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Riesz transforms on generalized Hardy spaces and a uniqueness theorem for the Navier-Stokes equations

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Abstract. The purpose of this paper is twofold. Let R_j $(j = 1, 2, \ldots, n)$ be Riesz transforms on $\mathbb{R}^n$. First we prove the convergence of truncated operators of $R_i R_j$ in generalized Hardy spaces. Our first result is an extension of the convergence in $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. Secondly, as an application of the first result, we show a uniqueness theorem for the Navier-Stokes equation. J. Kato (2003) established the uniqueness of solutions of the Navier-Stokes equations in the whole space when the velocity field is bounded and the pressure field is a BMO-valued locally integrable-in-time function for bounded initial data. We extend the part "BMO-valued" in his result to "generalized Campanato space valued". The generalized Campanato spaces include L^1, BMO and homogeneous Lipschitz spaces of order α $(0 < \alpha < 1)$.

Key words: Navier-Stokes equation, uniqueness, Campanato space, Hardy space

1. Introduction

Riesz transforms R_j $(j = 1, 2, \ldots, n)$ on n-dimensional Euclidean space $\mathbb{R}^n$, which are typical examples of singular integral operators, are very important tools for studying on partial differential equations. To study singular integral operators, the convergence of their truncated operators give us many useful information.

The purpose of this paper is twofold. First we prove the convergence of truncated operators of $R_i R_j$ by smooth cut off functions. We consider the convergence in generalized Hardy spaces introduced in [12]. It is known that, for some of singular integral operators, their truncated operators converge in $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. Our first result is one of its extension. Secondly, as an application of our first result, we show the uniqueness of nondecaying solutions for the Navier-Stokes equation in generalized Campanato spaces

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which are the dual of generalized Hardy spaces.

We are concerned in the Navier-Stokes equation,

$$u_t - \Delta u + (u, \nabla)u + \nabla p = 0 \quad \text{in } (0,T) \times \mathbb{R}^n, \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } (0,T) \times \mathbb{R}^n, \tag{1.2}$$

with initial data $u|_{t=0} = u_0$, where $u = u(t,x) = (u_1(t,x), \dots, u_n(t,x))$ ($n \geq 2$) and $p = p(t,x)$ stand for the unknown velocity vector field of the fluid and its pressure field respectively, while $u_0 = u_0(x) = (u_0^1(x), \dots, u_0^n(x))$ is the given initial velocity vector field.

J. Kato [4] established the uniqueness of solutions of the Navier-Stokes equations when the velocity field u and the pressure field p satisfy

$$u \in L^\infty((0,T) \times \mathbb{R}^n), \quad p \in L^1_{\mathrm{loc}}((0,T); \mathrm{BMO}), \tag{1.3}$$

for bounded initial data u_0. In this paper we extend the part "BMO" in his result to generalized Campanato spaces "$\mathcal{L}_{1,\phi}$", where ϕ is a function from $(0,\infty)$ to itself. If $\phi \equiv 1$, then $\mathcal{L}_{1,\phi} = \mathrm{BMO}$. If $\phi \geq 1$, then $\mathcal{L}_{1,\phi} \supset \mathrm{BMO}$. The definition of $\mathcal{L}_{1,\phi}$ is in the next section.

Galdi and Maremonti [1] showed that if u and ∇u are bounded in $(0,T) \times \mathbb{R}^3$, then the uniqueness of classical solutions holds provided that for some $C > 0$ and some $\epsilon > 0$ the inequality

$$|p(t,x)| \leq C(1 + |x|)^{1-\epsilon} \tag{1.4}$$

holds. See also [6] and [15].

The assumption (1.3) does not imply (1.4). If $0 < \alpha < 1$ and

$$\phi(r) = \begin{cases} 1 & \text{for} \quad 0 < r < 1, \\ r^\alpha & \text{for} \quad r \geq 1, \end{cases} \tag{1.5}$$

then our function space $\mathcal{L}_{1,\phi}$ includes BMO and contains functions f such that

$$|f(x)| \leq C\phi(1 + |x|) = C(1 + |x|)^\alpha \quad \text{for} \quad x \in \mathbb{R}^n.$$

Therefore, our result is an extension of both Kato's theorem and the result of Galdi and Maremonti. Moreover, our theorem holds for $\mathcal{L}_{1,\phi}$ with

$$\phi(r) = \begin{cases} r^{-n} & \text{for} \quad 0 < r < 1, \\ r(\log(1+r))^{-\beta} & \text{for} \quad r \geq 1, \end{cases} \tag{1.6}$$

where $\beta > 1$ if $n \geq 3$ and $\beta > 2$ if $n = 2$. In this case $\mathcal{L}_{1,\phi}$ includes $L^1 \cup \mathrm{BMO}$ and contains functions f such that

$$|f(x)| \leq C\phi(1+|x|) = C(1+|x|)(\log(2+|x|))^{-\beta} \quad \text{for} \quad x \in \mathbb{R}^n.$$

Giga, Inui and Matsui [3] pointed out that the equation (1.1), (1.2) has trivial non-constant solutions of the form $u(x,t) = b(t)$, $p(x,t) = -b'(t)x$. Therefore, the uniqueness fails for $\beta = 0$ and is still open for $0 < \beta \leq 1$ ($0 < \beta \leq 2$ if $n = 2$).

Recently, Koch, Nadirashvili, Seregin and Sverák [5] proved that, if a weak solution u is in $L^\infty((0,T) \times \mathbb{R}^n)$, then u takes a certain explicit form with the mild solution and a function $b(t)$ which is independent of x. In [5] they assumed the divergence free condition on the test functions. In this paper we don't assume the divergence free condition, since we are interested in not only velocity u but also pressure p.

Our first result for generalized Hardy spaces is an extension of Proposition 1 in Kato [4]. He proved the result for the Hardy space H^1 by using the maximal characterization of H^1. Then he proved his uniqueness theorem by the duality $(H^1)^* = \mathrm{BMO}$. The proof method for our second result is as the same as Kato's. However, we need a new result on generalized Hardy spaces to prove our first result. We use the atomic decomposition of functions by general atoms. It is known that $(H^p)^* = \mathrm{Lip}_\alpha$ when $\alpha = n(1/p - 1)$ and $n/(n+1) < p < 1$. Zorko [17] studied preduals of Morrey spaces. Our generalized Hardy spaces are a generalization of both the usual H^p and Zorko's predual.

