p-norm and (3), we have

$$
\dot b^{s+s'-\frac{n}{\zeta}}_{pp} \subset \dot b^s_{p\infty}\left(e^{s'}_{\zeta\zeta}\right) \subset \dot b^s_{p\infty}\left(e^{s'}_{\zeta\eta}\right) \subset \dot b^{s+s'-\frac{n}{\zeta}}_{p\infty}
$$

if $0<p\le\zeta\le\eta\le\infty$ and $s=n/p$. Therefore we obtain the first inclusion of (8). We will prove the second inclusion of (8). Similarly to (7) we see

$$
\dot b^{\frac{n}{p}}_{p\infty}\left(e^{s'}_{\zeta\zeta}\right) \subset \dot b^{s'-\frac{n}{\zeta}+\frac{n}{p}}_{pp}
$$

if $0<\zeta\le p\le\infty$. Hence we have from the embedding theorem of the Besov space $\dot b^s_{pq}$ and the first inclusion of (8),

$$
\dot b^{s'}_{\zeta\zeta} \subset \dot b^{\frac{n}{\zeta}}_{\zeta\infty}\left(e^{s'}_{\zeta\zeta}\right) \subset \dot b^{\frac{n}{p}}_{p\infty}\left(e^{s'}_{\zeta\zeta}\right) \subset \dot b^{s'-\frac{n}{\zeta}+\frac{n}{p}}_{pp}
$$

if $0<\zeta\le p\le\infty$. This completes the second inclusion of (8).

Let k_1, $k_2 \geq 0$ and $L > n$. We say that a matrix operator A associated with matrix $\{a_{QP}\}_{QP}$, indexed by dyadic cubes Q and P, is (k_1, k_2, L)-almost diagonal if the matrix $\{a_{QP}\}$ satisfies that

$$\sup_{Q,P} \frac{|a_{QP}|}{\omega(Q,P)} < \infty$$

where

$$\omega(Q,P) = \left(\frac{l(Q)}{l(P)}\right)^{k_1} \big(1 + l(P)^{-1}|x_Q - x_P|\big)^{-L} \quad \text{if } \ l(Q) < l(P),$$

$$\omega(Q,P) = \big(1 + l(P)^{-1}|x_Q - x_P|\big)^{-L} \quad \text{if } \ l(Q) = l(P),$$

and

$$\omega(Q,P) = \left(\frac{l(P)}{l(Q)}\right)^{k_2} \big(1 + l(Q)^{-1}|x_Q - x_P|\big)^{-L} \quad \text{if } \ l(Q) > l(P).$$

Lemma A *Let Q, P be dyadic cubes. Let k_1, $k_2 \in \mathbb{N}_0$, $L > n$ and $L_1 > n + k_1$, $L_2 > n + k_2$. Assume that ϕ_Q, φ_P are functions on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} \phi_Q(x) x^\gamma dx = 0 \qquad \textit{for } \ |\gamma| < k_1, \tag{A.1}$$

$$|\phi_Q(x)| \leq C\big(1 + l(Q)^{-1}|x - x_Q|\big)^{-\max(L,L_1)}, \tag{A.2}$$

$$\big|\partial^\gamma \phi_Q(x)\big| \leq Cl(Q)^{-|\gamma|}\big(1 + l(Q)^{-1}|x - x_Q|\big)^{-L} \qquad \textit{for } \ 0 < |\gamma| \leq k_2, \tag{A.3}$$

$$\int_{\mathbb{R}^n} \varphi_P(x) x^\gamma dx = 0 \qquad \textit{for } \ |\gamma| < k_2, \tag{A.4}$$

$$|\varphi_P(x)| \leq C\big(1 + l(P)^{-1}|x - x_P|\big)^{-\max(L,L_2)}, \tag{A.5}$$

$$\big|\partial^\gamma \varphi_P(x)\big| \leq Cl(P)^{-|\gamma|}\big(1 + l(P)^{-1}|x - x_P|\big)^{-L} \qquad \textit{for } \ 0 < |\gamma| \leq k_1. \tag{A.6}$$

(A.1) *and* (A.6) *are void when $k_1 = 0$, while* (A.3) *and* (A.4) *are void when $k_2 = 0$.*

Then we have

$$(1)\qquad l(Q)^{-n}|\langle\phi_Q,\varphi_P\rangle| \le C\left(\frac{l(Q)}{l(P)}\right)^{k_1}\big(1+l(P)^{-1}|x_Q-x_P|\big)^{-L}$$
$$\text{if } \ l(Q)\le l(P)$$

and

$$(2)\qquad l(P)^{-n}|\langle\phi_Q,\varphi_P\rangle| \le C\left(\frac{l(P)}{l(Q)}\right)^{k_2}\big(1+l(Q)^{-1}|x_Q-x_P|\big)^{-L}$$
$$\text{if } \ l(P)< l(Q).$$

Hence $\{l(Q)^{-n}\langle\phi_Q,\varphi_P\rangle\}_{QP}$ *is* (k_1,k_2+n,L)*-almost diagonal.*

Proof. See Corollary B.3 in [1].

Lemma B *Suppose that* s, $s'\in\mathbb{R}$ *and* $0<p$, q, ζ, $\eta\le\infty$. *Let* k_1, $k_2\in\mathbb{N}_0$, $L>n$ *and* $L_1>n+k_1$, $L_2>n+k_2$. *Assume that* ϕ_Q, φ_P *as in Lemma A.*

Then for $c\in\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$ *and a dyadic cube* Q, *we have* $\sum_P c_P\langle\varphi_P,\phi_Q\rangle$ *is convergent if* $k_1>s+s'-\frac{n}{p}-\frac{n}{\zeta}$, $k_2>J-n-s'$ *and* $L>J$ *where* $J=n/\min(1,\zeta,\eta)$ *in the case* $e^{s'}_{\zeta\eta}=f^{s'}_{\zeta\eta}$, *or* $J=n/\min(1,\ \zeta)$ *in the case* $e^{s'}_{\zeta\eta}=b^{s'}_{\zeta\eta}$.

Proof. We will prove this lemma using an argument similar to that for Lemmas 4.1 and 4.2 in [6].

Let Q be a dyadic cube and $c\in\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$. Then

$$\begin{aligned}\Big|\sum_P c_P\langle\varphi_P,\phi_Q\rangle\Big| &\le \sum_{l(Q)\le l(P)}|c_P||\langle\varphi_P,\phi_Q\rangle| + \sum_{l(Q)>l(P)}|c_P||\langle\varphi_P,\phi_Q\rangle|\\ &=\sigma_1+\sigma_2.\end{aligned}$$

Using Lemma A, we get

$$\sigma_1\le C\sum_{l(Q)\le l(P)}|c_P|l(Q)^n\left(\frac{l(Q)}{l(P)}\right)^{k_1}\big(1+l(P)^{-1}|x_Q-x_P|\big)^{-L}.$$

Using the fact that

$$|c_P|\le Cl(P)^{s-\frac{n}{p}}l(P)^{s'-\frac{n}{\zeta}}$$

since $c \in \dot{a}^{s}_{pq}\big(e^{s'}_{\zeta\eta}\big)$, we obtain that

$$\begin{aligned}\sigma_1 &\le C \sum_{j\ge \log_2 l(Q)} 2^{-j\left(k_1-s-s'+\frac{n}{p}+\frac{n}{\zeta}\right)} l(Q)^{n+k_1} \sum_{l(P)=2^j} \big(1+l(P)^{-1}|x_Q-x_P|\big)^{-L}\\ &\le C \sum_{j\ge \log_2 l(Q)} 2^{-j\left(k_1-s-s'+\frac{n}{p}+\frac{n}{\zeta}\right)} l(Q)^{n+k_1} < \infty\end{aligned}$$

because $k_1 > s+s'-\frac{n}{p}-\frac{n}{\zeta}$ and $L>n$.

To estimate σ_2, we will use the operator M_t defined by $M_t(f) \equiv M(f^t)^{1/t}$ for the maximal operator M. Using Lemma A and [6, Lemma 7.1] with $0<t\le 1$ and $L>n/t$, we get

$$\begin{aligned}\sigma_2 &\le C \sum_{l(P)<l(Q)} |c_P| l(P)^n \left(\frac{l(P)}{l(Q)}\right)^{k_2} \big(1+l(Q)^{-1}|x_Q-x_P|\big)^{-L}\\ &\le C \sum_{j\ge -\log_2 l(Q)} l(Q)^{n/t-k_2} 2^{-j(k_2-n/t+n+s')} M_t\Bigg(\sum_{l(P)=2^{-j}} 2^{js'}|c_P|\chi_P\Bigg)(x)\end{aligned}$$

for $x\in Q$. Using the monotonicity of the l^q-norm and Hölder's inequality, we get

$$\sigma_2 \le C l(Q)^{n/t-k_2}\Bigg(\sum_{j\ge -\log_2 l(Q)} \Bigg(M_t\Bigg(\sum_{l(P)=2^{-j}} 2^{js'}|c_P|\chi_P\Bigg)(x)\Bigg)^{\eta}\Bigg)^{1/\eta}$$

for $x\in Q$ where $0<\eta\le\infty$ if $k_2 > n/t-n-s'$. Taking $L^{\zeta}(Q)$ norm $(0<\zeta<\infty)$ and using the Fefferman-Stein inequality for $0<t<\min(\zeta,\eta)$, we have

$$\begin{aligned}\sigma_2 &= l(Q)^{-\frac{n}{\zeta}}\|\sigma_2\|_{L^{\zeta}(Q)}\\ &\le C l(Q)^{n/t-n/\zeta-k_2}\Bigg\|\Bigg(\sum_{j\ge -\log_2 l(Q)}\Bigg(M_t\Bigg(\sum_{l(P)=2^{-j}} 2^{js'}|c_P|\chi_P\Bigg)\Bigg)^{\eta}\Bigg)^{1/\eta}\Bigg\|_{L^{\zeta}(Q)}\\ &\le C l(Q)^{n/t-n/\zeta-k_2}\Bigg\|\Bigg(\sum_{j\ge -\log_2 l(Q)}\Bigg(\sum_{l(P)=2^{-j}} 2^{js'}|c_P|\chi_P\Bigg)^{\eta}\Bigg)^{1/\eta}\Bigg\|_{L^{\zeta}(Q)}\\ &\le C l(Q)^{n/t-n/\zeta-k_2+s-n/p} l(Q)^{-(s-n/p)} c_{f^{s'}_{\zeta\eta}(Q)} \le C\|c\|_{\dot{a}^{s}_{pq}\left(f^{s'}_{\zeta\eta}\right)} < \infty.\end{aligned}$$

Repeating the above argument, we will prove the case $\zeta = \infty$. Taking the $L^\eta(Q)$ norm, we have

$$\begin{aligned}
\sigma_2 &= l(Q)^{-\frac{n}{\eta}} \|\sigma_2\|_{L^\eta(Q)} \\
&\le Cl(Q)^{n/t-n/\eta-k_2} \left\| \left(\sum_{j \ge -\log_2 l(Q)} \left(M_t \left(\sum_{l(P)=2^{-j}} 2^{js'} |c_P| \chi_P \right) \right)^\eta \right)^{1/\eta} \right\|_{L^\eta(Q)} \\
&\le Cl(Q)^{n/t-k_2} l(Q)^{-\frac{n}{\eta}} \left\| \left(\sum_{j \ge -\log_2 l(Q)} \left(\sum_{l(P)=2^{-j}} 2^{js'} |c_P| \chi_P \right)^\eta \right)^{1/\eta} \right\|_{L^\eta(Q)} \\
&\le Cl(Q)^{n/t-k_2+s-n/p} l(Q)^{-(s-n/p)} c_{f^{s'}_{\infty\eta}(Q)} \le C\|c\|_{\dot{a}^s_{pq}(f^{s'}_{\infty\eta})} < \infty.
\end{aligned}$$

Using the above same argument, we can also prove in the case of B-type that

$$\sigma_2 \le Cl(Q)^{n/t-n/\zeta-k_2+s-n/p} l(Q)^{-(s-n/p)} c_{b^{s'}_{\zeta\eta}(Q)} \le C\|c\|_{\dot{a}^s_{pq}(b^{s'}_{\zeta\eta})} < \infty$$

if $k_2 > J - n - s'$ where $J = n/\min(1,\zeta)$. This completes the proof of the lemma.

The results about the boundedness of almost diagonal operators in [1], are generalized into the following conclusions.

