Hokkaido Mathematical Journal Vol. 40 (2011) p. 103–110

The structure of δ-stable minimal hypersurfaces in $\mathbb{R}^{n+1}$

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(Received December 21, 2009; Revised June 11, 2010)

Abstract. Let $M^n (n \geq 3)$ be a complete $\delta(> \frac{(n-1)^2}{n^2})$-stable minimal hypersurface in an $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. We prove that there are no nontrivial L^2 harmonic 1-forms on M and the first de Rham's cohomology group with compact support of M is trivial. As corollaries, M has only one end. This implies that if M has finite total curvature, then M is a hyperplane.

Key words: end, L^2 harmonic forms, minimal hypersurface, stability.

1. Introduction

Let M^n be a minimal hypersurface in $\mathbb{R}^{n+1}$. M is said to be stable if

$$0 \leq \int_M \left(|\nabla f|^2 - |A|^2 f^2\right), \quad \forall f \in C_0^\infty(M), \tag{1.1}$$

where $|A|$ is the norm of the second fundamental form of M. For some number $0 < \delta \leq 1$, it is defined that M is δ-stable if

$$0 \leq \int_M \left(|\nabla f|^2 - \delta|A|^2 f^2\right), \quad \forall f \in C_0^\infty(M). \tag{1.2}$$

Obviously, given $\delta_1 > \delta_2$, δ_1-stable implies δ_2-stable. So, that M is stable implies that M is δ-stable. By Lemma 1 in [8], M is δ-stable if and only if there is a positive function g satisfying the equation $(\Delta + \delta|A|^2)g = 0$ on M. Tam and Zhou showed that a catenoid in $\mathbb{R}^{n+1}(n \geq 3)$ is $\frac{n-2}{n}$-stable since its second fundamental form A satisfies $\Delta|A|^{\frac{n-2}{n}} + \frac{n-2}{n}|A|^2|A|^{\frac{n-2}{n}} = 0$ in [17].

2000 Mathematics Subject Classification : 53C42.

Supported by Mathematical Tianyuan Youth Foundation of National Natural Science Foundation of China (11026109), Youth Science Foundation of Education Department of Jiangxi Province of China (GJJ11044) and Jiangxi Province Natural Science Foundation of China (2010GZS).

There are some works on δ-stable complete minimal hypersurfaces in $\mathbb{R}^{n+1}$. It is known that a complete stable minimal surface in $\mathbb{R}^3$ must be a plane, which was proved by do Carmo and Peng, and Fischer-Cobrie and Schoen independently [5], [8]. Kawai proved that a $\delta(>\frac{1}{8})$-stable complete minimal surface in $\mathbb{R}^3$ must be a plane [9]. For higher dimension $n \geq 3$, Do Carmo and Peng [6] have showed that a stable complete minimal hypersurface with finite L^2-norm of the second fundamental form in $\mathbb{R}^{n+1}$ is a hyperplane. Shen and Zhu [16] have proved that a complete stable minimal hypersurface in $\mathbb{R}^{n+1}$ with finite total curvature, i.e., $\int_M |A|^n < +\infty$, is a hyperplane. Tam and Zhou [17] showed that an $\frac{n-2}{n}$-stable complete minimal hypersurface whose second fundamental form satisfies some decay conditions in $\mathbb{R}^{n+1}$ is either a hyperplane or a catenoid. In [4], Cheng and Zhou proved that if M is an $\frac{n-2}{n}$-stable complete minimal hypersurface in $\mathbb{R}^{n+1}$ and has bounded norm of the second fundamental from, then M must either have only one end or be a catenoid. Recently Li and the author generalize the above result [7]. However, not much is known for the geometric structure of stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ with $n \geq 3$. Cao-Shen-Zhu proved a topological obstruction for complete stable minimal hypersurface M^n of $\mathbb{R}^{n+1}$ with $n \geq 3$ that M must have only one end [2]. Its strategy was to utilize a result of Schoen-Yau asserting that a complete stable minimal hypersurface of $\mathbb{R}^{n+1}$ can not admit a non-constant harmonic function with finite Dirichlet integral [15]. Assuming that M^n has more than one end, they constructed a non-constant harmonic function with finite Dirichlet integral in [2]. According to the work of Li-Tam [11], Li-Wang modified this proof to show that each end of a complete immersed minimal submanifold M^n of $\mathbb{R}^{n+p}$ with $n \geq 3$ must be non-parabolic in [12]. Due to this connection with harmonic functions, this allows one to estimate the number of ends of the above hypersurface by estimating the dimension of the space of bounded harmonic functions with finite Dirichlet integral [11]. Since the exterior differential form of harmonic function with finite Dirichlet integral is an L^2 harmonic 1-form, the theory of L^2 harmonic forms give one to study minimal submanifolds in $\mathbb{R}^{n+1}$ [12], [18]. In this direction related with stable hypersurfaces, there are some known results. For instance, If M is a complete immersed stable minimal hypersurface in $\mathbb{R}^{n+1}$, then there exist no nontrivial L^2 harmonic 1-forms on M [13], [14].