In the next section we state the definitions of the function spaces. The first and second results are in Sections 3 and 4, respectively. Section 5 is to prove a lemma on generalized Hardy spaces which is used in the proof of the first result.

2. Definitions of function spaces

Let $\mathcal{S}$ be the space of rapidly decreasing functions in $\mathbb{R}^n$ and $\mathcal{S}'$ be the space of tempered distributions in the sense of Schwartz. The space $\mathcal{S}'$ is the topological dual of $\mathcal{S}$ and its canonical pairing is denoted by $\langle , \rangle$. We denote by $\|f\|_q$ the L^q norm of f for $1 \le q \le \infty$.

A function $\phi : (0,\infty) \to (0,\infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$C^{-1} \le \frac{\phi(r)}{\phi(s)} \le C \quad \text{for} \quad \frac{1}{2} \le \frac{r}{s} \le 2.$$

A function $\phi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that

$$\phi(r) \le C\phi(s) \quad (\phi(r) \ge C\phi(s)) \quad \text{for} \quad r \le s.$$

Let $\mathcal{G}$ be the set of all functions ϕ such that $\phi(r)r^n$ is almost increasing and $\phi(r)/r$ is almost decreasing. For example, $\phi(r) = r^\alpha$ is in $\mathcal{G}$ if $-n \le \alpha \le 1$. Each $\phi \in \mathcal{G}$ satisfies the doubling condition.

Let $\phi : (0,\infty) \to (0,\infty)$. For a ball $B = B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, we shall write $\phi(B)$ in place of $\phi(r)$. For a function $f \in L^1_{\mathrm{loc}}(X)$ and for a ball B, let $f_B = |B|^{-1}\int_B f(x)\,dx$, where $|B|$ is the measure of the ball B.

Definition 2.1 ($\mathcal{L}_{q,\phi}$) For $1 \le q < \infty$ and $\phi : (0,\infty) \to (0,\infty)$, a generalized Campanato space $\mathcal{L}_{q,\phi} = \mathcal{L}_{q,\phi}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^q_{\mathrm{loc}}$ such that $\|f\|_{\mathcal{L}_{q,\phi}} < \infty$, where

$$\|f\|_{\mathcal{L}_{q,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^q \, dx \right)^{1/q}.$$

In the above the supremum is taken over all balls B in $\mathbb{R}^n$.

We regard $\mathcal{L}_{q,\phi}$ as a space of functions modulo constants. Then $\mathcal{L}_{q,\phi}$ is a Banach space equipped with the norm $\|f\|_{\mathcal{L}_{q,\phi}}$. If $q = 1$ and $\phi(r) \equiv 1$, then $\mathcal{L}_{1,\phi}$ is the usual BMO. If $q = 1$ and $\phi(r) = r^\alpha$ $(0 < \alpha \le 1)$, then $\mathcal{L}_{1,\phi}$ coincides with the homogeneous Lipschitz space Lip_α. If ϕ is almost increasing, then $\mathcal{L}_{q,\phi} = \mathcal{L}_{1,\phi}$ for $1 \le q < \infty$. If $\phi(r) = r^{-\lambda}$ $(0 < \lambda \le n/q)$, then $\mathcal{L}_{q,\phi}$ coincides with the Morrey space. In particular, if $\phi(r) = r^{-n/q}$, then $\mathcal{L}_{q,\phi} = L^q$. If $1 < q_1 < q_2 < \infty$, then $\mathcal{L}_{1,\phi} \supset \mathcal{L}_{q_1,\phi} \supset \mathcal{L}_{q_2,\phi}$. If $\phi_1 \le \phi_2$, then $\mathcal{L}_{q,\phi_1} \subset \mathcal{L}_{q,\phi_2}$. Therefore, if

$$\phi(r) = \begin{cases} r^{-n} & \text{for} \quad 0 < r < 1, \\ r^\alpha & \text{for} \quad r \ge 1, \end{cases}$$

then $\mathcal{L}_{1,\phi} \supset L^1 \cup \mathrm{BMO} \cup \mathrm{Lip}_\alpha$. For the function space $\mathcal{L}_{q,\phi}$, see Peetre [16], Mizuhara [7], Nakai and Yabuta [13], [14], Nakai [8], [9], [10], [11], etc. We give a relation between $\mathcal{S}$ and $\mathcal{L}_{q,\phi}$ below.

Proposition 2.1 *Assume that $\phi(r)r^{n/q}$ is almost increasing and that $\phi(r)/r$ is almost decreasing. Then $\mathcal{S}$ is continuously embedded in $\mathcal{L}_{q,\phi}$. More precisely, there exists a positive constant C such that, for all $f \in \mathcal{S}$,*

$$\|f\|_{\mathcal{L}_{q,\phi}} \le C\big(\|(1+|x|^{n+1})f\|_\infty + \|\nabla f\|_\infty\big).$$

Proof. Let $B = B(z,r)$. If $r < 1$, then $1 \le c\,\phi(r)/r$ for some constant $c > 0$ and

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B |f(x) - f_B|^q\,dx\right)^{1/q}\\ &\qquad \le \sup_{x,y\in B} |f(x) - f(y)| \le 2r\|\nabla f\|_\infty \le 2c\phi(r)\|\nabla f\|_\infty.\end{aligned}$$

If $r \ge 1$, then $1 \le c'\phi(r)r^{n/q}$ for some constant $c' > 0$. Letting $V(x) = 1 + |x|^{n+1}$, we have

$$\begin{aligned}\left(\frac{1}{|B|}\int_B |f(x) - f_B|^q\,dx\right)^{1/q} &\le 2\left(\frac{1}{|B|}\int_B |f(x)|^q\,dx\right)^{1/q}\\ \le \frac{2}{|B|^{1/q}}\left(\int_B \left(\frac{\|Vf\|_\infty}{V(x)}\right)^q dx\right)^{1/q} &\le \frac{2c'\|1/V\|_q}{|B(0,1)|^{1/q}}\phi(r)\|Vf\|_\infty. \qquad \square\end{aligned}$$

Next we recall the definition of generalized Hardy spaces by using atoms which are introduced in [12].