Theorem 2 *Suppose that $\frac{p}{n} \le s < \infty$, $s' \in \mathbb{R}$ and $0 < p,\ q,\ \zeta,\ \eta \le \infty$. A (k_1, k_2, L)-almost diagonal operator is bounded on $\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$ if $k_1 > \max\big(s', s+s'-\frac{n}{\zeta}\big)$, $k_2 > J - s'$ and $L > J$ where $J = n/\min(1,\zeta,\eta)$ in the case $e^{s'}_{\zeta\eta} = f^{s'}_{\zeta\eta}$, or $J = n/\min(1,\zeta)$ in the case $e^{s'}_{\zeta\eta} = b^{s'}_{\zeta\eta}$.*

Proof. We assume that $A = (a_{PP'})$ is almost diagonal. Let $c = (c_Q) \in \dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$. For dyadic cubes Q and P with $P \subset Q$, we write $Ac = A_0c + A_1c + A_2c$ with

$$(A_0c)_P = \sum_{l(P) \le l(P') \le l(Q)} a_{PP'} c_{P'},$$

$$(A_1c)_P = \sum_{l(P') < l(P)} a_{PP'} c_{P'} \qquad \text{and}$$

$$(A_2c)_P = \sum_{l(P)\le l(Q)<l(P')} a_{PP'}c_{P'}.$$

We will consider the case of the F-type. Since A is almost diagonal, we see that for dyadic cubes Q with $l(Q) = 2^{-l}$,

$$\begin{aligned}
&(A_0c)_{f^{s'}_{\zeta\eta}(Q)} \\
&\quad = \left\| \left\{ \sum_{j\ge l} \sum_{l(P)=2^{-j}} \big(l(P)^{-s'}|(A_0c)_P|\big)^{\eta} \chi_P \right\}^{1/\eta} \right\|_{L^{\zeta}(Q)} \\
&\quad \le C \left\| \left\{ \sum_{j\ge l} \sum_{l(P)=2^{-j}} 2^{js'\eta} \left(\sum_{j\ge i\ge l} \sum_{l(P')=2^{-i}} |a_{PP'}||c_{P'}| \right)^{\eta} \chi_P \right\}^{1/\eta} \right\|_{L^{\zeta}(Q)} \\
&\quad \le C \left\| \left\{ \sum_{j\ge l} \sum_{l(P)=2^{-j}} 2^{js'\eta} \left(\sum_{j\ge i\ge l} \sum_{l(P')=2^{-i}} 2^{-(j-i)k_1} \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \cdot (1 + l(P')^{-1}|x_P - x_{P'}|)^{-L}|c_{P'}| \right)^{\eta} \chi_P \right\}^{1/\eta} \right\|_{L^{\zeta}(Q)}.
\end{aligned}$$

Using Lemma 3.1 of [10] for the maximal function Mf, we have

$$\begin{aligned}
&(A_0c)_{f^{s'}_{\zeta\eta}(Q)} \\
&\quad \le C \left\| \left\{ \sum_{j\ge l} \sum_{l(P)=2^{-j}} 2^{js'\eta} 2^{-jk_1\eta} \right.\right. \\
&\qquad\quad \left.\left. \cdot \left(\sum_{j\ge i\ge l} 2^{ik_1} M\left(\sum_{l(P')=2^{-i}} |c_{P'}|\chi_{P'} \right) \right)^{\eta} \chi_P \right\}^{1/\eta} \right\|_{L^{\zeta}(Q)} \\
&\quad \le C \left\| \left\{ \sum_{j\ge l} 2^{-j(k_1-s')\eta} \left(\sum_{j\ge i\ge l} 2^{ik_1} M\left(\sum_{l(P')=2^{-i}} |c_{P'}|\chi_{P'} \right) \right)^{\eta} \right\}^{1/\eta} \right\|_{L^{\zeta}(Q)} \\
&\quad \le C \left\| \left\{ \sum_{j\ge l} 2^{js'\eta} M\left(\sum_{l(P')=2^{-j}} |c_{P'}|\chi_{P'} \right)^{\eta} \right\}^{1/\eta} \right\|_{L^{\zeta}(Q)} \\
&\quad \le C \left\| \left\{ \sum_{j\ge l} 2^{js'\eta} \left(\sum_{l(P')=2^{-j}} |c_{P'}|\chi_{P'} \right)^{\eta} \right\}^{1/\eta} \right\|_{L^{\zeta}(Q)} = Cc_{f^{s'}_{\zeta\eta}(Q)}
\end{aligned}$$

where these inequalities follow from Hardy's inequality and the Fefferman-Stein vector valued inequality if $k_1 > s'$, $1 < \zeta < \infty$ and $1 < \eta \le \infty$. Similarly to the above A_0 case we can prove that

$$(A_1 c)_{f^{s'}_{\zeta\eta}(Q)} \le C c_{f^{s'}_{\zeta\eta}(Q)}$$

if $k_2 > n - s'$, $1 < \zeta < \infty$ and $1 < \eta \le \infty$. Note that we also get the same estimate for the case $\zeta = \infty$. Hence we have

$$\|A_i c\|_{\dot{a}^s_{pq}(f^{s'}_{\zeta\eta})} \le C\|c\|_{\dot{a}^s_{pq}(f^{s'}_{\zeta\eta})}, \quad i = 0, 1$$

if $k_1 > s'$, $k_2 > n - s'$, $1 < \zeta \le \infty$ and $1 < \eta \le \infty$.

For the B-type case we have

$$\begin{aligned}
&(A_0 c)_{b^{s'}_{\zeta\eta}(Q)} \\
&= \Bigg(\sum_{j \ge l} \Bigg\| \sum_{l(P)=2^{-j}} l(P)^{-s'} |(A_0 c)_P| \chi_P \Bigg\|^{\eta}_{L^\zeta(Q)} \Bigg)^{1/\eta} \\
&\le C \Bigg\{ \sum_{j \ge l} \Bigg\| \sum_{l(P)=2^{-j}} 2^{js'} \sum_{j \ge i \ge l} \sum_{l(P')=2^{-i}} 2^{-(j-i)k_1} \\
&\qquad\qquad \cdot \big(1 + l(P')^{-1}|x_P - x_{P'}|\big)^{-L} |c_{P'}| \chi_P \Bigg\|^{\eta}_{L^\zeta(Q)} \Bigg\}^{1/\eta} \\
&\le C \Bigg\{ \sum_{j \ge l} 2^{-j(k_1 - s')\eta} \Bigg\| \sum_{l(P)=2^{-j}} \sum_{j \ge i \ge l} 2^{ik_1} M \\
&\qquad\qquad \cdot \Bigg(\sum_{l(P')=2^{-i}} |c_{P'}| \chi_{P'} \Bigg) \chi_P \Bigg\|^{\eta}_{L^\zeta(Q)} \Bigg\}^{1/\eta} \\
&\le C \Bigg\{ \sum_{j \ge l} 2^{-j(k_1 - s')\eta} \Bigg(\sum_{j \ge i \ge l} 2^{ik_1} \Bigg\| M \Bigg(\sum_{l(P')=2^{-i}} |c_{P'}| \chi_{P'} \Bigg) \Bigg\|_{L^\zeta(Q)} \Bigg)^{\eta} \Bigg\}^{1/\eta} \\
&\le C \Bigg\{ \sum_{j \ge l} 2^{-j(k_1 - s')\eta} \Bigg(\sum_{j \ge i \ge l} 2^{ik_1} \Bigg\| \sum_{l(P')=2^{-i}} |c_{P'}| \chi_{P'} \Bigg\|_{L^\zeta(Q)} \Bigg)^{\eta} \Bigg\}^{1/\eta} \\
&\le C \Bigg(\sum_{j \ge l} 2^{js'\eta} \Bigg\| \sum_{l(P')=2^{-j}} |c_{P'}| \chi_{P'} \Bigg\|^{\eta}_{L^\zeta(Q)} \Bigg)^{1/\eta} = C c_{b^{s'}_{\zeta\eta}(Q)}
\end{aligned}$$

where the inequalities follow from Hardy's inequality and the boundedness of maximal operators if $k_1 > s'$, $1 < \zeta \le \infty$ and $0 < \eta \le \infty$. Similarly to the above A_0 case we obtain that

$$(A_1 c)_{b^{s'}_{\zeta\eta}(Q)} \le C c_{b^{s'}_{\zeta\eta}(Q)}$$

if $k_2 > n - s'$, $1 < \zeta \le \infty$ and $0 < \eta \le \infty$. Hence we have

$$\|A_i c\|_{\dot{a}^s_{pq}(b^{s'}_{\zeta\eta})} \le C\|c\|_{\dot{a}^s_{pq}(b^{s'}_{\zeta\eta})}, \quad i = 0, 1$$

if $k_1 > s'$, $k_2 > n - s'$, $1 < \zeta \le \infty$ and $0 < \eta \le \infty$.

Next, we will give the estimates for the A_2 case. By applying the boundedness of maximal operators on the L^ζ-space, we obtain, for $1 < \zeta < \infty$ and $0 < \eta \le \infty$,

$$\begin{aligned}
&(A_2 c)_{f^{s'}_{\zeta\eta}(Q)} \\
&\quad = \bigg\| \bigg\{ \sum_{j \ge l} \sum_{l(P) = 2^{-j}} \big(l(P)^{-s'} |(A_2 c)_P| \big)^\eta \chi_P \bigg\}^{1/\eta} \bigg\|_{L^\zeta(Q)} \\
&\quad \le C \bigg\| \bigg\{ \sum_{j \ge l} \sum_{l(P) = 2^{-j}} l(P)^{-s'\eta} \bigg(\sum_{l \ge i} \sum_{l(P') = 2^{-i}} 2^{-(j-i)k_1} \\
&\qquad\qquad \cdot \big(1 + l(P')^{-1}|x_P - x_{P'}|\big)^{-L} |c_{P'}| \bigg)^\eta \chi_P \bigg\}^{1/\eta} \bigg\|_{L^\zeta(Q)} \\
&\quad \le C \bigg\| \bigg\{ \sum_{j \ge l} 2^{-j(k_1 - s')\eta} \sum_{l(P) = 2^{-j}} \\
&\qquad\qquad \cdot \bigg(\sum_{l \ge i} 2^{ik_1} M\bigg(\sum_{l(P') = 2^{-i}} |c_{P'}|\chi_{P'} \bigg) \bigg)^\eta \chi_P \bigg\}^{1/\eta} \bigg\|_{L^\zeta(Q)} \\
&\quad \le C \bigg\| 2^{-l(k_1 - s')} \sum_{l \ge i} 2^{ik_1} M\bigg(\sum_{l(P') = 2^{-i}} |c_{P'}|\chi_{P'} \bigg) \bigg\|_{L^\zeta(Q)} \\
&\quad \le C 2^{-l(k_1 - s')} \sum_{l \ge i} 2^{ik_1} \bigg\| M\bigg(\sum_{l(P') = 2^{-i}} |c_{P'}|\chi_{P'} \bigg) \bigg\|_{L^\zeta(Q)}
\end{aligned}$$

$$\leq C2^{-l(k_1-s')}\sum_{l\geq i}2^{ik_1}\Big\|\sum_{l(P')=2^{-i}}|c_{P'}|\chi_{P'}\Big\|_{L^\zeta(Q)}$$

$$\leq C2^{-l\left(k_1-s'+\frac{n}{\zeta}\right)}\sum_{l\geq i}2^{ik_1}\sum_{l(P')=2^{-i},Q\subset P'}|c_{P'}|$$

if $k_1 > s'$. We also see the same estimate for the case $\zeta = \infty$.

