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On the Riesz bases, frames and minimal exponential systems in $L^2[-\pi,\pi]$

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Abstract. P. G. Casazza, O. Christensen, S. Li, and A. Lindner proved in [3] that some families of complex exponentials were either Riesz bases or not frames in $L^2[-\pi,\pi]$. First, we shall advance their results in this note. Sedletskii constructed in [9] an exponential system which is complete, minimal and not uniformly minimal with separable spectrum in $L^2[-\pi,\pi]$. Next, we shall construct a similar example with nonseparable spectrum in $L^2[-\pi,\pi]$.

Key words: Riesz basis, frame, minimal, uniformly minimal.

1. Introduction

A sequence $\{x_n\}$ in a Hilbert space H is said to be *complete* in H if the linear subspace $\mathrm{span}\{x_n\}$ spanned by the distinct elements x_n is dense in H, i.e., $\overline{\mathrm{span}}\{x_n\} = H$. The sequence $\{x_n\}$ is said to be *minimal* in H if each element of $\{x_n\}$ lies outside the closed linear span of the others, i.e., $x_k \notin \overline{\mathrm{span}}\{x_n\}_{n\neq k}$. If we write $M_k = \overline{\mathrm{span}}\{x_n\}_{n\neq k}$ for each k, this means that

$$d_k = \mathrm{dist}(x_k, M_k) = \inf_{x\in M_k} \|x_k - x\| > 0.$$

Also, it is said to be *uniformly minimal* if

$$d_k \geq \delta\|x_k\|$$

for each k, where δ is a positive constant independent of k. It is well known that if $\{x_n\}$ is basis, it is uniformly minimal (see [11, Theorem 3.1]). Also, $\{x_n\}$ is said to be ω-*independent* if $\sum_n \alpha_n x_n = 0$ implies that $\alpha_n = 0$ for all n. It is trivial that if $\{x_n\}$ is minimal, then it is ω-independent.

Next we say that $\{x_n\}$ is a *frame* for H if there exist constants $A, B > 0$ such that

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$$A\|x\|^2 \le \sum_{n=1}^{\infty} |(x, x_n)|^2 \le B\|x\|^2.$$

If we replace H with $\overline{\text{span}}\{x_n\}$, we say that $\{x_n\}$ is a *frame sequence.*

We say that $\{x_n\}$ is a *Riesz basis* for H if it is complete in H and there exist constants $A, B > 0$ such that

$$A \sum |c_n|^2 \le \left\| \sum c_n x_n \right\|^2 \le B \sum |c_n|^2$$

for all finite scalar sequences $\{c_n\}$. If we replace H with $\overline{\text{span}}\{x_n\}$, we say that $\{x_n\}$ is a *Riesz sequence.* It is well known that if $\{x_n\}$ is a frame and is ω-independent, then it is a Riesz basis (see the proof of Proposition 2.1).

A sequence $\mathbf{\Lambda} = \{\lambda_n\}_{n\in\mathbb{Z}}$ of complex numbers is said to be *separable* if

$$\inf_{n\ne m} |\lambda_n - \lambda_m| > 0.$$

In this note, we consider $H = L^2[-\pi, \pi]$ and $\{x_n\} = \{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ for a sequence $\mathbf{\Lambda} = \{\lambda_n\}_{n\in\mathbb{Z}}$ of distinct complex numbers with $\sup_n |\text{Im}\, \lambda_n| < \infty$. Then, $\mathbf{\Lambda}$ is said to be a *spectrum* with respect to the system $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$. And we raise the relationships between Riesz bases (Riesz sequences) and frames (frame sequences).

P. G. Casazza, O. Christensen, S. Li, and A. Lindner [3] obtained the next result using the result of Balan [2].

Theorem A ([3, Proposition 16.10]) *Let $\{\lambda_k\}_{k\in\mathbb{Z}}$ be a sequence of real numbers such that*

$$\sup_{k\in\mathbb{Z}} |\lambda_k - k| = \frac{1}{4}. \tag{1.1}$$

Then either $\{e^{i\lambda_k t}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi, \pi]$ or it is not a frame for $L^2[-\pi, \pi]$.

We shall obtain the result which except the condition (1.1) from Theorem A in Section 2.