Definition 2.2 ($[\phi,\infty]$-atom) Let $\phi \in \mathcal{G}$. A function a on $\mathbb{R}^n$ is called a $[\phi,\infty]$-atom if there exists a ball B such that

(i) $\operatorname{supp} a \subset B$,
(ii) $\|a\|_\infty \le \frac{1}{|B|\phi(B)}$,
(iii) $\int_{\mathbb{R}^n} a(x)\,dx = 0$.

We denote by $A[\phi,\infty]$ the set of all $[\phi,\infty]$-atoms.

If a is a $[\phi,\infty]$-atom and a ball B satisfies (1)–(3), then, for $g \in \mathcal{L}_{1,\phi}$,

$$
\begin{aligned}
\left| \int_{\mathbb{R}^n} a(x)g(x)\,dx \right| &= \left| \int_B a(x)(g(x) - g_B)\,dx \right| \\
&\le \|a\|_\infty \int_B |g(x) - g_B|\,dx \\
&\le \frac{1}{\phi(B)} \frac{1}{|B|} \int_B |g(x) - g_B|\,dx \le \|g\|_{\mathcal{L}_{1,\phi}}. \qquad (2.1)
\end{aligned}
$$

That is, the mapping $g \mapsto \int_{\mathbb{R}^n} ag\,dx$ is a bounded linear functional on $\mathcal{L}_{1,\phi}$ with norm not exceeding 1. Hence a is also in $\mathcal{S}'$ by Proposition 2.1.

Let $\mathcal{U}$ be the set of all continuous, concave, increasing and bijective functions from $[0,\infty)$ to itself such that

$$\sup_{0<s\le 1} \frac{U(rs)}{U(s)} \to 0 \quad (r \to 0). \qquad (2.2)$$

For example, $U(r) = r^p$ with $0 < p \le 1$ satisfies (2.2).

Definition 2.3 ($H_U^{[\phi,\infty]}$) For $\phi \in \mathcal{G}$ and $U \in \mathcal{U}$, the space $H_U^{[\phi,\infty]} \subset (\mathcal{L}_{1,\phi})^*$ is defined as follows:

> $f \in H_U^{[\phi,\infty]}$ if and only if there exist sequences $\{a_j\} \subset A[\phi,\infty]$ and positive numbers $\{\lambda_j\}$ such that
>
> $$f = \sum_j \lambda_j a_j \text{ in } (\mathcal{L}_{1,\phi})^* \quad \text{and} \quad \sum_j U(\lambda_j) < \infty. \qquad (2.3)$$

In the above the convergence in $(\mathcal{L}_{1,\phi})^*$ means in the weak* topology, that is, for every $g \in \mathcal{L}_{1,\phi}$,

$$\langle f, g \rangle = \sum_j \lambda_j \int a_j(x)g(x)\,dx.$$

From $U(0) = 0$ and the concavity of U it follows that

$$U(Cr) \le CU(r), \quad 1 \le C < \infty,\ 0 \le r < \infty, \qquad (2.4)$$

$$U(r+s) \le U(r) + U(s), \quad 0 \le r, s < \infty. \qquad (2.5)$$

Actually, for $0 \le t \le 1$,

$$t\,U(r) = (1-t)U(0) + t\,U(r) \le U((1-t)0 + tr) = U(tr).$$

Hence $(1/C)U(Cr) \le U(r)$ if $C \ge 1$. Moreover, for $t_1 = r/(r+s)$ and $t_2 = s/(r+s)$,

$$\begin{aligned} U(r+s) &= (t_1+t_2)U(r+s) \le U(t_1(r+s)) + U(t_2(r+s)) \\ &= U(r) + U(s). \end{aligned}$$

Then $H_U^{[\phi,\infty]}(X)$ is a linear space. Further, (2.5) implies

$$\sum_j \lambda_j \le U^{-1}\Big(\sum_j U(\lambda_j)\Big). \tag{2.6}$$

Therefore, if $\sum_j U(\lambda_j) < \infty$, then $\sum_j \lambda_j < \infty$ and $\sum_j \lambda_j a_j$ converges in $(\mathcal{L}_{1,\phi})^*$. In general, the expression (2.3) is not unique. Let

$$\|f\|_{H_U^{[\phi,\infty]}} = \inf \bigg\{ U^{-1}\Big(\sum_j U(\lambda_j)\Big)\bigg\},$$

where the infimum is taken over all expressions as in (2.3). Then $d(f,g) = U(\|f-g\|_{H_U^{[\phi,q]}})$ is a metric and $H_U^{[\phi,\infty]}$ is complete with respect to this metric. If $I(r) = r$, then $H_I^{[\phi,\infty]}$ is a Banach space equipped with the norm $\|f\|_{H_I^{[\phi,\infty]}}$.

In the case $\phi(r) = r^\alpha$ and $U(r) = r^p$ with $\alpha = n(1/p-1)$ and $n/(n+1) < p \le 1$, the space $H_U^{[\phi,\infty]}$ is the usual Hardy space H^p. Moreover, $\mathcal{L}_{1,\phi} = \mathrm{BMO}$ if $\alpha = 0$ (that is, $p = 1$) and $\mathcal{L}_{1,\phi} = \mathrm{Lip}_\alpha$ if $0 < \alpha < 1$ (that is, $n/(n+1) < p < 1$). It is known that $(H^1)^* = \mathrm{BMO}$ and $(H^p)^* = \mathrm{Lip}_\alpha$ with $\alpha = n(1/p-1)$ and $n/(n+1) < p < 1$.

3. Truncated operators of R_iR_j

In this section we state the first result on the convergence of truncated opetator of R_iR_j $(1 \le i,j \le n)$.

First we recall the definition of truncated operators $R_{i,j}^\epsilon$ of R_iR_j by Kato [4]. Let k denote the fundamental solution of $-\Delta$, i.e., $-\Delta k = \delta$. Its explicit form is

$$k(x) = \begin{cases} C_n |x|^{2-n} & \text{for } n \ge 3, \\ C_2 \log |x| & \text{for } n = 2, \end{cases}$$

where $1/C_n = (n-2)(2\pi^{n/2}/\Gamma(n/2))$ for $n \ge 3$ and $1/C_2 = -2\pi$. Let $\psi \in C^\infty(\mathbb{R}^n)$ be a radial function with $0 \le \psi \le 1$, $\psi(x) = 0$ for $|x| \le 1$, and $\psi(x) = 1$ for $|x| \ge 2$. We set $\lambda = 1 - \psi$. For $0 < \epsilon < 1/2$ we define $\psi_\epsilon(x) = \psi(x/\epsilon)$, $\lambda_\epsilon(x) = \lambda(\epsilon x)$, and $k_\epsilon = \psi_\epsilon \lambda_\epsilon k$ so that $\operatorname{supp} k_\epsilon \subset \{x : \epsilon \le |x| \le 2/\epsilon\}$.