Similarly, for the B-type case we have

$$\begin{aligned}
&(A_2c)_{b^{s'}_{\zeta\eta}(Q)}\\
&= \left(\sum_{j\geq l}\Big\|\sum_{l(P)=2^{-j}}l(P)^{-s'}|(A_2c)_P|\chi_P\Big\|^\eta_{L^\zeta(Q)}\right)^{1/\eta}\\
&\leq C\Bigg\{\sum_{j\geq l}\Big\|\sum_{l(P)=2^{-j}}2^{s'j}\sum_{l\geq i}\sum_{l(P')=2^{-i}}2^{-(j-i)k_1}\\
&\qquad\qquad \cdot\left(1+l(P')^{-1}|x_P-x_{P'}|\right)^{-L}|c_{P'}|\chi_P\Big\|^\eta_{L^\zeta(Q)}\Bigg\}^{1/\eta}\\
&\leq C\Bigg\{\sum_{j\geq l}2^{-j(k_1-s')\eta}\Big\|\sum_{l(P)=2^{-j}}\sum_{l\geq i}2^{ik_1}M\Big(\sum_{l(P')=2^{-i}}|c_{P'}|\chi_{P'}\Big)\chi_P\Big\|^\eta_{L^\zeta(Q)}\Bigg\}^{1/\eta}\\
&\leq C2^{-l(k_1-s')}\sum_{l\geq i}2^{ik_1}\Big\|M\Big(\sum_{l(P')=2^{-i}}|c_{P'}|\chi_{P'}\Big)\Big\|_{L^\zeta(Q)}\\
&\leq C2^{-l(k_1-s')}\sum_{l\geq i}2^{ik_1}\Big\|\sum_{l(P')=2^{-i}}|c_{P'}|\chi_{P'}\Big\|_{L^\zeta(Q)}\\
&\leq C2^{-l\left(k_1-s'+\frac{n}{\zeta}\right)}\sum_{l\geq i}2^{ik_1}\sum_{l(P')=2^{-i},\,Q\subset P'}|c_{P'}|
\end{aligned}$$

if $k_1 > s'$, $1 < \zeta \leq \infty$ and $0 < \eta \leq \infty$. Hence we have, for $p < \infty$,

$$\begin{aligned}
&\|A_2c\|_{\dot f^s_{pq}\left(e^{s'}_{\zeta\eta}\right)}\\
&\quad = \left\|\left(\sum_{l\in\mathbb{Z}}\sum_{l(Q)=2^{-l}}l(Q)^{-sq}(A_2c)^q_{e^{s'}_{\zeta\eta}(Q)}\chi_Q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

$$\leq C\Bigg\|\Bigg\{\sum_{l\in\mathbb{Z}}\sum_{l(Q)=2^{-l}} 2^{lsq}2^{-l\left(k_1-s'+\frac{n}{\zeta}\right)q}$$

$$\cdot\Bigg(\sum_{l\geq i}2^{ik_1}\sum_{l(P')=2^{-i},\,Q\subset P'}|c_{P'}|\Bigg)^q\chi_Q\Bigg\}^{1/q}\Bigg\|_{L^p(\mathbb{R}^n)}$$

$$\leq C\Bigg\|\Bigg\{\sum_{l\in\mathbb{Z}}2^{-l(k_1-s-s'+n/\zeta)q}\Bigg(\sum_{l\geq i}2^{ik_1}\sum_{l(P')=2^{-i}}|c_{P'}|\chi_{P'}\Bigg)^q\Bigg\}^{1/q}\|_{L^p(\mathbb{R}^n)}$$

$$\leq C\Bigg\|\Bigg\{\sum_{l\in\mathbb{Z}}2^{l(s'+s-n/\zeta)q}\Bigg(\sum_{l(P')=2^{-l}}|c_{P'}|\chi_{P'}\Bigg)^q\Bigg\}^{1/q}\Bigg\|_{L^p(\mathbb{R}^n)}$$

$$= C\|c\|_{\dot f^{s+s'-n/\zeta}_{pq}} \leq C\|c\|_{\dot f^s_{pq}\left(e^{s'}_{\zeta\eta}\right)}$$

where these inequalities follow from Hardy's inequality and (3) of Theorem 1 if $k_1 > s+s'-n/\zeta$. For the case $p=\infty$ we also get the same estimation. Similarly, for the B-type case we see

$$\|A_2c\|_{\dot b^s_{pq}\left(e^{s'}_{\zeta\eta}\right)} \leq C\|c\|_{\dot b^s_{pq}\left(e^{s'}_{\zeta\eta}\right)}$$

if $k_1 > s+s'-n/\zeta$. Thus we obtain the desired conclusion

$$\|Ac\|_{\dot a^s_{pq}\left(e^{s'}_{\zeta\eta}\right)} \leq C\|c\|_{\dot a^s_{pq}\left(e^{s'}_{\zeta\eta}\right)}$$

for $\min(\zeta,\eta)>1$ when $e^{s'}_{\zeta\eta}=f^{s'}_{\zeta\eta}$ or for $\zeta>1$ when $e^{s'}_{\zeta\eta}=b^{s'}_{\zeta\eta}$. For the other cases of ζ and η the desired conclusion follows by the usual routine. See [1] for details.

3. Definition

Let $\mathcal{S}=\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}_\infty = \mathcal{S}_\infty(\mathbb{R}^n)=\{f\in\mathcal{S} : \int_{\mathbb{R}^n} f(x)x^\gamma dx = 0, \text{ for all } \gamma\in\mathbb{N}_0^n\}$. $\mathcal{S}'_\infty=\mathcal{S}'_\infty(\mathbb{R}^n)=\mathcal{S}'/\mathfrak{P}$ denotes the topological dual of $\mathcal{S}_\infty$ where $\mathfrak{P}$ denotes the set of all polynomials on $\mathbb{R}^n$.

Let $r\in\mathbb{N}$ and $L>n$. We will use a family of (r,L)- smooth wavelets $\psi^{(i)}$ such that $\{2^{nj/2}\psi^{(i)}(2^jx-k)\ (i=1,\ldots,2^n-1,\ j\in\mathbb{Z},\ k\in\mathbb{Z}^n)\}$ forms an orthonormal basis of $L^2(\mathbb{R}^n)$ and satisfies that for $\gamma\in\mathbb{N}_0^n$,

$$|\psi^{(i)}(x)| \le C(1+|x|)^{-\max(L,L_0)} \quad \text{for some } L_0 > n+r, \tag{3.1}$$

$$|\partial^\gamma \psi^{(i)}(x)| \le C(1+|x|)^{-L} \quad \text{for } 0<|\gamma|\le r, \tag{3.2}$$

$$\int_{\mathbb{R}^n} \psi^{(i)}(x) x^\gamma dx = 0 \quad \text{for } |\gamma| < r. \tag{3.3}$$

We denote $\psi_Q(x) = \psi(l(Q)^{-1}(x-x_Q))$, $x_Q = 2^{-j}k$ for a dyadic cube $Q = [0,2^{-j})^n + 2^{-j}k$. We will forget to write the index i of the wavelet, which is of no consequence.

Definition Let s, s', $\in \mathbb{R}$, and $0 < p$, q, ζ, $\eta \le \infty$. We assume that

$$r > \max\left(s',\ s'+s-\frac{n}{\zeta},\ J-n-s'\right) \quad \text{and} \tag{3.4}$$

$$L > J. \tag{3.5}$$

where J is as in Theorem 2. We define

$$\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big) = \left\{ f = \sum_Q c_Q\psi_Q \in \mathcal{S}'_\infty : (c_Q) \in \dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) \right\}$$
$$\text{with } \|f\|_{\dot{A}^s_{pq}(E^{s'}_{\zeta\eta})} = \|c\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}.$$

Remark 1

(1) From Lemma B we observe that for every $(c_Q) \in \dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$, the series $\sum_Q c_Q\psi_Q$ converges in $\mathcal{S}'_\infty$, and for $f \in \dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)$ the representation $f = \sum_Q c_Q\psi_Q$ is unique. Furthermore, $\langle f, \psi_Q\rangle$ is well-defined and we have $c_Q = l(Q)^{-n}\langle f, \psi_Q\rangle$

(2) We can prove that the above definition is independent of the choice of any (r,L)-smooth wavelet ψ which satisfies (3.1) through (3.5). Indeed, suppose that ψ^1 and ψ^2 are (r,L)-smooth wavelets satisfying (3.1) through (3.5). Define $\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)(\psi^1)$ and $\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)(\psi^2)$ as the above definition, using ψ^1 and ψ^2 in the place of ψ. Notice that the wavelet expansion

$$\psi^1_P = \sum_Q l(Q)^{-n}\big\langle \psi^1_P, \psi^2_Q\big\rangle \psi^2_Q.$$

Then, for $\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)(\psi^1) \ni f = \sum_P c_P \psi^1_P \ \big(c \in \dot{a}^s_{pq}(e^{s'}_{\zeta\eta})\big)$, we have

$$f = \sum_P c_P \psi^1_P = \sum_Q (Ac)_Q \psi^2_Q$$

where $A = \{l(Q)^{-n}\langle\psi^1_P, \psi^2_Q\rangle\}_{QP}$. We may assume that $\frac{n}{p} \leq s < \infty$ by Theorem 1 (6). From Lemma A and Theorem 2, we see that $Ac \in \dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$. This shows that $f \in \dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)(\psi^2)$ and so $\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)(\psi^1) \subset \dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)(\psi^2)$. By the same argument, we also see that $\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)(\psi^2) \subset \dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)(\psi^1)$. This implies that the definition of $\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)$ is independent of the choice of ψ.

We have homeomorphism $\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) \cong \dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)$ from Remark 1 (1) as above. Hence we have

Theorem 3 *Theorem* 1 *holds with Besov-type or Triebel-Lizorkin type notations of the small letters $\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$, $\dot{a}^s_{pq}$ and $\dot{e}^s_{pq}$ replaced by the corresponding ones of the capital letters $\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)$, $\dot{A}^s_{pq}$ and $\dot{E}^s_{pq}$ respectively.*

Examples

(i) From (4) of Theorem 1 and Theorem 3 we see that $\dot{A}^0_{\infty\infty}(E^s_{pq}) = \dot{E}^s_{pq}$. Hence the above definition covers the classical class of Besov spaces $\dot{B}^s_{pq}$ and Triebel-Lizorkin spaces $\dot{F}^s_{pq}$.

(ii) The oscillation spaces $\mathcal{O}^{s,s'}_p$ introduced by S. Jaffard [4], are contained in our definition as that $\mathcal{O}^{s,s'}_p = \dot{B}^s_{p\infty}(B^{s'}_{\infty\infty})$.

(iii) The Besov type spaces $\dot{B}^{s,\tau}_{pq}$ and the Triebel-Lizorkin type spaces $\dot{F}^{s,\tau}_{pq}$ introduced by D. Yang and W. Yuan [10], [11], are contained in our definition as special cases that $\dot{B}^{s,\tau}_{pq} = \dot{B}^{n\tau}_{\infty\infty}(B^s_{pq})$ and $\dot{F}^{s,\tau}_{pq} = \dot{F}^{n\tau}_{\infty\infty}(F^s_{pq})$. See Theorem 5 as below.

(iv) The Triebel-Lizorkin-Morrey spaces $\dot{\mathcal{E}}^s_{upq}$ introduced by Y. Sawano and H. Tanaka [7], are contained in our definition as special cases, that is, $\dot{\mathcal{E}}^s_{upq} = \dot{F}^{n(\frac{1}{p}-\frac{1}{u})}_{\infty\infty}(F^s_{pq})$ if $0 < p \leq u < \infty$ and $0 < q \leq \infty$. Especially the Morrey space is $\mathcal{M}^u_p = \dot{F}^{n(\frac{1}{p}-\frac{1}{u})}_{\infty\infty}(F^0_{p2})$ if $1 < p \leq u < \infty$.

4. Characterizations

Let ϕ be a Schwartz function satisfying

$$\operatorname{supp}\hat{\phi} \subset \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\right\}, \tag{4.1}$$

$$\left|\hat{\phi}(\xi)\right| \ge C > 0 \quad \text{if } \frac{3}{5} \le |\xi| \le \frac{5}{3}. \tag{4.2}$$

Remark 2 (See [2]) Note that for a given ϕ as above there exists a Schwartz function φ satisfying the same conditions as ϕ such that

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}(2^j\xi)\hat{\phi}(2^j\xi) = 1 \quad \text{if } \xi \neq 0.$$

Assume that s, s', p, q, ζ, η are as in Section 3. We set $\phi_j(x) = 2^{jn}\phi(2^j x)$ and $\phi_Q(x) = \phi(l(Q)^{-1}(x - x_Q))$ for dyadic cubes Q. We define

$$D\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) = \left\{f = \sum_Q c_Q\phi_Q \in \mathcal{S}'_\infty : (c_Q) \in \dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)\right\}$$

with a (quasi-)norm

$$\|f\|_{D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} = \inf_{f=\sum_Q c_Q\phi_Q} \|c\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}$$

where the infimum is taken over all admissible representations $f = \sum_Q c_Q\phi_Q$.

By Lemma B, note that $f = \sum_Q c_Q\phi_Q$ is convergent in $\mathcal{S}'_\infty$.

