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Finite p-groups with a fixed-point-free automorphisms of order seven

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Abstract. We prove several properties of finite p-groups which are generated by two elements of prime order p and which have a fixed-point-free automorphism of order seven.

Key words: p-group, nilpotent class, fixed-point-free automorphism.

1. Introduction

In this paper, we study properties of finite p-groups which are generated by two elements of prime order p and which have a fixed-point-free automorphism of order seven.

An automorphism α of a group G is said to have a *fixed point* g in G if $g^\alpha = g$. $C_G(\alpha)$ denotes the subgroup of G consisting of all the elements fixed by α: $C_G(\alpha) := \{g \in G \mid g^\alpha = g\}$. If $C_G(\alpha) = 1$, then α is called *fixed-point-free* (for brevity, f.p.f.).

In [6] Thompson showed that if a finite group has a f.p.f. automorphism of prime order, then, it is nilpotent. In [3] Higman showed that if a finite nilpotent group has a f.p.f. automorphism of prime order q, then its nilpotent class is bounded by a function depending only on q. It is well-known results that if $q = 2$ then its nilpotent class is 1 and that if $q = 3$ then its one is less than 3. Without the aid of Lie algebra theory, the purpose of this paper is to prove the following theorem.

Theorem 1 *Let $p \geq 7$ be a prime and let P be a finite p-group which has two generators of prime order p. If P has a f.p.f. automorphism α of order 7, then it has nilpotent class less than 7.*

Moreover, suppose that $p \equiv 1 \pmod 7$ and let a, b be generators of P such that $a^\alpha = a^u w_1$, $b^\alpha = b^v w_2$ $(w_1, w_2 \in \Phi(P))$. Then

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1. *If* $v \equiv u \pmod{p}$, *then for all* $(x_1, \ldots, x_7) \in \{a, b\}^7$,

$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7] = 1.$$

2. *If* $v \equiv u^2 \pmod{p}$, $[a, b, b, b] = [a, b, b, a, a] = [a, b, a, a, a, a] = 1$.
3. *If* $v \equiv u^3 \pmod{p}$, $[a, b, b] = [a, b, a, a, a] = 1$.
4. *If* $v \equiv u^6$, $[a, b] = 1$.

Our notation is standard possibly except for the following:

$\Phi(G)$: Frattini subgroup of a group G,
C_{ij}: the commutator of x_i and x_j,
$C_{i\ldots k}$: the commutator $[x_i, \ldots, x_k]$ of $x_i, \ldots, x_k$,
$C_{ab\ldots z}$: the commutator $[a, b, \ldots, z]$ of $a, b, \ldots, z$,
$L_i(G)$: the lower central series of G.

We use the "bar" convertion for homomorphic images. Thus if G is a group, N is a normal subgroup and $\bar{G}$ denotes the factor group G/N, then, for any subset X of G, $\bar{X}$ denotes the image of X under the natural projection $G \to \bar{G}$.

The organization of the paper is as follows. Section 2 contains preliminary results. In Section 3, we discuss properties of finite p-groups which are generated by two elements of prime order p and which have nilpotent class 5 and 6. We prove our theorem in Section 4.

2. Preliminary results

In this section, we collect a number of preliminary lemmas to be used in later section.

The equations below are fundamental to commutator calculus. Let x, y, z be elements of a group. Then:

C1 $[x, y] = [y, x]^{-1}$
C2 $[xy, z] = [x, z]^y [y, z] = [x, z][x, z, y][y, z]$
C3 $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$
C4 $[x, y^{-1}] = ([x, y]^{y^{-1}})^{-1}$
C5 $[x^{-1}, y] = ([x, y]^{x^{-1}})^{-1}$

C6 $[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1$

The properties of the lower central series are listed here ([2]).

L1 $L_i(G)$ char G for all i.

L2 $L_{i+1}(G) \subseteq L_i(G)$.

L3 $L_i(G)/L_{i+1}(G)$ is included in the center of $G/L_{i+1}(G)$.

(Let $x, x' \in L_i(G), y, y' \in L_j(G), z \in L_k(G)$.)