Definition 3.1 ($R^\epsilon_{i,j}$) Let $1 \le i, j \le n$. For $0 < \epsilon < 1/4$, the operators $R^\epsilon_{i,j}$ are defined by $R^\epsilon_{i,j} f = (\partial_i \partial_j k_\epsilon) * f$ for $f \in \mathcal{S}'$.

We consider the following condition on ϕ and U.

$$\begin{cases} \displaystyle\int_1^\infty U\left(\frac{\phi(t)}{t}\right) \frac{dt}{t} < \infty, & \text{if } n \ge 3, \\ \displaystyle\int_1^\infty U\left(\frac{\phi(t)\log(1+t)}{t}\right) \frac{dt}{t} < \infty, & \text{if } n = 2. \end{cases} \tag{3.1}$$

Note that the functions ϕ in (1.5) and (1.6) satisfy this condition with $U(r) = r$. On the other hand $\phi(t) = t$ does not satisfy (3.1) for all $U \in \mathcal{U}$.

Then we have the following.

Theorem 3.1 *Let $\phi \in \mathcal{G}$ and $U \in \mathcal{U}$. Assume that* (3.1) *holds. If $\varphi \in \mathcal{S}$ and $\int \varphi = 0$, then*

$$\lim_{\epsilon \to 0} R^\epsilon_{i,j} \varphi = R_i R_j \varphi \quad \text{in } H_U^{[\phi,\infty]}.$$

In particular, $\lim_{\epsilon \to 0} (-\Delta) k_\epsilon * \varphi = \varphi$ *in* $H_U^{[\phi,\infty]}$.

Remark 3.1 Theorem 3.1 shows that, if $\varphi \in \mathcal{S}$ and $\int \varphi = 0$, then $R_i R_j \varphi \in H_U^{[\phi,\infty]}$.

Let $\phi(r) = r^{n(1/p-1)}$ and $U(r) = r^p$ with $n/(n+1) < p \le 1$ in the theorem above. Then we have the following.

Corollary 3.2 *Let $n/(n+1) < p \le 1$. If $\varphi \in \mathcal{S}$ and $\int \varphi = 0$, then*

$$\lim_{\epsilon \to 0} R^\epsilon_{i,j} \varphi = R_i R_j \varphi \quad \text{in } H^p.$$

In particular, $\lim_{\epsilon\to 0}(-\Delta)k_\epsilon * \varphi = \varphi$ *in* H^p.

The case $p = 1$ in the corollary was proved by Kato [4], using the maximal characterization of H^1.

Let $I(r) = r$. Then (3.1) with I instead of U is the following.

$$\begin{cases} \displaystyle\int_1^\infty \frac{\phi(t)}{t}\,\frac{dt}{t} < \infty, & \text{if } n \geq 3, \\ \displaystyle\int_1^\infty \frac{\phi(t)\log(1+t)}{t}\,\frac{dt}{t} < \infty, & \text{if } n = 2. \end{cases} \tag{3.2}$$

Note that, from (2.1) it follows that, if $f \in \mathcal{L}_{1,\phi}$, then f can be regarded as an element in $(H_I^{[\phi,\infty]})^*$. Then, using the equality

$$\lim_{\epsilon\to 0}\left\langle \sum_{j=1}^n R_{i,j}^\epsilon \partial_j f, \varphi \right\rangle = \lim_{\epsilon\to 0}\left\langle f, (-\Delta)k_\epsilon * \partial_i\varphi \right\rangle = \langle f, \partial_i\varphi\rangle$$

for all $\varphi \in \mathcal{S}$, we have the following.

Corollary 3.3 *Assume that* $\phi \in \mathcal{G}$ *satisfies* (3.2). *For* $f \in \mathcal{L}_{1,\phi}$,

$$\lim_{\epsilon\to 0}\sum_{j=1}^n R_{i,j}^\epsilon \partial_j f = -\partial_i f \quad in \quad \mathcal{S}'.$$

To prove Theorem 3.1 we state two lemmas. The first lemma gives a sufficient condition for functions to be in generalized Hardy spaces, which will be proved in Section 5 by using atoms.

For $U \in \mathcal{U}$, let

$$\overline{U}(r) = \sup_{0<s\leq 1} \frac{U(rs)}{U(s)}.$$

Then $U(rs) \leq \overline{U}(r)U(s)$ for $0 < s \leq 1$ and $\overline{U}(r) \to 0$ as $r \to 0$.

Lemma 3.4 *Let* $\phi \in \mathcal{G}$ *and* $U \in \mathcal{U}$. *Let* ℓ *be a continuous decreasing function from* $[0,\infty)$ *to* $(0,\infty)$ *such that* $\ell(r)r^\theta$ *is almost increasing for some* $\theta < 1$ *and that*

$$\int_1^\infty U\left(\frac{\phi(t)}{t\ell(t)}\right)\frac{dt}{t} < \infty.$$

Define

$$w(x) = (1+|x|)^{n+1}\ell(|x|) \quad \textit{for} \quad x \in \mathbb{R}^n.$$

If a function f satisfies

$$wf \in L^\infty \quad \textit{and} \quad \int_{\mathbb{R}^n} f = 0, \tag{3.3}$$

then $f \in H_U^{[\phi,\infty]}$. Moreover, there exist a constant $C > 0$ such that

$$\|f\|_{H_U^{[\phi,\infty]}} \le U^{-1}\big(C\overline{U}(C\|wf\|_\infty)\big), \tag{3.4}$$

where C is independent of f.

Using Lemma 2 in [4], we can prove the following lemma, which is a generalization of (4.15) in [4]. We omit the proof, since the method is the same as in [4].