Theorem 4 *For s, $s' \in \mathbb{R}$ and $0 < p, q, \zeta, \eta \le \infty$, we have*

$$\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big) = D\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$$

with equivalent (quasi-)norms

$$\|f\|_{\dot{A}^s_{pq}(E^{s'}_{\zeta\eta})} \approx \|f\|_{D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}.$$

Proof. We may assume that $\frac{n}{p} \le s < \infty$ by Theorem 1 (6). Let

$D\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) \ni f = \sum_Q c_Q\phi_Q$. Notice that the wavelet expansion

$$\phi_Q = \sum_P l(P)^{-n}\langle \phi_Q, \psi_P\rangle \psi_P.$$

Then we have

$$f = \sum_Q c_Q\phi_Q = \sum_P (A_0c)_P\psi_P$$

where $A_0 = \{l(Q)^{-n}\langle \phi_P, \psi_Q\rangle\}_{QP}$. Lemma A and Theorem 2 yield that

$$\|f\|_{\dot{A}^s_{pq}(E^{s'}_{\zeta\eta})} = \|A_0c\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} \le C\|c\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}$$

which implies $\|f\|_{\dot{A}^s_{pq}(E^{s'}_{\zeta\eta})} \le C\|f\|_{D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}$ and so $D\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) \subset \dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)$.

Conversely, let $\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big) \ni f = \sum_Q c_Q\psi_Q$. Notice, from [1, Lemma 2.1], that

$$\psi_Q = \sum_P l(P)^{-n}\langle \psi_Q, \varphi_P\rangle \phi_P.$$

where φ as in Remark 2. Hence we have

$$f = \sum_Q c_Q\psi_Q = \sum_P (A_1c)_P\phi_P$$

where $A_1 = \{l(Q)^{-n}\langle \psi_P, \varphi_Q\rangle\}_{QP}$. Applying Lemma A and Theorem 2, we obtain

$$\|f\|_{D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} \le \|A_1c\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} \le C\|c\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} = \|f\|_{\dot{A}^s_{pq}(E^{s'}_{\zeta\eta})}$$

which implies $\dot{A}^s_{pq}(E^{s'}_{\zeta\eta}) \subset D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})$ and so, the proof of the theorem follows.

Remark 3

(1) We see that the definition of $D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})$ is independent of the choice of $\phi \in \mathcal{S}$ satisfying the above conditions (4.1) and (4.2).
(2) We observe that the definition of $\dot{A}^s_{pq}(E^{s'}_{\zeta\eta})$ is independent of the choice of r, L satisfying (3.1) through (3.5) for the wavelets as in Section 3.

Let $s \in \mathbb{R}$, $0 < q \le \infty$ and ϕ satisfying (4.1) and (4.2) as above. For $f \in \mathcal{S}'_\infty$ we define some sequences indexed by dyadic cubes Q:

$$f_{b^s_{pq}(Q)^\phi} = \Bigg(\sum_{j \ge -\log_2 l(Q)} \big\| 2^{js} \phi_j * f \big\|^q_{L^p(Q)} \Bigg)^{1/q}, \qquad 0 < p \le \infty,$$

$$f_{f^s_{pq}(Q)^\phi} = \Bigg\| \Bigg(\sum_{j \ge -\log_2 l(Q)} \big(2^{js} |\phi_j * f| \big)^q \Bigg)^{1/q} \Bigg\|_{L^p(Q)}, \qquad 0 < p < \infty,$$

$$f_{f^s_{\infty q}(Q)^\phi} = l(Q)^{-\frac{n}{q}} \Bigg\| \Bigg(\sum_{j \ge -\log_2 l(Q)} \big(2^{js} |\phi_j * f| \big)^q \Bigg)^{1/q} \Bigg\|_{L^q(Q)},$$

with the usual modification for $q = \infty$. We define

$$T\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) = \Big\{ f \in \mathcal{S}'_\infty : \|f\|_{T\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} \equiv \big\| \big\{ f_{e^{s'}_{\zeta\eta}(Q)^\phi} \big\} \big\|_{\dot{a}^s_{pq}} < \infty \Big\}.$$

Then we have the following φ-transform characterization in the sense of Frazier-Jawerth [1].

Theorem 5 *For s, $s' \in \mathbb{R}$ and $0 < p$, q, ζ, $\eta \le \infty$, we have*

$$\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big) = T\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$$

with equivalent (quasi-)norms

$$\|f\|_{\dot{A}^s_{pq}(E^{s'}_{\zeta\eta})} \approx \|f\|_{T\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}.$$

Proof. By Theorem 4, it is sufficient to prove that $T\dot{a}^s_{pq}(e^{s'}_{\zeta\eta}) = D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})$. Let φ be as in Remark 2. Let $f \in D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})$. From Remark 3 (1), we see that $f = \sum_Q c_Q \varphi_Q : (c_Q) \in \dot{a}^s_{pq}(e^{s'}_{\zeta\eta})$. Notice that

$$\begin{aligned} |\phi_j * f(x)| &= \Big| \sum_Q c_Q \phi_j * \varphi_Q(x) \Big| = \Big| \sum_{l=j-1}^{j+1} \sum_{l(Q)=2^{-l}} c_Q \phi_j * \varphi_Q(x) \Big| \\ &\le C \sum_{l=j-1}^{j+1} \sum_{l(Q)=2^{-l}} |c_Q| \big(1 + 2^l |x - x_Q| \big)^{-L} \end{aligned}$$

for a large enough L. Then, using the argument similar to the proof of [1, Theorem 2.2], it is not difficult to show that

$$f_{e^{s'}_{\zeta\eta}(Q)^\phi} \le C c_{e^{s'}_{\zeta\eta}(Q)}.$$

Thus, we have

$$\|f\|_{T\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} = \big\|\{f_{e^{s'}_{\zeta\eta}(Q)^\phi}\}\big\|_{\dot{a}^s_{pq}} \le C\|c\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}.$$

This implies $D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta}) \subset T\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})$.

We will prove the converse inclusion. Let $T\dot{a}^s_{pq}(e^{s'}_{\zeta\eta}) \ni f$. Then from Lemma 2.1 in [1] we have the φ-transform $f = \sum_Q c_Q(f)\varphi_Q$ where $c_Q(f) = l(Q)^{-n}\langle f, \phi_Q\rangle$. Then we have

$$|c_Q(f)| = |\phi_j * f(x_Q)| \le \sup_Q f \equiv \sup_{y\in Q} |\phi_j * f(y)|$$

for a dyadic cube Q with $l(Q) = 2^{-j}$. Hence using the argument similar to the proof of [1, Lemma 2.5], we can prove that

$$c(f)_{e^{s'}_{\zeta\eta}(Q)} \le C f_{e^{s'}_{\zeta\eta}(Q)^\phi}.$$

Thus, by Remark 3 (1), $\|f\|_{D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} \le \|c(f)\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} \le C\|\{f_{e^{s'}_{\zeta\eta}(Q)^\phi}\}\|_{\dot{a}^s_{pq}} = C\|f\|_{T\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}$.

This implies $T\dot{a}^s_{pq}(e^{s'}_{\zeta\eta}) \subset D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})$. Hence we have $T\dot{a}^s_{pq}(e^{s'}_{\zeta\eta}) = D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})$. Furthermore, in the course of the above proof we proved

$$\|f\|_{D\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} \approx \|c(f)\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} \approx \|f\|_{T\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}.$$

Remark 4 Note that the definition of $T\dot{a}^s_{pq}(E^{s'}_{\zeta\eta})$ is independent of the choice of $\phi \in \mathcal{S}$ satisfying (4.1) and (4.2) as above.

We recall the definitions of smooth atoms and molecules.

Let $r_1, r_2 \in \mathbb{N}_0$ and $L > n$. We assume that r_1, r_2 and L satisfy

$$r_1 > \max\left(s', s + s' - \frac{n}{\zeta}\right), \tag{4.3}$$

$$r_2 > J - n - s', \tag{4.4}$$

$$L > J \tag{4.5}$$

where J is as in Theorem 2.

A family of functions $m = (m_Q)$ indexed by dyadic cubes Q is called a family of smooth molecules with (r_1, r_2, L) if

(i) $|m_Q(x)| \le (1 + l(Q)^{-1}|x - x_Q|)^{-\max(L,L_2)}$ for some $L_2 > n + r_2$

(ii) $|\partial^\gamma m_Q(x)| \le l(Q)^{-|\gamma|}(1 + l(Q)^{-1}|x - x_Q|)^{-L}$ for $0 < |\gamma| \le r_1$, and

(iii) $\displaystyle\int_{\mathbb{R}^n} x^\gamma m_Q(x)dx = 0$ for $|\gamma| < r_2$.

Note that (ii) is void when $r_1 = 0$ and (iii) is void when $r_2 = 0$.

A family of functions $a = (a_Q)$ indexed by dyadic cubes Q is called a family of smooth atoms with (r_1, r_2) if

(i) $\operatorname{supp} a_Q \subset 3Q$ for each dyadic cube Q,

where cQ denotes the cube obtained by expanding the cube Q with the factor c around its center,

(ii) $|\partial^\gamma a_Q(x)| \le l(Q)^{-|\gamma|}$ for $|\gamma| \le r_1$, and

(iii) $\displaystyle\int_{\mathbb{R}^n} x^\gamma a_Q(x)dx = 0$ for $|\gamma| < r_2$.

Note that (iii) is void when $r_2 = 0$.

We define

$$M\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) = \Big\{ f = \sum_Q c_Q m_Q \in \mathcal{S}'_\infty : (m_Q) \text{ smooth molecules with}$$

$$(r_1, r_2, L) \text{ satisfying (4.3), (4.4) and (4.5)}, (c_Q) \in \dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) \Big\}$$

with

$$\|f\|_{M\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} = \inf_{f = \sum_Q c_Q m_Q} \|c\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}$$

where the infimum is taken over all admissible representations $f =$

$\sum_Q c_Q m_Q$,

$$A\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) = \Big\{ f = \sum_Q c_Q a_Q \in \mathcal{S}'_\infty : (a_Q) \text{ smooth atoms with}$$
$$(r_1, r_2) \text{ satisfying (4.3) and (4.4)}, (c_Q) \in \dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) \Big\}$$

with

$$\|f\|_{A\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} = \inf_{f=\sum_Q c_Q a_Q} \|c\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}$$

where the infimum is taken over all admissible representations $f = \sum_Q c_Q a_Q$.

Remark 5 By Lemma B we remark that $f = \sum_Q c_Q m_Q$ or $f = \sum_Q c_Q a_Q$ is convergent in $\mathcal{S}'_\infty$.

We have the following molecular and atomic decomposition characterization:

Theorem 6 *Let* $s,\ s' \in \mathbb{R}$ *and* $0 < p,\ q,\ \zeta,\ \eta \le \infty$.

$$\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big) = M\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) = A\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$$

with equivalent (*quasi-*)*norms*

$$\|f\|_{\dot{A}^s_{pq}(E^{s'}_{\zeta\eta})} \approx \|f\|_{M\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} \approx \|f\|_{A\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}.$$

Proof. We may assume that $\frac{n}{p} \le s < \infty$ by Theorem 1 (6). From Lemma A and Theorem 2, it is easy to see that $A\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) \subset M\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) \subset \dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)$. Hence, by Theorem 5, in order to prove the theorem, it suffices to prove that $T\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big) \subset A\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$. Let ϕ, φ as in Remark 2 and $f \in T\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$. From [1, Lemma 2.1], we have the φ-transform

$$f = \sum_Q c_Q(f)\varphi_Q$$

with $c_Q(f) = l(Q)^{-n}\langle f, \phi_Q\rangle$ satisfying that $\|c(f)\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} \approx \|f\|_{T\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}$

by the proof of Theorem 5. Using the argument similar to the proof of [1, Theorem 4.1], we see that there exist a family of smooth atoms $\{a_Q\}$ and a sequence of coefficients $\{c_Q\}$ such that $f = \sum_Q c_Q a_Q$ and $\|c\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} \le C\|c(f)\|_{\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})} \approx C\|f\|_{T\dot{a}^s_{pq}(e^{s'}_{\zeta\eta})}$. Thus we get the desired result.

Remark 6 The definition of $M\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$ and $A\dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$ is independent of the choice of (r_1, r_2, L) satisfying (4.3), (4.4) and (4.5) for smooth molecules or of (r_1, r_2) satisfying (4.3) and (4.4) for smooth atoms.