L4 $[x, y] \in L_{i+j}(G)$.

L5 If $x \equiv x' \pmod{L_{i+1}(G)}$ and $y \equiv y' \pmod{L_{j+1}(G)}$, then $[x, y] \equiv [x', y'] \pmod{L_{i+j+1}(G)}$.

L6 $[xx', y] \equiv [x, y][x', y] \pmod{L_{i+j+1}(G)}$.

L7 $[x, yy'] \equiv [x, y][x, y'] \pmod{L_{i+j+1}(G)}$.

L8 $[x, y, z][y, z, x][z, x, y] \equiv 1 \pmod{L_{i+j+k+1}(G)}$.

L9 For any non-negative integer a, $[x, y]^a \equiv [x^a, y] \equiv [x, y^a] \pmod{L_{i+j+1}(G)}$.

Proposition 1 *Let p, q be prime numbers. If a non-abelian p-group P which is generated by two elements of P has a fixed-point-free automorphism α of order q, then $p \equiv 1 \pmod{q}$.*

Proof. α induces a fixed-point-free automorphism on $X := L_2(P)/L_3(P)$. Since X is cyclic, its subgroup Y of order p is characteristic. Hence α induces a fixed-point-free automorphism on Y. We get $p = 1 + qk \equiv 1 \pmod{q}$. □

Lemma 1 *Let P be a group. Let l, m be natural numbers, and $y_\lambda \in L_{k_1}(P)$ $(1 \le \lambda \le l)$, $z_\mu \in L_{k_2}(P)$ $(1 \le \mu \le m)$, $w_1 \in L_{k_1+1}(P)$ and $w_2 \in L_{k_2+1}(P)$. We get the following equation*

$$\left[\left(\prod_{1\le\lambda\le l} y_\lambda\right) w_1, \left(\prod_{1\le\mu\le m} z_\mu\right) w_2\right] \equiv \prod_{\substack{1\le\lambda\le l\\1\le\mu\le m}} [y_\lambda, z_\mu] \pmod{L_{k_1+k_2+1}(P)}.$$

Proof. By induction on $l + m$, we have

$$
\begin{aligned}
&\Big[\Big(\prod_{1\le\lambda\le l} y_\lambda\Big)w_1, \Big(\prod_{1\le\mu\le m} z_\mu\Big)w_2\Big] \\
&= \Big[\Big(\prod_{1\le\lambda\le l} y_\lambda\Big)w_1, w_2\Big]\Big[\Big(\prod_{1\le\lambda\le l} y_\lambda\Big)w_1, \prod_{1\le\mu\le m} z_\mu\Big]^{w_2} && \text{(by C3)} \\
&\equiv \Big[\Big(\prod_{1\le\lambda\le l} y_\lambda\Big)w_1, \prod_{1\le\mu\le m} z_\mu\Big] \pmod{L_{k_1+k_2+1}(P)} && \text{(by L4 and L3)} \\
&\equiv \Big[\prod_{1\le\lambda\le l} y_\lambda, \prod_{1\le\mu\le m} z_\mu\Big]^{w_1}\Big[w_1, \prod_{1\le\mu\le m} z_\mu\Big] \pmod{L_{k_1+k_2+1}(P)} && \text{(by C2)} \\
&\equiv \Big[\prod_{1\le\lambda\le l} y_\lambda, \prod_{1\le\mu\le m} z_\mu\Big] \pmod{L_{k_1+k_2+1}(P)} && \text{(by L4 and L3)} \\
&\equiv \Big[\prod_{1\le\lambda\le l-1} y_\lambda, \prod_{1\le\mu\le m} z_\mu\Big]\Big[y_l, \prod_{1\le\mu\le m} z_\mu\Big] \pmod{L_{k_1+k_2+1}(P)} && \text{(by L6)} \\
&\equiv \Big[\prod_{1\le\lambda\le l-1} y_\lambda, \prod_{1\le\mu\le m-1} z_\mu\Big]\Big[\prod_{1\le\lambda\le l-1} y_\lambda, z_m\Big] \\
&\qquad \cdot \Big[y_l, \prod_{1\le\mu\le m-1} z_\mu\Big][y_l, z_m] \pmod{L_{k_1+k_2+1}(P)} && \text{(by L7)} \\
&\equiv \Big(\prod_{\substack{1\le\lambda\le l-1\\ 1\le\mu\le m-1}} [y_\lambda, z_\mu]\Big)\Big(\prod_{1\le\lambda\le l-1} [y_\lambda, z_m]\Big)\Big(\prod_{1\le\mu\le m-1} [y_l, z_\mu]\Big)[y_l, z_m] \\
&\qquad \pmod{L_{k_1+k_2+1}(P)} && \text{(by induction)} \\
&\equiv \prod_{\substack{1\le\lambda\le l\\ 1\le\mu\le m}} [y_\lambda, z_\mu] \pmod{L_{k_1+k_2+1}(P)}
\end{aligned}
$$