Next, Sedletskii constructed in [9, Theorem 3] an exponential system which is complete, minimal and not uniformly minimal with separable spectrum. We shall construct a similar example with nonseparable spectrum in

Section 3.

2. Riesz bases and frames

In this section, we examine relations of Riesz bases (Riesz sequences) and frames (frame sequences) aiming at generalizing Theorem A.

Proposition 2.1 *If $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is ω-independent, then it is either a Riesz sequence or not a frame sequence in $L^2[-\pi,\pi]$.*

Proof. We suppose that $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a frame sequence. If we refer to the proof of Theorem 12 in Young [13, Ch.4, Section 7], we see that if a frame is ω-independent, then it is a Riesz basis. Consequently, $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a Riesz sequence since it is ω-independent. □

The next result is well known.

Theorem B (Schwarz, 1943; see Alexander and Redheffer [1, p. 61, Remark 4]) *If $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is incomplete, then it is minimal.*

This result is one of the intrinsic properties of complex exponential systems. If it is not a complex exponential system, we easily have a counterexample.

Example 2.1 The system

$$\left\{e^{it}+e^{i2t},\ e^{it},\ e^{i2t},\ e^{i3t},\dots,e^{int},\dots\right\}$$

is incomplete, but not minimal in $L^2[-\pi,\pi]$.

Corollary 2.1 *If $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is incomplete, then it is either a Riesz sequence or not a frame sequence in $L^2[-\pi,\pi]$.*

Proof. If $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is incomplete, it is minimal by Theorem B. Consequently, the result follows from Proposition 2.1. □

Now, we define the notion of excess introduced by Paley and Wiener (see Redheffer [7, p. 2]). We say the system $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ has *excess N* if it remains complete and becomes minimal when N terms $e^{i\lambda_n t}$ are removed and we define

$$E(\mathbf{\Lambda}) = N.$$

Conversely we define the excess

$$E(\mathbf{\Lambda}) = -N$$

if it becomes complete and minimal when N terms

$$e^{i\mu_1 t}, \dots, e^{i\mu_N t}$$

are adjoined. By convention we define $E(\mathbf{\Lambda}) = \infty$ if arbitrarily many terms can be removed without losing completeness and $E(\mathbf{\Lambda}) = -\infty$ if arbitrarily many terms can be adjoined without getting completeness. We see that $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is minimal if and only if $E(\mathbf{\Lambda}) \leq 0$ from Theorem B. And it is obvious that $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is to be complete and minimal if and only if $E(\mathbf{\Lambda}) = 0$.

Corollary 2.2 *Let a, b be nonnegative constants and $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ be a complex sequence such that*

$$\varepsilon_0 = 0, \quad \sup_{n\neq 0} |\mathrm{Re}\,\varepsilon_n| < \frac{1}{4}, \quad \sup_{n\neq 0} |\mathrm{Im}\,\varepsilon_n| < \infty.$$

If we define the sequence $\mathbf{\Lambda} = \{\lambda_n\}_{n\in\mathbb{Z}}$ as follows,

$$\lambda_n = \begin{cases} n + \varepsilon_n + a, & n > 0, \\ 0, & n = 0, \\ n + \varepsilon_n - b, & n < 0, \end{cases}$$

then $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is either a Riesz sequence or not a frame sequence in $L^2[-\pi, \pi]$.

Proof. If

$$\begin{cases} \mu_0 = 0 \\ \mu_n = n + \varepsilon_n, \quad n \neq 0, \end{cases}$$

then $\{e^{i\mu_n t}\}_{n\in\mathbb{Z}}$ is a Riesz basis ([13, p. 164, Corollary 2]). By [4], we have

$$E(\mathbf{\Lambda}) \leq E(\boldsymbol{\mu}) = 0.$$

Hence $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is minimal. The conclusion follows from Proposition 2.1. □

We are particularly interested in the case of which $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ is a real sequence and $a = b = 1/4$ in Corollary 2.2, i.e.,

$$\lambda_n = \begin{cases} n + \frac{1}{4} + \varepsilon_n, & n > 0, \\ 0, & n = 0, \\ n - \frac{1}{4} + \varepsilon_n, & n < 0, \end{cases} \tag{2.1}$$

with $\sup_{n\neq 0} |\varepsilon_n| < \frac{1}{4}$. Now let α be a real number and we consider the isometric isomorphism

$$\phi(t) \longmapsto e^{i\alpha t}\phi(t) \tag{2.2}$$

on $L^2[-\pi,\pi]$. Besides we know the next result:

Proposition A (e.g. [6, Corollary 1.1]) *We suppose that* $\sup_n |\mathrm{Im}\ \lambda_n| < \infty$ *and* $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ *is a basis. If we replace finitely many points* λ_n *by the same number of points* $\mu_n \notin \{\lambda_n\}$, $\mu_n \neq \mu_m$, $n \neq m$, *then the basis property of* $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ *is not violated. Consequently the same applies to any Riesz basis.*

Using the above isomorphism and Proposition A, if we consider the next sequence,

$$\mu_n = \begin{cases} n - \frac{1}{4} + \varepsilon_n, & n > 0, \\ n + \frac{1}{4} + \varepsilon_n, & n < 0, \end{cases}$$

then we see that $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ with the λ_n given by (2.1) has the same basis properties as $\{e^{i\mu_n t}\}_{n\neq 0}$. Hence we see by Corollary 2.2 that $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ and $\{e^{i\mu_n t}\}_{n\neq 0}$ are either Riesz sequences or not frame sequences in $L^2[-\pi,\pi]$. If we choose $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ in (2.1) such that

$$\varepsilon_n \begin{cases} \leq 0, & n > 0, \\ \geq 0, & n < 0, \end{cases}$$

and $\inf_{n\neq 0} |\varepsilon_n| = 0$, then we again obtain Theorem A.

Besides Redheffer and Young [8] proved the next result for the case $\varepsilon_n = \beta/\log n$ $(n \geq 2)$.

Theorem C ([8, Theorem 3]) *Let*

$$\mu_n = \begin{cases} 0, & n = 0, \\ 1, & n = 1, \\ n + \frac{1}{4} + \frac{\beta}{\log n}, & n \geq 2, \\ -\mu_{-n}, & n < 0, \end{cases}$$

then $\{e^{i\mu_n t}\}_{n\in\mathbb{Z}}$ *is complete in* $L^2[-\pi, \pi]$ *if* $0 \leq \beta \leq 1/4$ *and not if* $\beta > 1/4$.

We remark that the sequence $\boldsymbol{\mu} = \{\mu_n\}_{n\in\mathbb{Z}}$ does not satisfy the condition (1.1) in Theorem A for $\beta > 0$. We have proved in [6, Theorem 2.1 and Section 3] that $\{e^{i\mu_n t}\}_{n\in\mathbb{Z}}$ is not a Riesz basis for $0 \leq \beta \leq 1/4$ and not a Riesz sequence for $\beta > 1/4$. Consequently, we conclude by Corollary 2.2 that $\{e^{i\mu_n t}\}_{n\in\mathbb{Z}}$ is not a frame for $0 \leq \beta \leq 1/4$ and not a frame sequence for $\beta > 1/4$.

The next result is proved inductively by using Young [13, p. 156, Lemma 6].

Proposition B (see [13, p. 156, Lemma 6]) *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a frame in a Hilbert space* H *and* I *be a finite subset of* $\mathbb{N}$,

$$I = \{n_1, n_2, \ldots, n_m\}.$$

Then $\{f_n\}_{n\in\mathbb{N}-I}$ *leaves either a frame or an incomplete set.*

We have the next result from Proposition B.

Theorem 2.1 *Let* $\boldsymbol{\Lambda} = \{\lambda_n\}_{n\in\mathbb{Z}}$ *be a sequence of complex numbers satisfying*

$$|\lambda_n - n| \leq L, \tag{2.3}$$

where L *is a positive constant and assume that* $E(\boldsymbol{\Lambda}) = m \geq 1$, *and let*

$$I = \{n_1, n_2, \ldots, n_m\} \subset \mathbb{Z}.$$

Then $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}-I}$ *is a Riesz basis if and only if* $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ *is a frame.*