5. Calderòn-Zygmund operators and trace theorems

Let $\mathcal{D}$ be the space of Schwartz test functions and $\mathcal{D}'$ its dual. For an arbitrary r_1, $r_2 \in \mathbb{N}_0$ the Calderòn-Zygmund operator T with an exponent $\epsilon > 0$ is a continuous linear operator $\mathcal{D} \to \mathcal{D}'$ such that its kernel K off the diagonal $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ satisfies that

(i) $|\partial_1^\gamma K(x, y)| \le C|x - y|^{-(n+|\gamma|)}$ for $|\gamma| \le r_1$,
(ii) $|K(x, y) - K(x, y')| \le C|y - y'|^{r_2+\epsilon}|x - y|^{-(n+r_2+\epsilon)}$ if $2|y' - y| \le |x - y|$,
(iii) $|\partial_1^\gamma K(x, y) - \partial_1^\gamma K(x, y')| \le C|y - y'|^\epsilon |x - y|^{-(n+|\gamma|+\epsilon)}$ if $2|y' - y| \le |x - y|$ for $0 < |\gamma| \le r_1$ (where this statement is void when $r_1 = 0$),

$$\big|\partial_1^\gamma K(x, y) - \partial_1^\gamma K(x', y)\big| \le C|x' - x|^\epsilon |x - y|^{-(n+|\gamma|+\epsilon)}$$
$$\text{if } 2|x' - x| \le |x - y| \text{ for } |\gamma| \le r_1,$$

(where the subindex 1 stands for derivatives in the first variable)

(iv) T is bounded on $L^2(\mathbb{R}^n)$.

We obtain the following theorem.

Theorem 7 *For $\frac{n}{p} \le s < \infty$, $s' \in \mathbb{R}$ and $0 < p, q, \zeta, \eta \le \infty$ and J as in Theorem 2, the Calderòn-Zygmund operator T with an exponent $\epsilon > J - n$ satisfying $T(x^\gamma) = 0$ for $|\gamma| \le r_1$ and $T^*(x^\gamma) = 0$ for $|\gamma| < r_2$, is bounded on $\dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)$ if $r_1 > \max\big(s', s + s' - \frac{n}{\zeta}\big)$ and $r_2 > J - n - s'$.*

Proof. The proof is similar to ones of [3]. Let $f \in \dot{A}^s_{pq}\big(E^{s'}_{\zeta\eta}\big)$ with a wavelet expansion $f = \sum_Q c_Q \psi_Q : (c_Q) \in \dot{a}^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$. We suppose that the wavelet ψ is compactly supported with large enough smoothness by Remark 1 (2) and Remark 3 (2). Let $\operatorname{supp} \psi_Q \subset cQ$ for every dyadic cube Q.

We claim that $Tf = \sum_Q c_Q(T\psi_Q)$ is convergent in $\mathcal{S}'_\infty$ and $\|Tf\|_{\dot{A}^s_{pq}(E^{s'}_{\zeta\eta})} \le C\|f\|_{\dot{A}^s_{pq}(E^{s'}_{\zeta\eta})}$. To see this, by Theorem 6, it suffices to prove that $T\psi_Q$ is a constant multiple of smooth molecule with $(r_1, r_2, n+\epsilon)$ satisfying (4.3), (4.4) and (4.5) for a dyadic cube Q with $l(Q) = 2^{-l}$. The zero moment condition follows from the assumption $T^*x^\gamma = 0$ for $|\gamma| < r_2$. We choose a suitable large constant C_0. From [3, Corollary 2.14], when $|x - x_Q| < 2C_0 2^{-l}$, we see that

$$
\begin{aligned}
\left|\partial^\gamma T\psi_Q(x)\right| &\le \left\|\partial^\gamma T\psi_Q\right\|_\infty \le C \sum_{|\alpha|\le|\gamma|+1} 2^{l(|\gamma|-|\alpha|)}\left\|\partial^\alpha \psi_Q\right\|_\infty \\
&\le C2^{l|\gamma|} \le Cl(Q)^{-|\gamma|}\left(1 + l(Q)^{-1}|x - x_Q|\right)^{-L}
\end{aligned}
$$

for any $L \ge 0$ and $|\gamma| \le r_1$. When $|x - x_Q| \ge 2C_0 2^{-l}$, using the condition (ii) as above, we obtain

$$
\begin{aligned}
|T\psi_Q(x)| &= \left|\int_{\mathbb{R}^n} K(x,y)\psi_Q(y)dy\right| \\
&= \left|\int_{\mathbb{R}^n} \big(K(x,y) - K(x,x_Q)\big)\psi_Q(y)dy\right| \\
&\le C\int_{cQ} |K(x,y) - K(x,x_Q)||\psi_Q(y)|dy \\
&\le C\int_{|y-x_Q|\le C_0 2^{-l}} |y - x_Q|^{r_2+\epsilon}|x - x_Q|^{-(n+r_2+\epsilon)}dy \\
&\le C(2^l|x - x_Q|)^{-(n+r_2+\epsilon)} \le C(1 + 2^l|x - x_Q|)^{-(n+r_2+\epsilon)}.
\end{aligned}
$$

Moreover, using the condition (iii) as above for $0 < |\gamma| \le r_1$, we have

$$
\begin{aligned}
\left|\partial^\gamma T\psi_Q(x)\right| &\le C\int_{cQ} \left|\partial_1^\gamma K(x,y) - \partial_1^\gamma K(x,x_Q)\right||\psi_Q(y)|dy \\
&\le C\int_{|y-x_Q|\le C_0 2^{-l}} |y - x_Q|^{\epsilon}|x - x_Q|^{-(n+|\gamma|+\epsilon)}dy \\
&\le C2^{-l(n+\epsilon)}|x - x_Q|^{-(n+|\gamma|+\epsilon)} \\
&\le C2^{l|\gamma|}(1 + 2^l|x - x_Q|)^{-(n+\epsilon)}.
\end{aligned}
$$

Hence we observe that $T\psi_Q$ is a constant multiple of smooth molecule with $(r_1, r_2, n+\epsilon)$ satisfying (4.3), (4.4) and (4.5).

We put $x = (x', x_n) \in \mathbb{R}^n$ where $x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. The trace operator is defined by $\operatorname{Tr} f(x') = f(x', 0)$ for $f \in \mathcal{S}(\mathbb{R}^n)$. The trace theorem for the function spaces is as follows.

Theorem 8 *Let $n \geq 2$ and $\frac{n}{p} \leq s < \infty$, $s' \in \mathbb{R}$ and $0 < p,\ q,\ \zeta,\ \eta \leq \infty$. Assume that*

$$s' - \frac{1}{\zeta} > (n-1)\left(\frac{1}{\min(1,\zeta)} - 1\right).$$

Then the trace operator Tr *extends to a linear continuous surjective operator such that*

(1) $\operatorname{Tr} \dot{B}^s_{pq}\big(B^{s'}_{\zeta\eta}\big)(\mathbb{R}^n) = \dot{B}^{s-\frac{1}{p}}_{pq}\left(B^{s'-\frac{1}{\zeta}}_{\zeta\eta}\right)(\mathbb{R}^{n-1})$,

(2) $\operatorname{Tr} \dot{B}^s_{pq}\big(F^{s'}_{\zeta\eta}\big)(\mathbb{R}^n) = \dot{B}^{s-\frac{1}{p}}_{pq}\left(F^{s'-\frac{1}{\zeta}}_{\zeta\zeta}\right)(\mathbb{R}^{n-1})$ *if* $0 < \zeta < \infty$,

(3) $\operatorname{Tr} \dot{F}^s_{pq}\big(B^{s'}_{\zeta\eta}\big)(\mathbb{R}^n) = \dot{F}^{s-\frac{1}{p}}_{pp}\left(B^{s'-\frac{1}{\zeta}}_{\zeta\eta}\right)(\mathbb{R}^{n-1})$ *if* $0 < p < \infty$,

(4) $\operatorname{Tr} \dot{F}^s_{pq}\big(F^{s'}_{\zeta\eta}\big)(\mathbb{R}^n) = \dot{F}^{s-\frac{1}{p}}_{pp}\left(F^{s'-\frac{1}{\zeta}}_{\zeta\zeta}\right)(\mathbb{R}^{n-1})$ *if* $0 < p,\ \zeta < \infty$.

Proof. The proof is similar to ones of [8]. We will only prove (1) and (2) since the proofs of (3) and (4) are as same as the proofs of (1) and (2).

(I) Proof of (1): Let $f \in \dot{B}^s_{pq}\big(B^{s'}_{\zeta\eta}\big)(\mathbb{R}^n)$. By Theorem 6, we have $f = \sum_Q c_Q a_Q$, where a_Q is a smooth atom with (r_1, r_2) satisfying (4.3) and (4.4) in $\mathbb{R}^n$, and $c \in \dot{b}^s_{pq}\big(b^{s'}_{\zeta\eta}\big)(\mathbb{R}^n)$.

We claim that $\operatorname{Tr} f(x') = \sum_Q c_Q a_Q(x', 0)$ is convergent in $\mathcal{S}'_\infty(\mathbb{R}^{n-1})$ and

$$\|\operatorname{Tr} f\|_{\dot{B}^{s-\frac{1}{p}}_{pq}\left(B^{s'-\frac{1}{\zeta}}_{\zeta\eta}\right)(\mathbb{R}^{n-1})} \leq C\|f\|_{\dot{B}^s_{pq}\left(B^{s'}_{\zeta\eta}\right)(\mathbb{R}^n)}.$$

Since $\operatorname{supp} a_Q \subset 3Q$ for any dyadic cube Q, there is a positive integer N such that

$$\sum_Q c_Q a_Q(x',0) = \sum_{i=-N}^{N} \sum_{Q'} c_{Q'\times[il(Q'),(i+1)l(Q'))} a_{Q'\times[il(Q'),(i+1)l(Q'))}(x',0)$$

$$\equiv \sum_{i=-N}^{N} \sum_{Q'} \tilde{c}_{Q',i} a_{Q',i}(x')$$

where $\tilde{c}_{Q',i} = c_{Q'\times[il(Q'),(i+1)l(Q'))}$ and $a_{Q',i}(x') = a_{Q'\times[il(Q'),(i+1)l(Q'))} \cdot(x',0)$. From the assumption it is straightforward to see that $a_{Q',i}(x')$ is a smooth atom with $(r_1,0)$ satisfying (4.3) and (4.4) for $\mathbb{R}^{n-1}$ because $r_1 > \max\left(s', s+s'-\frac{n}{\zeta}\right) \geq \max\left(s'-\frac{1}{\zeta}, s-\frac{1}{p}+s'-\frac{1}{\zeta}-\frac{n-1}{\zeta}\right)$, and $0 > \frac{n-1}{\min(1,\zeta)} - (n-1) - \left(s'-\frac{1}{\zeta}\right)$. In order to show that $\operatorname{Tr} f = \sum_{i=-N}^{N} \sum_{Q'} \tilde{c}_{Q',i} a_{Q',i}(x')$ converges in $\mathcal{S}'_\infty(\mathbb{R}^{n-1})$ it suffices to see $(\tilde{c}_{Q',i}) \in \dot{b}_{pq}^{s-\frac{1}{p}}\left(b_{\zeta\eta}^{s'-\frac{1}{\zeta}}\right)(\mathbb{R}^{n-1})$. By similarity, we only consider the case $\tilde{c}_{Q'} \equiv \tilde{c}_{Q',0}$. Then we have for a dyadic cube Q' with $l(Q') = 2^{-l}$ in $\mathbb{R}^{n-1}$,

$$\begin{aligned}
\tilde{c}_{b_{\zeta\eta}^{s'-\frac{1}{\zeta}}(Q')} &= \left\{ \sum_{j\geq l} \left\| \sum_{l(P')=2^{-j}} l(P')^{-\left(s'-\frac{1}{\zeta}\right)} |\tilde{c}_{P'}| \chi_{P'} \right\|_{L^\zeta(Q')}^\eta \right\}^{1/\eta} \\
&\leq C\Bigg\{ \sum_{j\geq l} \Bigg(\int_{Q'\times[0,2^{-l})} \\
&\qquad \cdot \Bigg(\sum_{l(P')=2^{-j}} 2^{js'} \left| c_{P'\times[0,2^{-j})} \right| \chi_{P'\times[0,2^{-j})} \Bigg)^\zeta dx \Bigg)^{\eta/\zeta} \Bigg\}^{1/\eta} \\
&\leq C c_{b_{\zeta\eta}^{s'}(Q'\times[0,2^{-l}))}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\|\tilde{c}\|_{\dot{b}_{pq}^{s-\frac{1}{p}}\left(b_{\zeta\eta}^{s'-\frac{1}{\zeta}}\right)(\mathbb{R}^{n-1})} &= \left\{ \sum_l \left(\sum_{l(Q')=2^{-l}} 2^{l\left(s-\frac{1}{p}-\frac{n-1}{p}\right)p}\, \tilde{c}^p_{b_{\zeta\eta}^{s'-\frac{1}{\zeta}}(Q')} \right)^{q/p} \right\}^{1/q} \\
&\leq C\left\{ \sum_l \left(\sum_{l(Q')=2^{-l}} 2^{l\left(s-\frac{n}{p}\right)p}\, c^p_{b_{\zeta\eta}^{s'}(Q'\times[0,2^{-l}))} \right)^{q/p} \right\}^{1/q} \\
&\leq C\|c\|_{\dot{b}_{pq}^{s}\left(b_{\zeta\eta}^{s'}\right)(\mathbb{R}^n)}.
\end{aligned}$$

This implies that $\tilde{c} \in \dot{b}_{pq}^{s-\frac{1}{p}}\big(b_{\zeta\eta}^{s'-\frac{1}{\zeta}}\big)(\mathbb{R}^{n-1})$ and $\|\operatorname{Tr} f\|_{\dot{B}_{pq}^{s-\frac{1}{p}}\big(B_{\zeta\eta}^{s'-\frac{1}{\zeta}}\big)(\mathbb{R}^{n-1})} \le C\|f\|_{\dot{B}^s_{pq}(B^{s'}_{\zeta\eta})(\mathbb{R}^n)}$.