Lemma 3.5 *Let ℓ be a continuous decreasing function from $[0,\infty)$ to $(0,\infty)$ such that $\ell(r) \ge (1+r)^{-n-1}$ and that*

$$\lim_{r\to\infty} \ell(r) = 0 \ \textit{if}\ n \ge 3, \qquad \lim_{r\to\infty} \ell(r)\log r = 0 \ \textit{if}\ n = 2.$$

Define

$$w(x) = (1+|x|)^{n+1}\ell(|x|) \quad \textit{for} \quad x \in \mathbb{R}^n.$$

If $\varphi \in \mathcal{S}$ and $\int \varphi = 0$, then

$$\lim_{\epsilon\to 0} \big\|(R^{\epsilon}_{i,j}\varphi - R_iR_j\varphi)w\big\|_\infty = 0.$$

Proof of Theorem 3.1. If (3.1) holds, then there exists a continuous decreasing function m such that $\lim_{r\to\infty} m(r) = 0$ and that

$$\begin{cases} \displaystyle\int_1^\infty U\left(\frac{\phi(t)}{tm(t)}\right)\frac{dt}{t} < \infty, & \text{if } n \ge 3, \\ \displaystyle\int_1^\infty U\left(\frac{\phi(t)\log(1+t)}{tm(t)}\right)\frac{dt}{t} < \infty, & \text{if } n = 2. \end{cases}$$

Actually, if $\int_1^\infty U(F(t))\,\frac{dt}{t} < \infty$, $F(t) = \phi(t)/t$ or $\phi(t)\log(1+r)/t$, then we can take a positive increasing sequence $\{r_j\}$ and a continuous decreasing function m such that

$$\int_{r_j}^\infty U(F(t))\,\frac{dt}{t} \le \frac{1}{j^3}, \quad \text{for} \quad j = 1, 2, \dots,$$

and

$$1 \ge m(t) \ge \frac{1}{j} \quad \text{for} \quad r_j \le t \le r_{j+1}.$$

Then, by (2.4),

$$\begin{aligned} \int_{r_1}^\infty U\Big(\frac{F(t)}{m(t)}\Big)\,\frac{dt}{t} &\le \int_{r_1}^\infty \frac{U(F(t))}{m(t)}\,\frac{dt}{t} \\ &= \sum_{j=1}^\infty \int_{r_j}^{r_{j+1}} \frac{U(F(t))}{m(t)}\,\frac{dt}{t} \le \sum_{j=1}^\infty \frac{1}{j^2} < \infty. \end{aligned}$$

We may assume that $m(r)r^\nu$ is almost increasing for some small $\nu > 0$. Let ℓ be a continuous decreasing function from $[0,\infty)$ to $(0,\infty)$ such that, for $r \ge 1$,

$$\ell(r) = \begin{cases} m(r), & \text{if } n \ge 3, \\ m(r)/\log(1+r), & \text{if } n = 2. \end{cases}$$

Then ℓ satisfies the assumption of both Lemmas 3.4 and 3.5.

Let $\varphi \in \mathcal{S}$ and $\int \varphi = 0$. Then $R^\epsilon_{i,j}\varphi \in \mathcal{S}$ and $\int R^\epsilon_{i,j}\varphi = 0$. Hence $R^\epsilon_{i,j}\varphi \in H_U^{[\phi,\infty]}$ by Lemma 3.4. Note that $\int R_iR_j\varphi = 0$. Then $\int (R^\epsilon_{i,j}\varphi - R_iR_j\varphi) = 0$. By Lemma 3.5 we have $\|(R^\epsilon_{i,j}\varphi - R_iR_j\varphi)w\|_\infty < \infty$. Hence $R^\epsilon_{i,j}\varphi - R_iR_j\varphi \in H_U^{[\phi,\infty]}$ by Lemma 3.4. It follows that $R_iR_j\varphi \in H_U^{[\phi,\infty]}$. Moreover, using both Lemmas 3.4 and 3.5, we have

$$\big\|R^\epsilon_{i,j}\varphi - R_iR_j\varphi\big\|_{H_U^{[\phi,\infty]}} \le U^{-1}\big(C\overline{U}(C\|(R^\epsilon_{i,j}\varphi - R_iR_j\varphi)w\|_\infty)\big) \to 0,$$

as $\epsilon \to 0$. $\square$

4. Uniqueness theorem

In this section, we show the uniqueness theorem for the Navier-Stokes equation. It is well known (see [3]) that for initial data $u_0 \in L^\infty(\mathbb{R}^n)$ the equations (1.1), (1.2) admit a unique time-local (regular) solution u with

$$p = \sum_{i,j=1}^{n} R_i R_j u_i u_j \quad \text{(modulo constants)}.$$

Following J. Kato [4], by *a solution in the distribution sense* we mean a weak solution in the following sense.

Definition 4.1 We call (u, p) the solution of the Navier-Stokes equations (1.1), (1.2) on $(0, T) \times \mathbb{R}^n$ with initial data u_0 in the distribution sense if (u, p) satisfy $\operatorname{div} u = 0$ in $\mathcal{S}'$ for a.e. t and

$$\int_0^T \big\{\langle u(s), \partial_s \Phi(s)\rangle + \langle u(s), \Delta\Phi(s)\rangle + \langle (u \otimes u)(s), \nabla\Phi(s)\rangle \\ + \langle p(s), \operatorname{div}\Phi(s)\rangle\big\}\, ds = -\langle u_0, \Phi(0)\rangle \tag{4.1}$$

for $\Phi \in C^1([0, T] \times \mathbb{R}^n)$ satisfying $\Phi(s, \cdot) \in \mathcal{S}(\mathbb{R}^n)$ for $0 \le s \le T$, and $\Phi(T, \cdot) \equiv 0$, where $\langle (u \otimes u), \nabla\Phi\rangle = \sum_{i,j=1}^n \langle u_i u_j, \partial_i \Phi_j\rangle$.

Now we state the second result.

Theorem 4.1 *Assume that $\phi \in \mathcal{G}$ satisfies* (3.2). *Let $u_0 \in L^\infty$ with $\operatorname{div} u_0 = 0$. Suppose that (u, p) is the solution of* (1.1), (1.2) *in the distribution sense satisfying*

$$u \in L^\infty((0, T) \times \mathbb{R}^n), \quad p \in L^1_{\mathrm{loc}}((0, T); \mathcal{L}_{1,\phi}). \tag{4.2}$$

Then $(u, \nabla p)$ is uniquely determined by the initial data u_0. Moreover, $\nabla p = \sum_{i,j=1}^n \nabla R_i R_j u^i u^j$ in $\mathcal{S}'$ for a.e. t.