In this paper, we study an n-dimensional complete $\delta\big(>\frac{(n-1)^2}{n^2}\big)$-stable minimal hypersurface in an $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. Let

$H^1(L^2(M))$ denote the space of L^2 harmonic 1-forms on M, $H_0^1(M)$ denote the first de Rham's cohomology group with compact support of M and Δ denote the Laplacian on M.

Now we can mention our results as follows.

Theorem 1.1 *If $M^n (n \geq 3)$ is a complete $\delta\big(> \frac{(n-1)^2}{n^2}\big)$-stable minimal hypersurface in $\mathbb{R}^{n+1}$, then $H^1(L^2(M)) = 0$.*

Corollary 1.2 *If $M^n (n \geq 3)$ is a complete $\delta\big(> \frac{(n-1)^2}{n^2}\big)$-stable minimal hypersurface in $\mathbb{R}^{n+1}$, then M has only one end. Moreover, $H_0^1(M) = 0$.*

A theorem due to Anderson [1] says that the n-dimensional complete minimal submanifold with only one end and finite total curvature in $\mathbb{R}^{n+p}$ with $n \geq 3$ is an affine plane. Hence by Corollary 1.2 we have the following

Corollary 1.3 *If $M^n (n \geq 3)$ is a complete $\delta\big(> \frac{(n-1)^2}{n^2}\big)$-stable minimal hypersurface with finite total curvature in $\mathbb{R}^{n+1}$, then M is a hyperplane.*

Remark 1.4 Theorem 1.1, Corollary 1.2 and Corollary 1.3 can be regarded as generalization of main theorem in [13], [14], [2] and [16] respectively.

2. Proof of the theorem

Before proving our results, we list some known facts we need.

Definition 2.1 Let $D \subset M$ be a compact subset of M. An end E of M with respect to D is a connected unbounded component of $M \backslash D$. When we say that E is an end, it is implicitly assumed that E is an end with respect to some compact subset $D \subset M$.

Definition 2.2 A manifold is said to be parabolic if it does not admit a positive Green's function. Conversely, a nonparabolic manifold is one which admits a positive Green's function. An end E of a manifold is said to be nonparabolic if it admits a positive Green's function with Neumann boundary condition on ∂E. Otherwise, it is said to be parabolic.

Lemma 2.3 ([11]) *Let M be a complete manifold. Let $\mathcal{H}_D^0(M)$ denote the space of bounded harmonic functions with finite Dirichlet integral. Then the number of non-parabolic ends of M is at most the dimension of $\mathcal{H}_D^0(M)$.*

Lemma 2.4 ([12]) *Let M^n be a complete minimal submanifold of $\mathbb{R}^m$. If*

$n \geq 3$, then each end of M must be nonparabolic.

Before we prove Theorem 1.1, we need the generalized Kato's inequality. Although it was proved in [12], for completeness, we also write it out.

Lemma 2.5 *Let ω be an L^2 harmonic 1-form on a Riemannian manifold M of dimension n. Then*

$$\frac{n}{n-1}\big|\nabla|\omega|\big|^2 \leq |\nabla\omega|^2. \tag{2.1}$$

Proof. By choosing an orthonormal co-frame $\{\omega_1, \ldots, \omega_n\}$ such that $|\omega|\omega_1 = \omega$, we have $\omega = \sum_{i=1}^n a_i\omega_i$. Since ω is an L^2 harmonic 1-form, it must be both closed, i.e., $a_{i,j} = a_{j,i}$, and co-closed, i.e., $\sum_{i=1}^n a_{i,i} = 0$. So we get

$$\begin{aligned}
|\nabla\omega|^2 = a_{i,j}^2 &\geq a_{1,1}^2 + \sum_{i=2}^n a_{i,i}^2 + 2\sum_{i=2}^n a_{1,i}^2 \\
&\geq a_{1,1}^2 + \frac{\big(\sum_{i=2}^n a_{i,i}\big)^2}{n-1} + 2\sum_{i=2}^n a_{1,i}^2 \\
&\geq \frac{n}{n-1}\left(a_{1,1}^2 + \sum_{i=2}^n a_{1,i}^2\right) \\
&= \frac{n}{n-1}\big|\nabla|\omega|\big|^2.
\end{aligned}$$

This completes the proof of Lemma 2.5. □

Proof of Theorem 1.1. For each $\omega \in H^1(L^2(M))$, we have the following well-known Bochner formula.