Next, let us show that the trace operator is onto. Let $f \in \dot{B}_{pq}^{s-\frac{1}{p}}\big(B_{\zeta\eta}^{s'-\frac{1}{\zeta}}\big) \cdot(\mathbb{R}^{n-1})$. By Theorem 6 and Remark 6, we have $f=\sum_{Q'} c_{Q'}a_{Q'}$ where $a_{Q'}$ is a smooth atom with large enough (r_1,r_2) satisfying (4.3) and (4.4) in $\mathbb{R}^{n-1}$, and $c \in \dot{b}_{pq}^{s-\frac{1}{p}}\big(b_{\zeta\eta}^{s'-\frac{1}{\zeta}}\big)(\mathbb{R}^{n-1})$. Let $\tilde{\phi} \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\tilde{\phi} \subset \big(-\frac{1}{2},\frac{1}{2}\big)$ and $\tilde{\phi}(0)=1$ and $\|\partial^\gamma\tilde{\phi}\|_\infty \le 1$ for $|\gamma| \le r_1$. We set $\tilde{\phi}_l(t)=\tilde{\phi}(2^l t)$, $\tilde{a}_{Q'\times[0,2^{-l})} = a_{Q'} \otimes \tilde{\phi}_l$ and $c'_{Q'\times[0,2^{-l})} = c_{Q'}$. We define $F=\sum_l\sum_{l(Q')=2^{-l}} c'_{Q'\times[0,2^{-l})} \cdot\tilde{a}_{Q'\times[0,2^{-l})}$. It is easy to see that $\tilde{a}_{Q'\times[0,2^{-l})}$ is a smooth atom satisfying (4.3) and (4.4) in $\mathbb{R}^n$. We will prove that $\{c'_{Q'\times[0,2^{-l})}\} \in \dot{b}^s_{pq}\big(b^{s'}_{\zeta\eta}\big)(\mathbb{R}^n)$. We see the following estimates: for a dyadic cube Q' with $l(Q')=2^{-l}$ in $\mathbb{R}^{n-1}$,

$$
\begin{aligned}
&c'_{b^{s'}_{\zeta\eta}(Q'\times[0,2^{-l}))}\\
&\quad=\Bigg\{\sum_{j\ge l}\Bigg\|\sum_{l(P')=2^{-j}} 2^{js'}\big|c'_{P'\times[0,2^{-j})}\big|\chi_{P'\times[0,2^{-j})}\Bigg\|^\eta_{L^\zeta(Q'\times[0,2^{-l}))}\Bigg\}^{1/\eta}\\
&\quad\le C\Bigg\{\sum_{j\ge l}\Bigg(\int_{Q'}\Bigg(\sum_{l(P')=2^{-j}} 2^{j\left(s'-\frac{1}{\zeta}\right)}|c_{P'}|\chi_{P'}(x')\Bigg)^\zeta dx'\Bigg)^{\eta/\zeta}\Bigg\}^{1/\eta}\\
&\quad\le Cc_{b_{\zeta\eta}^{s'-\frac{1}{\zeta}}(Q')}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\|c'\|_{\dot{b}^s_{pq}\left(b^{s'}_{\zeta\eta}\right)(\mathbb{R}^n)} &= \Bigg\{\sum_l\Bigg\|\sum_{l(Q')=2^{-l}} 2^{ls}c'_{b^{s'}_{\zeta\eta}(Q'\times[0,2^{-l}))}\chi_{Q'\times[0,2^{-l})}\Bigg\|^q_{L^p(\mathbb{R}^n)}\Bigg\}^{1/q}\\
&\le C\Bigg\{\sum_l\Bigg\|\sum_{l(Q')=2^{-l}} 2^{ls}c_{b_{\zeta\eta}^{s'-\frac{1}{\zeta}}(Q')}\chi_{Q'\times[0,2^{-l})}\Bigg\|^q_{L^p(\mathbb{R}^n)}\Bigg\}^{1/q}\\
&\le C\Bigg\{\sum_l\Bigg\|\sum_{l(Q')=2^{-l}} 2^{l\left(s-\frac{1}{p}\right)}c_{b_{\zeta\eta}^{s'-\frac{1}{\zeta}}(Q')}\chi_{Q'}\Bigg\|^q_{L^p(\mathbb{R}^{n-1})}\Bigg\}^{1/q}\\
&\le C\|c\|_{\dot{b}_{pq}^{s-\frac{1}{p}}\left(b_{\zeta\eta}^{s'-\frac{1}{\zeta}}\right)(\mathbb{R}^{n-1})}.
\end{aligned}
$$

By Theorem 6, this implies that $F \in \dot{B}^s_{pq}\big(B^{s'}_{\zeta\eta}\big)(\mathbb{R}^n)$, and Tr $F = f$, which shows that the trace operator is onto.

(II) Proof of (2): The proof is similar to (I). We use the same notation as in (I). In order to show that the trace operator is continuous, it suffices to prove that for a dyadic cube Q' with $l(Q') = 2^{-l}$ in $\mathbb{R}^{n-1}$,

$$\tilde{c}_{f^{s'-\frac{1}{\zeta}}_{\zeta\zeta}(Q')} \le C c_{f^{s'}_{\zeta\eta}(Q'\times[0,2^{-l}))}$$

if $0 < \zeta < \infty$. We see that

$$\begin{aligned}\tilde{c}_{f^{s'-\frac{1}{\zeta}}_{\zeta\zeta}(Q')} &= \left\| \left\{ \sum_{j\ge l} \left(\sum_{l(P')=2^{-j}} l(P')^{-(s'-\frac{1}{\zeta})} |\tilde{c}_{P'}|\chi_{P'} \right)^{\zeta} \right\}^{1/\zeta} \right\|_{L^\zeta(Q')} \\ &\le C\left\{ \int_{Q'\times[0,2^{-l})} \sum_{j\ge l} \sum_{l(P')=2^{-j}} 2^{js'\zeta} |c_{P'\times[0,2^{-j})}|^{\zeta} \chi_{P'\times[2^{-(j+1)},2^{-j})} dx \right\}^{1/\zeta}.\end{aligned}$$

Since $\{P' \times [2^{-(j+1)}, 2^{-j})\}$ forms a disjoint family, we have

$$\begin{aligned}&\tilde{c}_{f^{s'-\frac{1}{\zeta}}_{\zeta\zeta}(Q')} \\ &\quad\le C\left\{ \int_{Q'\times[0,2^{-l})} \left(\sum_{j\ge l} \sum_{l(P')=2^{-j}} 2^{js'} |c_{P'\times[0,2^{-j})}| \chi_{P'\times[2^{-(j+1)},2^{-j})} \right)^{\zeta} dx \right\}^{1/\zeta} \\ &\quad\le C\left\{ \int_{Q'\times[0,2^{-l})} \left(\sum_{j\ge l} \sum_{l(P')=2^{-j}} 2^{js'} |c_{P'\times[0,2^{-j})}| \chi_{P'\times[2^{-(j+1)},2^{-j})} \right)^{\eta\frac{\zeta}{\eta}} dx \right\}^{1/\zeta} \\ &\quad\le C\left\{ \int_{Q'\times[0,2^{-l})} \left(\sum_{j\ge l} \left(\sum_{l(P')=2^{-j}} 2^{js'} |c_{P'\times[0,2^{-j})}| \chi_{P'\times[2^{-(j+1)},2^{-j})} \right)^{\eta} \right)^{\frac{\zeta}{\eta}} dx \right\}^{1/\zeta} \\ &\quad\le C\left\{ \int_{Q'\times[0,2^{-l})} \left(\sum_{j\ge l} \left(\sum_{l(P')=2^{-j}} 2^{js'} |c_{P'\times[0,2^{-j})}| \chi_{P'\times[0,2^{-j})} \right)^{\eta} \right)^{\frac{\zeta}{\eta}} dx \right\}^{1/\zeta} \\ &\quad\le C\left\| \left\{ \sum_{j\ge l} \left(\sum_{l(P')=2^{-j}} 2^{js'} |c_{P'\times[0,2^{-j})}| \chi_{P'\times[0,2^{-j})} \right)^{\eta} \right\}^{\frac{1}{\eta}} \right\|_{L^\zeta(Q'\times[0,2^{-l}))} \\ &\quad\le C c_{f^{s'}_{\zeta\eta}(Q'\times[0,2^{-l}))}.\end{aligned}$$

Hence we obtain that the trace operator is continuous.

In order to prove that the trace operator is onto, it is sufficient to show, for a dyadic cube Q' with $l(Q') = 2^{-l}$ in $\mathbb{R}^{n-1}$,

$$c'_{f^{s'}_{\zeta\eta}(Q'\times[0,2^{-l}))} \leq C c_{f^{s'-\frac{1}{\zeta}}_{\zeta\zeta}(Q')}$$

if $0 < \zeta < \infty$. For this we note that

$$c'_{f^{s'}_{\zeta\eta}(Q'\times[0,2^{-l}))}$$
$$= \left\| \left\{ \sum_{j\geq l} \left(\sum_{l(P')=2^{-j}} 2^{js'} \left| c'_{P'\times[0,2^{-j})} \right| \chi_{P'\times[0,2^{-j})} \right)^{\eta} \right\}^{1/\eta} \right\|_{L^{\zeta}(Q'\times[0,2^{-l}))}.$$

From the proof of Theorem 1.4 in [8], it follows that for $t > 0$,

$$\sum_{l(P')=2^{-j}} 2^{js'} |c_{P'}| \chi_{P'\times[0,2^{-j})} \leq C M_t \bigg(\sum_{l(P')=2^{-j}} 2^{js'} |c_{P'}| \chi_{P'\times[2^{-(j+1)},2^{-j})} \bigg).$$

Hence we denote $M_t(f) \equiv M(f^t)^{1/t}$ for the maximal operator M. Then from the Fefferman-Stein inequality for $0 < t < \min(\zeta, \eta)$ and $0 < \zeta < \infty$, we obtain

$$c'_{f^{s'}_{\zeta\eta}(Q'\times[0,2^{-l}))}$$
$$\leq C \left\| \left\{ \sum_{j\geq l} \left(M_t \bigg(\sum_{l(P')=2^{-j}} 2^{js'} |c_{P'}| \chi_{P'\times[2^{-(j+1)},2^{-j})} \bigg) \right)^{\eta} \right\}^{1/\eta} \right\|_{L^{\zeta}(Q'\times[0,2^{-l}))}$$
$$\leq C \left\| \left\{ \sum_{j\geq l} \left(\sum_{l(P')=2^{-j}} 2^{js'} |c_{P'}| \chi_{P'\times[2^{-(j+1)},2^{-j})} \right)^{\eta} \right\}^{1/\eta} \right\|_{L^{\zeta}(Q'\times[0,2^{-l}))}$$
$$\leq C \left\| \sum_{j\geq l} \sum_{l(P')=2^{-j}} 2^{js'} |c_{P'}| \chi_{P'\times[2^{-(j+1)},2^{-j})} \right\|_{L^{\zeta}(Q'\times[0,2^{-l}))}$$
$$\leq C \left\{ \int_{Q'} \sum_{j\geq l} \left(\sum_{l(P')=2^{-j}} 2^{j\left(s'-\frac{1}{\zeta}\right)} |c_{P'}| \chi_{P'} \right)^{\zeta} dx' \right\}^{1/\zeta} \leq C c_{f^{s'-\frac{1}{\zeta}}_{\zeta\zeta}(Q')},$$

which completes the proof of (2).