We give examples of $\phi \in \mathcal{G}$ satisfying (3.2). To get a larger class of $\mathcal{L}_{1,\phi}$ we need to choose a bigger $\phi \in \mathcal{G}$, since $\mathcal{L}_{1,\phi_1} \subset \mathcal{L}_{1,\phi_2}$ for $\phi_1 \le \phi_2$.

Let

$$\phi(r) = \begin{cases} r^{-n} & \text{for} \quad 0 < r < 1, \\ r(\log(1+r))^{-\beta} & \text{for} \quad r \ge 1, \end{cases}$$

where $\beta > 1$ if $n \geq 3$ and $\beta > 2$ if $n = 2$. Then ϕ satisfies (3.2) and the theorem holds. In this case $\mathcal{L}_{1,\phi} \supset L^1 \cup \mathrm{BMO}$ and $\mathcal{L}_{1,\phi}$ contains functions f such that

$$|f(x)| \leq C\phi(1+|x|) = C(1+|x|)(\log(2+|x|))^{-\beta} \quad \text{for} \quad x \in \mathbb{R}^n.$$

Moreover, if $\gamma > 1$ and

$$\phi(r) = \begin{cases} r^{-n} & \text{for } \ 0 < r < 1, \\ r(\log(1+r))^{-1}(\log(1+\log(1+r)))^{-\gamma} & \text{for } \ r \geq 1 \ \text{ if } \ n \geq 3, \\ r(\log(1+r))^{-2}(\log(1+\log(1+r)))^{-\gamma} & \text{for } \ r \geq 1 \ \text{ if } \ n = 2, \end{cases}$$

then ϕ satisfies (3.2) and the theorem holds. In this case $\mathcal{L}_{1,\phi} \supset L^1 \cup \mathrm{BMO}$ and $\mathcal{L}_{1,\phi}$ contains functions f such that

$$|f(x)| \leq C\phi(1+|x|) \quad \text{for} \quad x \in \mathbb{R}^n.$$

To prove Theorem 4.1 we need the following.

Theorem 4.2 (J. Kato [4]) *Let $1 \leq i, j, l \leq n$.*

(i) *For $f \in L^\infty$,*

$$\lim_{\epsilon \to 0} \left\langle R^\epsilon_{i,j} f, \varphi \right\rangle = \langle R_i R_j f, \varphi \rangle$$

for all $\varphi \in \mathcal{S}$ with $\int \varphi = 0$. Moreover,

$$\lim_{\epsilon \to 0} R^\epsilon_{i,j} \partial_l f = \partial_l R_i R_j f \quad \textit{in} \quad \mathcal{S}'.$$

(ii) *For $f \in \mathcal{S}'$ with $\operatorname{div} f = 0$, $0 < \epsilon < 1/4$,*

$$\sum_{j=1}^n R^\epsilon_{i,j} f_j = 0 \quad \textit{in} \quad \mathcal{S}'.$$

(iii) *For $f \in \mathrm{BMO}$,*

$$\lim_{\epsilon \to 0} \sum_{j=1}^n R^\epsilon_{i,j} \partial_j f = -\partial_i f \quad \textit{in} \quad \mathcal{S}'.$$

In [4] Kato proved his uniqueness theorem by using Theorem 4.2. In the same way we can prove Theorem 4.1 by using Theorem 4.2 (i), (ii) and Corollary 3.3 instead of Theorem 4.2 (iii). See [4] for the detail.

5. Proof of the lemma on generalized Hardy spaces

In this section we prove Lemma 3.4. Let

$$MO(f, B(x,r)) = \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|\, dy.$$

Then (Lemma 2.4 in [14]) there exists a constant $C > 0$ dependent only on n such that

$$|f_{B(x,r)} - f_{B(x,s)}| \le C \int_r^{2s} \frac{MO(f, B(x,t))}{t}\, dt \qquad \text{for } x \in \mathbb{R}^n,\ 0 < r < s. \tag{5.1}$$

In general, if $\Theta : (0,\infty) \to (0,\infty)$ satisfies the doubling condition, then

$$\Theta(r) = (\log 2)^{-1} \int_r^{2r} \frac{\Theta(r)}{t}\, dt \sim \int_r^{2r} \frac{\Theta(t)}{t}\, dt. \tag{5.2}$$

Proof of Lemma 3.4. Let f satisfy the condition (3.3). First we show that fg is integrable for all $g \in \mathcal{L}_{1,\phi}$ and that $f \in (\mathcal{L}_{1,\phi})^*$. Next we show that $f \in H_U^{[\phi,\infty]}$ and that (3.4) holds.

Part 1. For $g \in \mathcal{L}_{1,\phi}$, we show

$$\int_{\mathbb{R}^n} |f(x)(g(x) - g_{B_0})|\, dx \le C\|wf\|_\infty \|g\|_{\mathcal{L}_{1,\phi}}, \tag{5.3}$$

where $B_0 = B(0,1)$. Then, combining with the integrability of f, we have the integrability of fg. Moreover, from $\int f = 0$ it follows that

$$\left| \int_{\mathbb{R}^n} f(x)g(x)\, dx \right| = \left| \int_{\mathbb{R}^n} f(x)(g(x) - g_{B_0})\, dx \right| \le C\|wf\|_\infty \|g\|_{\mathcal{L}_{1,\phi}}.$$

This implies that $f \in (\mathcal{L}_{1,\phi})^*$.