$$\Delta|\omega|^2 = 2\big(|\nabla\omega|^2 + Ric_M(\omega^\sharp, \omega^\sharp)\big), \tag{2.2}$$

where Ric_M denotes the Ricci curvature of M and $\omega^\sharp$ denotes the vector field dual to ω. On the other hand, we have

$$\Delta|\omega|^2 = 2\big(|\omega|\Delta|\omega| + \big|\nabla|\omega|\big|^2\big). \tag{2.3}$$

From (2.1), (2.2) and (2.3), we obtain

$$|\omega|\Delta|\omega| \geq Ric_M(\omega^\sharp, \omega^\sharp) + \frac{1}{n-1}\big|\nabla|\omega|\big|^2. \tag{2.4}$$

By the Gauss equation, we have

$$\begin{aligned}
|A|^2 &\geq h_{11}^2 + \sum_{i=2}^{n} h_{ii}^2 + 2\sum_{i=2}^{n} h_{1i}^2 \\
&\geq h_{11}^2 + \frac{\left(\sum_{i=2}^{n} h_{ii}\right)^2}{n-1} + 2\sum_{i=2}^{n} h_{1i}^2 \\
&\geq \frac{n}{n-1}\left(h_{11}^2 + \sum_{i=2}^{n} h_{1i}^2\right) \\
&= -\frac{n}{n-1} Ric_M(e_1, e_1).
\end{aligned}$$

Then we get

$$Ric_M \geq -\frac{n-1}{n}|A|^2.$$

Combining with (2.4), we have

$$\begin{aligned}
|\omega|\Delta|\omega| &\geq \frac{1}{n-1}\big|\nabla|\omega|\big|^2 - \frac{n-1}{n}|A|^2|\omega^\sharp|^2 \\
&\geq \frac{1}{n-1}\big|\nabla|\omega|\big|^2 - \frac{n-1}{n}|A|^2|\omega|^2
\end{aligned} \tag{2.5}$$

for $|\omega| = |\omega^\sharp|$.

Fixing a point $p \in M$ and for $r > 0$, we choose a C^1 cut-off function η satisfying $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_r(p) \subset M$, $\eta \equiv 0$ on $M \backslash B_{2r}(p)$, and $|\nabla \eta| \leq \frac{1}{r}$ on $B_{2r}(p) \backslash B_r(p) \subset M$. Multiplying (2.5) by η^2 and integrating by parts over M, we get

$$\begin{aligned}
0 &\leq \int_M \left(\eta^2|\omega|\Delta|\omega| - \frac{1}{n-1}\eta^2\big|\nabla|\omega|\big|^2 + \frac{n-1}{n}|A|^2\eta^2|\omega|^2\right) \\
&= -2\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| - \frac{n}{n-1}\int_M \eta^2\big|\nabla|\omega|\big|^2 + \frac{n-1}{n}\int_M \eta^2|A|^2|\omega|^2.
\end{aligned} \tag{2.6}$$

Choosing $f = \eta|\omega|$ in the stability inequality (1.2), we obtain

$$\delta \int_M |A|^2 \eta^2 |\omega|^2 \leq \int_M |\nabla(\eta|\omega|)|^2.$$

Substituting the above inequality into (2.6), then we have

$$\begin{aligned}
0 &\leq -2\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| - \frac{n}{n-1}\int_M \eta^2\big|\nabla|\omega|\big|^2 + \frac{n-1}{n\delta}\int_M |\nabla(\eta|\omega|)|^2 \\
&\leq -2\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| + \frac{n-1}{n\delta}\int_M \left(|\omega|^2|\nabla\eta|^2 + \eta^2\big|\nabla|\omega|\big|^2\right) \\
&\quad + \frac{2(n-1)}{n\delta}\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| - \frac{n}{n-1}\int_M \eta^2\big|\nabla|\omega|\big|^2 \\
&\leq 2\left(\frac{n-1}{n\delta} - 1\right)\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega| + \frac{(n-1)^2 - n^2\delta}{n(n-1)\delta}\int_M \eta^2\big|\nabla|\omega|\big|^2 \\
&\quad + \frac{n-1}{n\delta}\int_M |\omega|^2|\nabla\eta|^2. \qquad (2.7)
\end{aligned}$$

Using Schwarz inequality, we get

$$2\left|\int_M \eta\langle\nabla\eta, \nabla|\omega|\rangle|\omega|\right| \leq \epsilon\int_M \eta^2\big|\nabla|\omega|\big|^2 + \frac{1}{\epsilon}\int_M |\omega|^2|\nabla\eta|^2. \qquad (2.8)$$