6. Inhomogeneous cases

Since almost all of our methods and results of the previous sections so far easily adapt to the inhomogeneous case, except for few notational inconveniences, we restrict ourselves to highlighting the only differences.

In Section 1 the inhomogeneous version of sequence spaces will be indexed by the set of dyadic cubes Q with $l(Q) \le 1$. We use notations a^s_{pq} and $a^s_{pq}\bigl(e^{s'}_{\zeta\eta}\bigr)$ replacing $\dot{a}^s_{pq}$ and $\dot{a}^s_{pq}\bigl(e^{s'}_{\zeta\eta}\bigr)$. Then the inhomogeneous versions of Theorem 1 hold except the argument of (4) and (6) which is replaced by

(4)′ $e^s_{pq} \subset a^0_{\infty\infty}(e^s_{pq})$,

(6)′ When $s < \frac{n}{p}$,

$$b^{s'+\frac{n}{p}-\frac{n}{\zeta}}_{pp} \subset b^s_{p\infty}\bigl(e^{s'}_{\zeta\eta}\bigr) \subset b^{s+s'-\frac{n}{\zeta}}_{p\infty} \quad \text{if } 0 < p \le \zeta \le \eta \le \infty.$$

The inhomogeneous versions of Lemma A, Lemma B and Theorem 2 with $s \in \mathbb{R}$ hold.

We assume that s, s', p, q, ζ, η, r, L, are as in Section 3.

We will use a family of smooth wavelets $\{\psi_0, \psi^{(i)}\}$ for the inhomogeneous case such that $\{\psi_0(x-k)\ (k \in \mathbb{Z}^n),\ 2^{n(j-1)/2}\psi^{(i)}(2^{j-1}x-k)\ (i = 1, \dots, 2^n - 1, j \in \mathbb{N},\ k \in \mathbb{Z}^n)\}$ forms an orthonormal basis of $L^2(\mathbb{R}^n)$, and $\psi^{(i)}$ satisfies (3.1), (3.2) and (3.3), and a scaling function ψ_0 satisfies (3.1) and (3.2), but does not satisfy the vanishing moment condition (3.3). We will forget to write the index i of the wavelet, which is of no consequence.

We put $\psi_Q(x) = \psi_0(x-k)$ if $Q = [0,1)^n + k$, $k \in \mathbb{Z}^n$, $\psi_Q(x) = \psi(2^{l-1}x - k)$ if $Q = [0, 2^{-l})^n + 2^{-l}k$, $l \in \mathbb{N}$, $k \in \mathbb{Z}^n$. We suppose that r and L satisfy (3.4) and (3.5). We define the inhomogeneous version of the new function spaces given by

$$A^s_{pq}\bigl(E^{s'}_{\zeta\eta}\bigr) = \Bigl\{ f = \sum_{l(Q)\le 1} c_Q \psi_Q \in \mathcal{S}' : (c_Q) \in a^s_{pq}\bigl(e^{s'}_{\zeta\eta}\bigr) \Bigr\}$$
$$\text{with } \|f\|_{A^s_{pq}(E^{s'}_{\zeta\eta})} = \|c\|_{a^s_{pq}(e^{s'}_{\zeta\eta})}.$$

From Lemma B, note that $f = \sum_{l(Q)\le 1} c_Q \psi_Q$ is convergent in $\mathcal{S}'$ for $(c_Q) \in a^s_{pq}\bigl(e^{s'}_{\zeta\eta}\bigr)$. Then we get the inhomogeneous version of Theorem 3.

We select a function $\phi_0 \in \mathcal{S}$ satisfying

(i) $\operatorname{supp}\hat{\phi}_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\}$,

(ii) $|\hat{\phi}_0(\xi)| \ge C > 0$ if $|\xi| \le \frac{5}{3}$.

Let $\phi \in \mathcal{S}$ satisfying (4.1) and (4.2) and $\phi_j(x) = 2^{nj}\phi(2^j x)$, $j \in \mathbb{N}$. We put $\phi_Q(x) = \phi_0(x-k)$ if $Q = [0,1)^n + k$, $k \in \mathbb{Z}^n$, $\phi_Q(x) = \phi(2^l x - k)$ if $Q = [0, 2^{-l})^n + 2^{-l}k$, $l \in \mathbb{N}$, $k \in \mathbb{Z}^n$. Using $\{\phi_Q\}_{l(Q)\le 1}$ and $\{\phi_j\}_{j\in\mathbb{N}_0}$, we define the inhomogeneous version of $Da^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$ and $Ta^s_{pq}\big(e^{s'}_{\zeta\eta}\big)$ as in Section 4. Then we obtain the inhomogeneous analogues of Theorem 4 and Theorem 5.

In the inhomogeneous case we define a family of smooth molecules m_Q with (r_1, r_2, L) satisfying (4.3), (4.4) and (4.5) as in Section 4 if $l(Q) < 1$. If $l(Q) = 1$, we assume

$$\big|\partial^\gamma m_Q(x)\big| \le l(Q)^{-|\gamma|}\big(1 + l(Q)^{-1}|x - x_Q|\big)^{-L}, \quad |\gamma| \le r_1$$

with (r_1, L) satisfying (4.3) and (4.5), but we do not assume the vanishing moment conditions if $l(Q) = 1$. For the smooth atom we also do not assume the vanishing moment conditions if $l(Q) = 1$. Then we obtain the inhomogeneous analogue of Theorem 6.

Theorem 8 of the trace theorem with $s \in \mathbb{R}$ also carries over to the inhomogeneous case under some appropriate inhomogeneous modifications.

In the inhomogeneous case we have the following characterization of local polynomial approximation (cf. S. Jaffard [5]).

Theorem 9

(i) *Let $s,\ s' \in \mathbb{R}$ and let $1 \le p \le \infty$, $0 < q \le \infty$ in the B-type case or $1 \le p < \infty$, $1 \le q \le \infty$ in the F-type case. We assume $s' > 0$, $s + s' - \frac{n}{p} > 0$ and $s + s' - \frac{n}{p} \notin \mathbb{N}$. If $f \in A^s_{\infty\infty}\big(E^{s'}_{pq}\big)$, then*

$$\sup_{l(Q)\le 1} l(Q)^{-s} \inf_{\deg \mathcal{P} < s+s'-\frac{n}{p}} \|f - \mathcal{P}\|_{E^{s'}_{pq}(Q)} < \infty$$

where the infimum is taken over all polynomials $\mathcal{P}$ of degree $< s+s'-\frac{n}{p}$ and $E^{s'}_{pq}(Q)$ denotes either Besov spaces or Triebel-Lizorkin spaces on Q.

(ii) *Conversely, let $s,\ s' \in \mathbb{R}$ and $0 < p,\ q \le \infty$ and $r > \max(s', J - n - s')$ with J as in Theorem 2. We assume that $f \in A^0_{\infty\infty}\big(E^{s'}_{pq}\big)$ and for each $c > 1$,*

$$\sup_{l(Q)\le c^{-1}} l(Q)^{-s} \inf_{\deg \mathcal{P}<r} \|f-\mathcal{P}\|_{E^{s'}_{pq}(cQ)} < \infty$$

where the infimum is taken over all polynomials $\mathcal{P}$ of degree $< r$. Then we have $f \in A^s_{\infty\infty}\big(E^{s'}_{pq}\big)$.

Proof. We may assume that the smooth wavelets are compactly supported because of independence of wavelet basis choice by Remark 1 (2). Therefore, we assume that there exists $c > 1$ such that supp $\psi_Q \subset cQ$ for any dyadic cube Q.

(i) Let $f \in A^s_{\infty\infty}\big(B^{s'}_{pq}\big)$ and the wavelet expansion $f = \sum_{j\ge 0}\sum_{l(Q)=2^{-j}} \cdot c_Q\psi_Q$ with $(c_Q) \in a^s_{\infty\infty}\big(b^{s'}_{pq}\big)$. Let $\triangle_h f(x) = f(x+h) - f(x)$.

We recall that for $k > s' > 0$ and $0 < q \le \infty$, $1 \le p \le \infty$,

$$\|f\|_{B^{s'}_{pq}(Q)} = \|f\|_{L^p(Q)} + \left\{ \sum_{j\ge 0} \Big(2^{js'} \sup_{|h|\le 2^{-j}} \big\|\triangle^k_h f\big\|_{L^p(Q(kh))}\Big)^q \right\}^{1/q}$$

where Q is a dyadic cube with $l(Q) = 2^{-l} \le 1$, and

$$Q(kh) = \{x \in \mathbb{R}^n : [x, x+kh] \subset Q\}.$$

Let $r' \in \mathbb{N}$ such that $r'-1 < s+s'-\frac{n}{p} < r'$. Let $\mathcal{P}_j(x) = \sum_{|\alpha|<r'} \frac{\partial^\alpha g_j(x_0)}{\alpha!}(x-x_0)^\alpha$ denote the Taylor polynomial of $g_j = \sum_{l(R)=2^{-j}} c_R\psi_R$ of degree $r'-1$ at some point $x_0 \in Q$. We put $\mathcal{P}(x) = \sum_{j\ge 0}\mathcal{P}_j(x)$. Notice that $|\partial^\alpha\psi_R(x)| \le Cl(R)^{-|\alpha|}$ and $|c_R| \le Cl(R)^{s+s'-\frac{n}{p}}$. Hence, it follows that $|\partial^\alpha g_j(x)| \le C2^{-j\left(s+s'-\frac{n}{p}-|\alpha|\right)}$ for $|\alpha| < r'$. Therefore the series $\mathcal{P}(x)$ converges and it is a polynomial of degree $r'-1$. We put

$$f-\mathcal{P} = \sum_{j\ge 0}(g_j-\mathcal{P}_j) = \sum_{0\le j\le l}(g_j-\mathcal{P}_j) + \sum_{j>l} g_j - \sum_{j>l}\mathcal{P}_j \equiv f_1 + f_2 - f_3.$$

Then we claim that $\|f_i\|_{B^{s'}_{pq}(Q)} \le Cl(Q)^s$, $i = 1, 2, 3$. Since

$$g_j(x) - \mathcal{P}_j(x) = \sum_{|\alpha|=r'} \int_0^1 \frac{r'}{\alpha!}\partial^\alpha g_j(x_0+(x-x_0)t)(1-t)^{r'-1}(x-x_0)^\alpha dt,$$

and

$$\triangle_h^k(g_j-\mathcal{P}_j)(x)=\int_{-\infty}^{\infty}N_k(t)\sum_{|\alpha|=k}\frac{k!}{\alpha!}\partial^\alpha(g_j-\mathcal{P}_j)(x+th)h^\alpha dt$$

where N_k is the B-spline of order k, we see that

$$\begin{aligned}\left|\triangle_h^k(g_j-\mathcal{P}_j)(x)\right| &\le C\sum_{0\le|\alpha|\le\min(k,r')}|h|^k2^{-l(r'-|\alpha|)}2^{(k+r'-|\alpha|)j}2^{-j\left(s+s'-\frac{n}{p}\right)}\\ &\le C\sum_{0\le|\alpha|\le\min(k,r')}2^{-\nu k}2^{-l(r'-|\alpha|)}2^{(k+r'-|\alpha|)j}2^{-j\left(s+s'-\frac{n}{p}\right)}\end{aligned}$$

if $|h|\le 2^{-\nu}$ and $x\in Q(kh)$. Therefore we have

$$\begin{aligned}&\left\{\sum_{l\le\nu}\left(2^{\nu s'}\sup_{|h|\le2^{-\nu}}\left\|\triangle_h^kf_1\right\|_{L^p(Q(kh))}\right)^q\right\}^{1/q}\\ &\le C\left\{\sum_{l\le\nu}\left(2^{\nu s'}\sum_{0\le j\le l}\sup_{|h|\le2^{-\nu}}\left\|\triangle_h^k(g_j-\mathcal{P}_j)\right\|_{L^p(Q(kh))}\right)^q\right\}^{1/q}\\ &\le C\left\{\sum_{l\le\nu}\left(2^{\nu s'}\sum_{0\le j\le l}\sum_{0\le|\alpha|\le\min(k,r')}\right.\right.\\ &\qquad\left.\left.\cdot\,2^{-\nu k}2^{-l(r'-|\alpha|)}2^{-j\left(s+s'-\frac{n}{p}\right)}2^{j(r'+k-|\alpha|)}\right)^q\right\}^{1/q}2^{-nl/p}\\ &\le C\left\{\sum_{l\le\nu}2^{\nu(s'-k)q}\left(\sum_{0\le|\alpha|\le\min(k,r')}2^{-l(r'-|\alpha|)}\right.\right.\\ &\qquad\left.\left.\cdot\sum_{0\le j\le l}2^{j\left(r'-s-s'+\frac{n}{p}\right)}2^{j(k-|\alpha|)}\right)^q\right\}^{1/q}2^{-nl/p}\\ &\le C2^{-nl/p}2^{-(k-s')l}2^{l\left(r'-s-s'+\frac{n}{p}\right)}\sum_{0\le|\alpha|\le\min(k,r')}2^{-l(r'-|\alpha|)}2^{l(k-|\alpha|)}\\ &\le C2^{-ls}\end{aligned}$$

if $k>s'>0$ and $r'>s+s'-\frac{n}{p}$.