To show (5.3), let $B_j = B(0, 2^j)$ and let

$$L_0 = B_0, \quad L_j = B_j \setminus B_{j-1} \ (j = 1, 2, \dots).$$

Then, using (5.1), (5.2) and $MO(g, B(0,t)) \le \phi(t)\|g\|_{\mathcal{L}^{1,\phi}}$, we have

$$\begin{aligned}
\frac{1}{|B_j|}\int_{B_j} |g(x) - g_{B_0}|\,dx &\le \frac{1}{|B_j|}\int_{B_j} |g(x) - g_{B_j}|\,dx + |g_{B_j} - g_{B_0}| \\
&\le \left(\phi(2^j) + C\int_1^{2^{j+1}} \frac{\phi(t)}{t}\,dt\right)\|g\|_{\mathcal{L}^{1,\phi}} \\
&\le C\int_1^{2^{j+1}} \frac{\phi(t)}{t}\,dt\ \|g\|_{\mathcal{L}^{1,\phi}}. \qquad (5.4)
\end{aligned}$$

Let $W(r) = (1+r)^{n+1}\ell(r)$. Then $w(x) = W(|x|)$. Note that W is almost increasing and satisfies the doubling condition. Then there exists a constant $C_W > 0$ such that

$$\operatorname*{esssup}_{x\in L_j} |f(x)| \le C_W \frac{\|wf\|_\infty}{W(2^j)}.$$

Hence, by (5.4),

$$\begin{aligned}
&\int_{\mathbb{R}^n} |f(x)(g(x) - g_{B_0})|\,dx \\
&\quad = \sum_{j=1}^{\infty} \int_{L_j} |f(x)(g(x) - g_{B_0})|\,dx \\
&\quad \le \sum_{j=1}^{\infty} \left(\operatorname*{esssup}_{x\in L_j} |f(x)|\right)|B_j|\left(\frac{1}{|B_j|}\int_{B_j} |g(x) - g_{B_0}|\,dx\right) \\
&\quad \le CC_W \sum_{j=1}^{\infty} \frac{2^{jn}}{W(2^j)} \int_1^{2^{j+1}} \frac{\phi(t)}{t}\,dt\ \|wf\|_\infty \|g\|_{\mathcal{L}_{1,\phi}}.
\end{aligned}$$

In the following, we prove

$$\sum_{j=1}^{\infty} \frac{2^{jn}}{W(2^j)} \int_1^{2^{j+1}} \frac{\phi(t)}{t}\,dt < \infty. \qquad (5.5)$$

Since W, ϕ and ℓ satisfy the doubling condition, we have by (5.2)

$$\sum_{j=1}^{\infty} \frac{2^{jn}}{W(2^j)} \int_1^{2^{j+1}} \frac{\phi(t)}{t}\,dt \sim \sum_{j=1}^{\infty} \int_{2^{j+1}}^{2^{j+2}} \left(\frac{s^n}{W(s)} \int_1^s \frac{\phi(t)}{t}\,dt \right) \frac{ds}{s}$$
$$\le \int_1^{\infty} \left(\frac{s^n}{W(s)} \int_1^s \frac{\phi(t)}{t}\,dt \right) \frac{ds}{s} = \int_1^{\infty} \frac{\phi(t)}{t} \left(\int_t^{\infty} \frac{s^n}{W(s)} \frac{ds}{s} \right) dt$$
$$\le \int_1^{\infty} \frac{\phi(t)}{t} \left(\int_t^{\infty} \frac{1}{\ell(s)s} \frac{ds}{s} \right) dt = \int_1^{\infty} \frac{\phi(t)}{t} \left(\int_t^{\infty} \frac{1}{\ell(s)s^{\theta}} \frac{ds}{s^{2-\theta}} \right) dt$$
$$\le C \int_1^{\infty} \frac{\phi(t)}{\ell(t)t^{1+\theta}} \left(\int_t^{\infty} \frac{ds}{s^{2-\theta}} \right) dt = C \int_1^{\infty} \frac{\phi(t)}{t\ell(t)} \frac{dt}{t}.$$

Note that there exists a positive constant C such that

$$\frac{\phi(t)}{t\ell(t)} \le C\,U\!\left(\frac{\phi(t)}{t\ell(t)}\right) \quad \text{for} \quad t \ge 1.$$

Actually, letting

$$c^* = \min\left(1, \left(\sup_{t\ge 1} \frac{\phi(t)}{t\ell(t)}\right)^{-1}\right), \tag{5.6}$$

we have, by the concavity and increasingness of U,

$$\frac{c^*\phi(t)}{t\ell(t)}\,U(1) \le U\!\left(\frac{c^*\phi(t)}{t\ell(t)}\right) \le U\!\left(\frac{\phi(t)}{t\ell(t)}\right),$$

since

$$\frac{c^*\phi(t)}{t\ell(t)} \le 1 \quad \text{and} \quad c^* \le 1.$$

Hence

$$\sum_{j=1}^{\infty} \frac{2^{jn}}{W(2^j)} \int_1^{2^{j+1}} \frac{\phi(t)}{t}\,dt \le C \int_1^{\infty} U\!\left(\frac{\phi(t)}{t\ell(t)}\right) \frac{dt}{t} < \infty.$$

Part 2. Now we show that $f \in H_U^{[\phi,\infty]}$ and that (3.4) holds. Define λ_j

and $a_j(x)$, $j = 0, 1, 2, \ldots$, as

$$\lambda_j = c_1 \frac{\|wf\|_\infty}{W(2^j)} |B_j| \phi(B_j),$$
$$\lambda_j a_j(x) = (f(x) - f_j)\chi_{L_j}(x),$$

where

$$c_1 = 2C_W, \quad f_j = |L_j|^{-1} \int_{L_j} f(x)\, dx.$$

Then

$$\int a_j(x)\, dx = 0, \quad \operatorname{supp} a_j \subset B_j,$$

and

$$\begin{aligned}\|a_j\|_\infty &\le {\lambda_j}^{-1} \Big(\operatorname*{esssup}_{x \in L_j} |f(x)| + |f_j| \Big) \le {\lambda_j}^{-1} \Big(2 \operatorname*{esssup}_{x \in L_j} |f(x)| \Big) \\ &\le {\lambda_j}^{-1} \left(2C_W \frac{\|wf\|_\infty}{W(2^j)} \right) = \frac{1}{|B_j| \phi(B_j)}.\end{aligned}$$

Hence a_j, $j = 1, 2, \ldots$, are $[\phi, \infty]$-atoms and the equality

$$f(x)\chi_{B_m}(x) = \sum_{j=0}^{m} \lambda_j a_j(x) + \sum_{j=0}^{m} f_j \, \chi_{L_j}(x)$$

holds. Next, to decompose the second term of the right hand side into atoms, let $\{\eta_j\}$ be the sequence defined as

$$\eta_0 = \int_{\mathbb{R}^n} f(x)\, dx = 0, \quad \eta_{j+1} = \int_{\mathbb{R}^n \setminus B_j} f(x)\, dx, \; j = 0, 1, 2, \ldots.$$