Proof. First, we remark that $E(\mathbf{\Lambda})$ is finite under the condition (2.3) by [7, Theorem 47]. We suppose that $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}-I}$ is a Riesz basis. We prove only the case $m=1$ since the arguement of the case $m\geq 2$ is similar to one of $m=1$. If $\{e^{i\lambda_n t}\}_{n\neq n_1}$ is a Riesz basis, then it is a frame with same bounds. Hence there exist positive constants A, B such that

$$A\|f\|^2 \leq \sum_{n\neq n_1} \left|(f, e^{i\lambda_n t})\right|^2 \leq B\|f\|^2$$

for $\forall f \in L^2[-\pi,\pi]$. It is trivial that

$$\left|(f, e^{i\lambda_{n_1} t})\right| \leq C\|f\|,$$

where C is a positive constant. Consequently we have

$$\begin{aligned} A\|f\|^2 &\leq \sum_{n\neq n_1} \left|(f, e^{i\lambda_n t})\right|^2 + \left|(f, e^{i\lambda_{n_1} t})\right|^2 \\ &\leq (B+C^2)\|f\|^2. \end{aligned}$$

Hence $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a frame.

Conversely, we suppose that $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is a frame. Since $E(\mathbf{\Lambda}) = m$, $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}-I}$ is complete. Then it is a frame by Proposition B. Besides it is also minimal, hence $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}-I}$ is a Riesz basis. □

Remark 2.1 Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers such that

$$\sup_{n\in\mathbb{Z}} |\lambda_n - n| = \frac{1}{4},$$

then we have $E(\mathbf{\Lambda}) = 0$ or 1 by [7, Theorem 47]. If $E(\mathbf{\Lambda}) = 1$, $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is not a Riesz basis. Hence it is not a frame by Theorem A. Consequently, $\{e^{i\lambda_n t}\}_{n\neq n_1}$ for any $n_1 \in \mathbb{Z}$ is not a Riesz basis by Theorem 2.1.

Finally, we give some examples.

Example 2.2 Let

$$\mu_n = \begin{cases} n - \frac{1}{4}, & n > 0, \\ n + \frac{1}{4}, & n < 0, \end{cases}$$

then the system $\{e^{i\mu_n t}\}_{n\neq 0}$ is complete and minimal by Levinson [5, p. 67] (see [13, Remark, p. 105]) and it is not a basis (Young [12, Theorem 2]). Therefore, since it is also not a Riesz basis, it is not a frame by Proposition 2.1.

The following example is also given by [3, Example 16.11].

Example 2.3 Let

$$\lambda_n = \begin{cases} n - \frac{1}{4}, & n > 0, \\ 0, & n = 0, \\ n + \frac{1}{4}, & n < 0, \end{cases}$$

then $E(\mathbf{\Lambda}) = 1$ and $\{e^{i\lambda_n t}\}_{n\neq 0}$ is not a Riesz basis as shown by the above example. Consequently, $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is not a frame by Theorem 2.1. Incidentally, using the result which the system $\{e^{i\mu_n t}\}_{n\neq 0}$ in Example 2.2 is not a basis, if we follow the arguement in Singer [11, Ch.I, Section 6, Example 6.1-b)], we see that $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is an example which is ω-independent, but not minimal.

3. Minimal exponential system with nonseparable spectrum

In this section, we construct an exponential system which is complete and minimal with nonseparable spectrum, hence not uniformly minimal.

Proposition C (see Sedletskii [10, p. 3569]) *If* $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ *is complete in* $L^2[-\pi, \pi]$*, then the following statements are equivalent*:

- (i) $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ *is uniformly minimal.*
- (ii) $\inf_k d_k = \inf_k \operatorname{dist}(e^{i\lambda_k t}, M_k) > 0$,
  where $M_k = \overline{\operatorname{span}}\{e^{i\lambda_n t}\}_{n\neq k}$.
- (iii) *There exists a sequence of coefficient functionals* $\{f_n\}_{n\in\mathbb{Z}} \subset L^2[-\pi, \pi]$ *associated to* $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ *such that* $\sup_n \|f_n\| < \infty$.

Since $\sup_n |\mathrm{Im}\,\lambda_n| < \infty$, we have

$$0 < \inf_n \|e^{i\lambda_n t}\| \le \sup_n \|e^{i\lambda_n t}\| < \infty.$$

Hence, it is trivial that (i) is equivalent to (ii). The equivalence of (ii) and (iii) follows from the Hahn-Banach theorem and the fact that the sequence $\{f_n\}_{n\in\mathbb{Z}}$ of coefficient functionals associated to $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is uniquely determined.

Using Proposition C, we have the next result. Sedletskii obtained a same result in [10, p. 3569].