On the other hand we have

$$\begin{aligned}\|f_1\|_{L^p(Q)} &\le \Big\| \sum_{0\le j\le l} (g_j - \mathcal{P}_j) \Big\|_{L^p(Q)} \le C \Big\| \sum_{0\le j\le l} 2^{-j\left(s+s'-\frac{n}{p}-r'\right)} 2^{-lr'} \Big\|_{L^p(Q)} \\ &\le C \sum_{0\le j\le l} 2^{j\left(r'-s-s'+\frac{n}{p}\right)} 2^{-lr'} 2^{-ln/p} \le C 2^{l\left(r'-s-s'+\frac{n}{p}\right)} 2^{-lr'} 2^{-ln/p} \\ &\le C 2^{-(s+s')l} \le C 2^{-ls}\end{aligned}$$

if $s' > 0$ and $r' > s + s' - \frac{n}{p}$. Hence we get $\|f_1\|_{B^{s'}_{pq}(Q)} \le Cl(Q)^s$.

We will next give estimates of f_2.

$$\begin{aligned}f_2(x) &= \sum_{j>l} g_j(x) = \sum_{j>l} \sum_{l(R)=2^{-j}} c_R \psi_R(x) \\ &= \sum_{j>l} \sum_{l(R)=2^{-j}, R\cap cQ \ne \emptyset} c_R \psi_R(x) \equiv \tilde{g}_l(x)\end{aligned}$$

for $x \in Q$. Then we have

$$\begin{aligned}\|\tilde{g}_l\|_{B^{s'}_{pq}(\mathbb{R}^n)} &\le \left\{ \sum_{j>l} \Big\| \sum_{l(R)=2^{-j}, R\cap cQ\ne\emptyset} l(R)^{-s'} |c_R| \chi_R \Big\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q} \\ &\le \left\{ \sum_{j>l} \Big\| \sum_{l(R)=2^{-j}} l(R)^{-s'} |c_R| \chi_R \Big\|^q_{L^p(cQ)} \right\}^{1/q} \\ &\le C \sum_{l(\bar{Q})=2^{-l}, cQ\cap\bar{Q}\ne\emptyset} \left\{ \sum_{j>l} \Big\| \sum_{l(R)=2^{-j}} l(R)^{-s'} |c_R| \chi_R \Big\|^q_{L^p(\bar{Q})} \right\}^{1/q}\end{aligned}$$

From the above we obtain

$$\begin{aligned}&\|f_2\|_{B^{s'}_{pq}(Q)} = \inf_{f_2 = g|_Q} \|g\|_{B^{s'}_{pq}(\mathbb{R}^n)} \le \|\tilde{g}_l\|_{B^{s'}_{pq}(\mathbb{R}^n)} \\ &\le C \sum_{l(\bar{Q})=2^{-l}, cQ\cap\bar{Q}\ne\emptyset} \left\{ \sum_{j>l} \Big\| \sum_{l(R)=2^{-j}} l(R)^{-s'} |c_R| \chi_R \Big\|^q_{L^p(\bar{Q})} \right\}^{1/q} \\ &\le C 2^{-ls} \sum_{l(\bar{Q})=2^{-l}, cQ\cap\bar{Q}\ne\emptyset} l(\bar{Q})^{-s} \left\{ \sum_{j>l} \Big\| \sum_{l(R)=2^{-j}} l(R)^{-s'} |c_R| \chi_R \Big\|^q_{L^p(\bar{Q})} \right\}^{1/q} \\ &\le C 2^{-ls} \|c\|_{a^s_{\infty\infty}\left(b^{s'}_{pq}\right)} \le C 2^{-ls}.\end{aligned}$$

On the other hand we have

$$\|f_2\|_{L^p(Q)} \leq \Big\| \sum_{j>l} g_j \Big\|_{L^p(Q)} \leq C \Big\| \sum_{j>l} 2^{-j\left(s+s'-\frac{n}{p}\right)} \Big\|_{L^p(Q)}$$

$$\leq C 2^{-l\left(s+s'-\frac{n}{p}\right)} 2^{-ln/p} \leq C 2^{-l(s+s')} \leq C 2^{-ls}$$

if $s+s'-\frac{n}{p}>0$ and $s'>0$. Hence we get $\|f_2\|_{B^{s'}_{pq}(Q)} \leq Cl(Q)^s$.

Since for $k \geq r'$ we have $\triangle^k_h f_3 = 0$, we may assume that $k<r'$. Since

$$\left|\triangle^k_h \mathcal{P}_j(x)\right| \leq \left| \int_{-\infty}^{\infty} N_k(t) \sum_{|\alpha|=k} \frac{k!}{\alpha!} \partial^\alpha \mathcal{P}_j(x+th) h^\alpha dt \right|$$

$$\leq C \sum_{k\leq|\alpha|<r'} 2^{j|\alpha|} 2^{-j\left(s+s'-\frac{n}{p}\right)} 2^{-l(|\alpha|-k)} |h|^k$$

if $x \in Q(kh)$, we have

$$\left\{ \sum_{\nu\geq l} \Big(2^{\nu s'} \sup_{|h|\leq 2^{-\nu}} \left\|\triangle^k_h f_3\right\|_{L^p(Q(kh))} \Big)^q \right\}^{1/q}$$

$$\leq C \left\{ \sum_{\nu\geq l} \Big(2^{\nu s'} \sum_{j>l} \sum_{k\leq|\alpha|<r'} 2^{j|\alpha|} 2^{-j\left(s+s'-\frac{n}{p}\right)} 2^{-l(|\alpha|-k)} 2^{-\nu k} 2^{-ln/p} \Big)^q \right\}^{1/q}$$

$$\leq C \left\{ \sum_{\nu\geq l} 2^{-\nu(k-s')q} \Big(\sum_{k\leq|\alpha|<r'} 2^{-l(|\alpha|-k)} \sum_{j>l} 2^{-j\left(s+s'-\frac{n}{p}-|\alpha|\right)} \Big)^q \right\}^{1/q} 2^{-ln/p}$$

$$\leq C 2^{-l(k-s')} \sum_{k\leq|\alpha|<r'} 2^{-l(|\alpha|-k)} 2^{-l\left(s+s'-\frac{n}{p}-|\alpha|\right)} 2^{-ln/p} \leq C 2^{-ls}$$

if $s+s'-\frac{n}{p} > r'-1$ and $k>s'$. On the other hand we have

$$\|f_3\|_{L^p(Q)} \leq \Big\| \sum_{j>l} \mathcal{P}_j \Big\|_{L^p(Q)}$$

$$\leq C \Big\| \sum_{j>l} \sum_{|\alpha|<r'} 2^{-j\left(s+s'-\frac{n}{p}-|\alpha|\right)} 2^{-l|\alpha|} \|_{L^p(Q)}$$

$$\leq C \sum_{|\alpha|<r'} \sum_{j>l} 2^{-j\left(s+s'-\frac{n}{p}-|\alpha|\right)} 2^{-l|\alpha|} 2^{-nl/p}$$

$$\leq C \sum_{|\alpha|<r'} 2^{-l\left(s+s'-\frac{n}{p}-|\alpha|\right)} 2^{-l|\alpha|} 2^{-nl/p} \leq C2^{-l(s+s')} \leq C2^{-ls}$$

if $s' > 0$ and $s + s' - \frac{n}{p} > r' - 1$. Hence we get $\|f_3\|_{B^{s'}_{pq}(Q)} \leq Cl(Q)^s$. For the F-type case the above argument also holds.

(ii) We shall treat only the case of the B-type. For the case of the F-type, the result follows similarly. We assume that $f \in A^0_{\infty\infty}\left(B^{s'}_{pq}\right)$ and the wavelet expansion $f = \sum_{l(Q)\leq 1} c_R(f)\psi_R$ with $(c_R(f)) \in a^0_{\infty\infty}\left(b^{s'}_{pq}\right)$ for an inhomogeneous (r, L)-smooth compactly supported wavelet ψ_Q where $c_R(f) = l(R)^{-n}\langle f, \psi_R\rangle$. Let supp $\psi_Q \subset cQ$ for any dyadic cube Q. Let $\mathcal{P}$ be any polynomial of degree $< r$ and Q a dyadic cube with $l(Q) = 2^{-l} < 1$. We choose $g \in B^{s'}_{pq}(\mathbb{R}^n)$ such that $g = f - \mathcal{P}$ on cQ. Since $c_R(f) = l(R)^{-n}\langle f, \psi_R\rangle = l(R)^{-n}\langle f - \mathcal{P}, \psi_R\rangle = l(R)^{-n}\langle g, \psi_R\rangle = c_R(g)$ for $R \subset Q$, we have, for $0 < p \leq \infty$,

$$c_{b^{s'}_{pq}(Q)} = \left\{ \sum_{j\geq l} \left\| \sum_{l(R)=2^{-j}} l(R)^{-s'} |c_R(f)|\chi_R \right\|^q_{L^p(Q)} \right\}^{1/q}$$

$$= \left\{ \sum_{j\geq l} \left\| \sum_{l(R)=2^{-j}} l(R)^{-s'} |c_R(g)|\chi_R \right\|^q_{L^p(Q)} \right\}^{1/q}$$

$$\leq \left\{ \sum_{j\geq 0} \left\| \sum_{l(R)=2^{-j}} l(R)^{-s'} |c_R(g)|\chi_R \right\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q} = \|g\|_{B^{s'}_{pq}(\mathbb{R}^n)}.$$

Hence we see $c_{b^{s'}_{pq}(Q)} \leq \|f - \mathcal{P}\|_{B^{s'}_{pq}(cQ)}$ for any polynomial $\mathcal{P}$ of degree $< r$, that is, $c_{b^{s'}_{pq}(Q)} \leq \inf_{\deg \mathcal{P}<r} \|f - \mathcal{P}\|_{B^{s'}_{pq}(cQ)}$. When $s > 0$, we have

$$\|c\|_{a^s_{\infty\infty}\left(b^{s'}_{pq}\right)} = \sup_{l(Q)\leq 1} l(Q)^{-s} c_{b^{s'}_{pq}(Q)}$$

$$\leq \sup_{l(Q)\leq c^{-1}} l(Q)^{-s} c_{b^{s'}_{pq}(Q)} + \sup_{c^{-1}<l(Q)\leq 1} l(Q)^{-s} c_{b^{s'}_{pq}(Q)}$$

$$\leq \sup_{l(Q)\leq c^{-1}} l(Q)^{-s} \inf_{\deg \mathcal{P}<r} \|f - \mathcal{P}\|_{B^{s'}_{pq}(cQ)} + c^s \|c\|_{a^0_{\infty\infty}\left(b^{s'}_{pq}\right)} < \infty.$$

This implies $f \in A^{s}_{\infty\infty}\left(B^{s'}_{pq}\right)$. When $s \le 0$, it is obvious that $f \in A^{s}_{\infty\infty}\left(B^{s'}_{pq}\right)$ because $A^{0}_{\infty\infty}\left(B^{s'}_{pq}\right) \subset A^{s}_{\infty\infty}\left(B^{s'}_{pq}\right)$. This concludes the proof.

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