This sequence satisfies

$$\eta_j - \eta_{j+1} = \int_{L_j} f(x)\, dx = |L_j| f_j.$$

Therefore

$$\begin{aligned}
&\sum_{j=0}^{m} f_j \chi_{L_j}(x) \\
&\quad = \sum_{j=0}^{m} (\eta_j - \eta_{j+1}) |L_j|^{-1} \chi_{L_j}(x) \\
&\quad = \sum_{j=1}^{m} \eta_j \big(|L_j|^{-1} \chi_{L_j}(x) - |L_{j-1}|^{-1} \chi_{L_{j-1}}(x) \big) - \eta_{m+1} |L_m|^{-1} \chi_{L_m}(x).
\end{aligned}$$

If $t \geq 1$, then

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus B(0,t)} \frac{1}{w(x)} \, dx &\sim \int_t^\infty \frac{s^{n-1}}{W(s)} \, ds = \int_t^\infty \frac{1}{\ell(s) s^\theta} \frac{ds}{s^{2-\theta}} \\
&\leq \frac{C}{\ell(t) t^\theta} \int_t^\infty \frac{ds}{s^{2-\theta}} \sim \frac{1}{\ell(t) t} \sim \frac{t^n}{W(t)}.
\end{aligned}$$

Hence there exists a constant C'_W such that

$$\int_{\mathbb{R}^n \setminus B_j} \frac{1}{w(x)} \, dx \leq C'_W \frac{|B_j|}{W(2^j)}.$$

Therefore

$$|\eta_j| \leq \int_{\mathbb{R}^n \setminus B_{j-1}} |f(x)| \, dx \leq \|wf\|_\infty \int_{\mathbb{R}^n \setminus B_{j-1}} \frac{1}{w(x)} \, dx \leq C'_W \frac{\|wf\|_\infty |B_{j-1}|}{W(2^{j-1})}.$$

Define λ'_j and $a'_j(x)$, $j = 1, 2, \ldots$, as

$$\begin{aligned}
\lambda'_j &= c_2 \frac{\|wf\|_\infty}{W(2^{j-1})} |B_j| \phi(B_j), \\
\lambda'_j a'_j(x) &= \eta_j \big(|L_j|^{-1} \chi_{L_j}(x) - |L_{j-1}|^{-1} \chi_{L_{j-1}}(x) \big),
\end{aligned}$$

where

$$c_2 = C'_W |B_{j-1}| |L_{j-1}|^{-1} = C'_W \frac{2^n}{2^n - 1}.$$

Then

$$\int a_j'(x)\,dx = 0, \quad \operatorname{supp} a_j' \subset B_j,$$

and

$$\|a_j'\|_\infty \le {\lambda_j'}^{-1}|\eta_j||L_{j-1}|^{-1} \le \frac{1}{|B_j|\phi(B_j)}.$$

Hence a_j', $j = 1, 2, \dots$, are $[\phi, \infty]$-atoms and the equality

$$f(x)\chi_{B_m}(x) = \sum_{j=0}^{m} \lambda_j a_j(x) + \sum_{j=0}^{m} \lambda_j' a_j'(x) - \eta_{m+1}|L_m|^{-1}\chi_{L_m}(x)$$

holds. Note that

$$\frac{2^{jn}}{W(2^j)} \int_1^{2^{j+1}} \frac{\phi(t)}{t}\,dt \to 0 \quad \text{as} \quad j \to 0,$$

since the sum (5.5) converges. Then, for all $g \in \mathcal{L}_{1,\phi}$, we have by (5.4)

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \eta_{m+1}|L_m|^{-1}\chi_{L_m}(x)g(x)\,dx \right| \\
&\quad \le |\eta_{m+1}| \left(\frac{C}{|B_m|} \int_{B_m} |g(x)|\,dx \right) \\
&\quad \le CC_W' \frac{\|wf\|_\infty |B_m|}{W(2^m)} \left(\int_1^{2^{m+1}} \frac{\phi(t)}{t}\,dt\ \|g\|_{\mathcal{L}_{1,\phi}} + |g_{B_0}| \right) \to 0 \quad (m \to \infty).
\end{aligned}$$

This shows

$$f = \sum_{j=0}^{\infty} \lambda_j a_j + \sum_{j=1}^{\infty} \lambda_j' a_j' \quad \text{in} \quad (\mathcal{L}_{1,\phi})^*.$$

Moreover, letting c^* be as in (5.6), we have

$$\begin{aligned}
&\sum_{j=0}^{\infty} U(\lambda_j) + \sum_{j=1}^{\infty} U(\lambda'_j) \\
&\quad = \sum_{j=0}^{\infty} U\left(c_1 \|wf\|_\infty \frac{|B_j|\phi(B_j)}{W(2^j)}\right) + \sum_{j=1}^{\infty} U\left(c_2 \|wf\|_\infty \frac{|B_j|\phi(B_j)}{W(2^{j-1})}\right) \\
&\quad \le 2 \sum_{j=0}^{\infty} U\left(C \|wf\|_\infty \frac{c^* \phi(2^j)}{2^j \ell(2^j)}\right),
\end{aligned}$$

for some constant C dependent on only ϕ, ℓ and n. From

$$\frac{c^* \phi(2^j)}{2^j \ell(2^j)} \le 1 \ (j = 0, 1, 2, \dots) \quad \text{and} \quad c^* \le 1,$$

it follows that

$$\begin{aligned}
U\left(C \|wf\|_\infty \frac{c^* \phi(2^j)}{2^j \ell(2^j)}\right) &\le \overline{U}(C \|wf\|_\infty) U\left(\frac{c^* \phi(2^j)}{2^j \ell(2^j)}\right) \\
&\le \overline{U}(C \|wf\|_\infty) U\left(\frac{\phi(2^j)}{2^j \ell(2^j)}\right).
\end{aligned}$$

By the property (5.2) we have

$$\sum_{j=0}^{\infty} U\left(\frac{\phi(2^j)}{2^j \ell(2^j)}\right) \sim \int_1^{\infty} U\left(\frac{\phi(s)}{s \ell(s)}\right) \frac{ds}{s}.$$

Hence

$$\sum_{j=0}^{\infty} U(\lambda_j) + \sum_{j=1}^{\infty} U(\lambda'_j) \le C \overline{U}(C \|wf\|_\infty).$$

This shows $f \in H_U^{[\phi,\infty]}$ and (3.4).

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