Proposition 3.1 *If* $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ *is uniformly minimal, the spectrum* $\boldsymbol{\Lambda} = \{\lambda_n\}_{n\in\mathbb{Z}}$ *is separable.*

Proof.

$$\begin{aligned} \big\|e^{i\lambda_k t} - e^{i\lambda_n t}\big\| &= \big\|\big\{e^{i(\lambda_k-\lambda_n)t} - 1\big\}e^{i\lambda_n t}\big\| \\ &\le C\big\|e^{i(\lambda_k-\lambda_n)t} - 1\big\| \qquad (3.1) \\ &= C\bigg\|\sum_{j=1}^{\infty} \frac{\{i(\lambda_k-\lambda_n)\}^j}{j!} t^j\bigg\| \\ &\le C\sum_{j=1}^{\infty} \frac{|\lambda_k-\lambda_n|^j}{j!}\pi^j \\ &\le C\big(e^{|\lambda_k-\lambda_n|\pi} - 1\big). \end{aligned}$$

Now, since $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$ is uniformly minimal, we have by Proposition C,

$$\big\|e^{i\lambda_k t} - e^{i\lambda_n t}\big\| \ge d_k \ge \delta \qquad (3.2)$$

for a positive constant δ. However, if

$$\inf_{n\ne k} |\lambda_k - \lambda_n| = 0,$$

then we obtain by (3.1)

$$\inf_{n\ne k} \big\|e^{i\lambda_k t} - e^{i\lambda_n t}\big\| = 0.$$

This contradicts (3.2). □

Sedletskii showed in [9] that the converse of Proposition 3.1 did not hold. He constructed the exponential system with real *separable spectrum* such that it was *complete and minimal, but not uniformly minimal.*

Theorem D ([9, Theorem 3]) *Let* V *be the sequence of all integers in the intervals*

$$I_s = [2^s, 2^s + [\log s]], \quad s \geq 3.$$

Let

$$\mathbf{\Lambda} = (n : n < 0,\ n \in V) \cup \left(n - \frac{1}{2} : n \in \mathbb{N}\backslash V\right).$$

Then $\{e^{i\lambda_n t}\}_{n\in\mathbf{\Lambda}}$ *is complete and minimal, but not uniformly minimal in* $L^2[-\pi,\pi]$.

This example can be written as follows:

$$\lambda_n = \begin{cases} n - \frac{1}{2}, & n \in \mathbb{N}\backslash V, \\ n, & n < 0 \text{ or } n \in V. \end{cases} \tag{3.3}$$

By the way, Redheffer and Young showed that $\{e^{i\mu_n t}\}_{n\in\mathbb{Z}}$ with the spectrum given by the following (3.4) was complete and minimal in [8, Lemma 1 and Remark]. Moreover, they showed that the coefficient functionals of the system were uniformly bounded, consequently the system were uniformly minimal.

Theorem E ([8, Theorem 5]) *Let*

$$\mu_n = \begin{cases} n + \frac{1}{4}, & n > 0, \\ 0, & n = 0, \\ n - \frac{1}{4}, & n < 0, \end{cases} \tag{3.4}$$

then the sequence $\{f_n\}$ *of coefficient functionals associated to the system* $\{e^{i\mu_n t}\}_{n\in\mathbb{Z}}$ *satisfies* $\sup_n \|f_n\| < \infty$.

Appling the isomorphism defined by (2.2) to the system $\{e^{i\mu_n t}\}_{n\in\mathbb{Z}}$ in Theorem E, we see that the exponential system $\{e^{i\gamma_n t}\}_{n\neq 0}$ with the following spectrum $\{\gamma_n\}_{n\neq 0}$,

$$\gamma_n = \begin{cases} n - \frac{1}{2}, & n > 0, \\ n, & n < 0, \end{cases} \tag{3.5}$$

is also uniformly minimal. We remark that the example of Sedletskii given by (3.3) is gained by moving a part of the spectrum given by (3.5). Now, the spectrum $\mathbf{\Lambda} = \{\lambda_n\}$ in (3.3) is separable. Hence we construct a similar example with nonseparable spectrum. We state the known results which we need.