From (2.7) and (2.8), we obtain

$$\begin{aligned}
&\left(\frac{n^2\delta - (n-1)^2}{n(n-1)\delta} - \frac{(n(1-\delta)-1)\epsilon}{n\delta}\right)\int_M \eta^2\big|\nabla|\omega|\big|^2 \\
&\quad \leq \frac{n(1-\delta) - 1 + (n-1)\epsilon}{n\delta\epsilon}\int_M |\omega|^2|\nabla\eta|^2 \\
&\quad \leq \frac{n(1-\delta) - 1 + (n-1)\epsilon}{n\delta\epsilon}\frac{1}{r^2}\int_{B_{2r}(p)} |\omega|^2.
\end{aligned}$$

Since $\delta > \frac{(n-1)^2}{n^2}$ and $\int_M |\omega|^2 < \infty$, choosing $\epsilon > 0$ sufficiently small and letting $r \to \infty$, we get $\nabla|\omega| = 0$ on M, i.e., $|\omega|$ is constant. Since $\int_M |\omega|^2 < \infty$, and the volume of M is infinite by Lemma 1 of [2], we have $\omega = 0$. Hence $H^1(L^2(M)) = 0$. $\square$

Proof of Corollary 1.2. Observe that if f is a harmonic function with finite Dirichlet integral then its exterior df is an L^2 harmonic 1-form. Moreover, $df = 0$ if and only if f is identically constant. Hence

$$\dim \mathcal{H}_D^0(M) \leq \dim H^1(L^2(M)) + 1.$$

Due to Lemma 2.3 and Theorem 1.1, we conclude that M has at most a non-parabolic end. By Lemma 2.4, M has only one end. Moreover, we obtain $H_0^1(M) = 0$ according to Lemma 2.3 in [3]. □

Acknowledgment The author would like to thank the referees for some helpful suggestions.

References

[1] Anderson M. T., *The compactification of a minimal submanifold in Euclidean space by the Gauss map.* preprint (final version in Dept. of Math., California Institute of Technology, Pasadena, CA 91125), 1986.

[2] Cao H. D., Shen Y. and Zhu S., *The structure of stable minimal hypersurfaces in R^{n+1}.* Math. Res. Lett. **4** (1997), 637–644.

[3] Carron G., *L^2 harmonic forms on non compact manifolds.* arXiv:0704.3194vl.

[4] Cheng X. and Zhou D. T., *Manifolds with weighted Poincaré inequality and uniqueness of minimal hypersurfaces.* Comm. Anal. and Geom. **17** (2009), 1–16.

[5] do Carmo M. and Peng C. K., *Stable complete minimal surfaces in R^3 are planes.* Bull. Amer. Math. Soc. **1** (1979), 903–906.

[6] do Carmo M. and Peng C. K., *Stable complete minimal hypersurfaces.* Proceeding of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 3 (Beijing, 1980), 1349–1358, Science Press, Beijing, 1982.

[7] Fu H. P. and Li Z. Q., *The structure of complete manifolds with weighted Poincaré inequality and minimal hypersurfaces.* International Journal of Mathematics. **21** (2010), 1421–1428.

[8] Fischer-Colbrie D. and Schoen R., *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature.* Comm. Pure. Appl. Math. **33** (1980), 199–211.

[9] Kawai S., *Operator $\Delta - aK$ on surface.* Hokkaido Math. J. **17** (1988), 147–150.

[10] Li P., *Curvature and function theory on Riemannian manifolds*, In Honor

of Atiyah, Bott, Hirzebruch, and Singer, 71–111, Survey in Differential Geometry vol. VII, International Press, Cambridge, 2000.

[11] Li P. and Tam L. F., *Harmonic functions and the structure of complete manifolds*. J. Diff. Geom. **35** (1992), 359–383.

[12] Li P. and Wang J. P., *Minimal hypersurfaces with finite index*. Math. Res. Lett. **9** (2002), 95–103.

[13] Miyaoka R., *L^2 harmonic 1-forms on a complete stable minimal hypersurfaces*. Geometry and Global Analysis (Sendai 1993), Tohoku Univ., 289–293.

[14] Palmer B., *Stability of minimal hypersurfaces*. Comment. Math. Helv. **66** (1991), 185–188.

[15] Schoen R. and Yau S. T., *Harmonic maps and the topology of stable hypersurfaces and manifolds with non-negative Ricci curvature*. Comm. Math. Helv. **51** (1976), 333–341.

[16] Shen Y. B. and Zhu X. H., *On the stable complete minimal hypersurfaces in $\mathbb{R}^{n+1}$*. Amer. J. Math. **120** (1998), 103–116.

[17] Tam L. F. and Zhou D. T., *Stability properties for the higher dimensional catenoid in $\mathbb{R}^{n+1}$*. Proc. Amer. Math. Soc. **137** (2009), 3451–3461.

[18] Yun G., *Total scalar curvature and L^2 harmonic 1-forms on a minimal hypersurface in Euclidean space*. Geom. Dedicata **89** (2002), 135–141.

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