Theorem F (Schwarz, 1959; see [13, p. 117, Theorem 15]) *If $\mathbf{\Lambda} = \{\lambda_n\}$ is a sequence of real numbers such that $\sum 1/|\lambda_n| < \infty$, then $\{e^{i\lambda_n t}\}$ fails to be complete in $L^2[-A, A]$ for any positive number A.*

Theorem G ([7, Theorem 47]) *For $-\infty < n < \infty$, let $\mathbf{\Lambda} \equiv \{\lambda_n\}$ be a sequence of complex numbers satisfying $|\lambda_n - n| \le h$ where h is a positive constant. Then $E(\mathbf{\Lambda})$ satisfies*

$$-\left(4h + \frac{1}{2}\right) < E(\mathbf{\Lambda}) \le 4h + \frac{1}{2}.$$

Next, we denote by $n(t)$ the number of points λ_n inside the disk $|z| \le t$ and we put

$$N(r) = \int_1^r \frac{n(t)}{t} dt.$$

Levinson obtained the following result in [5].

Theorem H ([5, Theorem III]; see [13, p. 99, Theorem 3]) *The set $\{e^{i\lambda_n t}\}$ is complete in $L^2[-\pi, \pi]$ whenever*

$$\limsup_{r\to\infty} \left(N(r) - 2r + \frac{1}{2}\log r\right) > -\infty.$$

Lemma 3.1 *We define the sequnce $\mathbf{\Lambda} = \{\lambda_n\}$ as follows:*

$$\mathbf{\Lambda} = \left\{2, 2 + \frac{1}{2}, 2^2, 2^2 + \frac{1}{2^2}, \ldots, 2^n, 2^n + \frac{1}{2^n}, \ldots\right\}.$$

Then $\{e^{i\lambda_n t}\}$ is minimal and $\mathbf{\Lambda}$ is nonseparable.

Proof. Since

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}=\sum_{n=1}^{\infty}\frac{1}{2^n}+\sum_{n=1}^{\infty}\frac{1}{2^n+\dfrac{1}{2^n}}<\infty,$$

we see that $\{e^{i\lambda_n t}\}$ is minimal by Theorem B and Theorem F. $\mathbf{\Lambda}$ is nonseparable obviously. □

Using this lemma, we obtain the next result.

Theorem 3.1 *The system $\{e^{i\lambda_n t}\}$ in Lemma 3.1 can be extended to a complete, minimal and not uniformly minimal exponential system $\{e^{i\mu_n t}\}$ with nonseparable spectrum in $L^2[-\pi,\pi]$.*

Proof. We define the sequence $\boldsymbol{\mu}=\{\mu_n\}$ as follows:

$$\mu_n=\begin{cases}2^k+\dfrac{1}{2^k}, & n=2^k+1,\\ n, & n\neq 2^k+1,\end{cases}$$

for $n=0,1,2,\dots$ and $k=1,2,\dots$ and let

$$\mu_{-n}=-\mu_n,\ n=1,2,\dots$$

Obviously, $\mathbf{\Lambda}=\{\lambda_n\}\subset\{\mu_n\}$ and we have

$$|\mu_n-n|\le 1,\ \forall n.$$

Consequently, by Theorem G, the excess $E(\boldsymbol{\mu})$ satisfies

$$-4\le E(\boldsymbol{\mu})\le 4.$$

Let denote by $n_1(t)$ and $n_2(t)$ the number of integers in $|z|\le t$ and the number of points μ_n in $|z|\le t$, respectively. We put

$$N_1(r)=\int_1^r\frac{n_1(t)}{t}dt,\quad N_2(r)=\int_1^r\frac{n_2(t)}{t}dt.$$

Since $N_1(r)\le N_2(r)$, we have

$$\limsup_{r\to\infty}\left(N_2(r)-2r+\frac{1}{2}\log r\right)\geq\limsup_{r\to\infty}\left(N_1(r)-2r+\frac{1}{2}\log r\right)>-\infty.$$

Hence, we see that $\{e^{i\mu_n t}\}$ is complete by Theorem H, consequently,

$$0\leq E(\boldsymbol{\mu})\leq 4.$$

If we delete 4 μ_n's $\notin \boldsymbol{\Lambda}$ from $\boldsymbol{\mu}$ at most, the rest system $\{e^{i\mu_n t}\}$ is complete and minimal in $L^2[-\pi,\pi]$. Since $\boldsymbol{\mu}$ is nonseparable, $\{e^{i\mu_n t}\}$ is not uniformly minimal by Proposition 3.1. □

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