□

Lemma 2 *Let $y_1 \in L_{k_1}(P)$, $y_2 \in L_{k_2}(P)$, $w_1 \in L_{k_1+1}(P)$, $w_2 \in L_{k_2+1}(P)$ and n_1, n_2 natural numbers. We obtain*

$$\big[y_1^{n_1} w_1, y_2^{n_2} w_2\big] \equiv [y_1, y_2]^{n_1 n_2} \pmod{L_{k_1+k_2+1}(P)}.$$

Proof. In lemma 1, we put $l = n_1$, $m = n_2$, $y_\lambda = y_1$ and $z_\mu = y_2$. □

Lemma 3 *Let* $y_i \in L_1(P)$*,* $w_i \in L_2(P)$ *and let* n_i *be natural numbers* $(1 \le i \le t)$*. One gets*

$$\left[y_1^{n_1}w_1, y_2^{n_2}w_2\right] \equiv [y_1, y_2]^{n_1n_2} \pmod{L_3(P)},$$
$$\left[y_1^{n_1}w_1, y_2^{n_2}w_2, y_3^{n_3}w_3\right] \equiv [y_1, y_2, y_3]^{n_1n_2n_3} \pmod{L_4(P)},$$
$$\left[y_1^{n_1}w_1, y_2^{n_2}w_2, \dots, y_t^{n_t}w_t\right] \equiv [y_1, y_2, \dots, y_t]^{n_1n_2\dots n_t} \pmod{L_{t+1}(P)}.$$

Proof.

$$\begin{aligned}
\left[y_1^{n_1}w_1, y_2^{n_2}w_2\right] &\equiv [y_1, y_2]^{n_1n_2} \pmod{L_3(P)} \qquad \text{(by Lemma 1)}\\
\left[y_1^{n_1}w_1, y_2^{n_2}w_2, y_3^{n_3}w_3\right] &= \left[[y_1, y_2]^{n_1n_2}z_1, y_3^{n_3}w_3\right] \quad (z_1 \in L_3(P))\\
&\equiv [y_1, y_2, y_3]^{n_1n_2n_3} \pmod{L_4(P)} \quad \text{(by Lemma 1)}\\
\left[y_1^{n_1}w_1, y_2^{n_2}w_2, \dots, y_t^{n_t}w_t\right] &= \left[[y_1, y_2, \dots, y_{t-1}]^{n_1n_2\dots n_{t-1}}z_t, y_t^{n_t}w_t\right]\\
&\qquad (z_t \in L_t(P))\\
&\equiv [y_1, y_2, \dots, y_t]^{n_1n_2\dots n_t} \pmod{L_{t+1}(P)}\\
&\qquad \text{(by Lemma 1)} \quad \square
\end{aligned}$$

Lemma 4 *Let* $y_1,\ y_2 \in L_{k_1}(P)$*,* $y_3 \in L_{k_2}(P)$ *and let* n_1, n_2, n_3 *be natural numbers. Then one obtains*

$$\left[y_1^{n_1}y_2^{n_2}, y_3^{n_3}\right] \equiv [y_1, y_3]^{n_1n_3}[y_2, y_3]^{n_2n_3} \pmod{L_{k_1+k_2+1}(P)}.$$

Proof.

$$\begin{aligned}
\left[y_1^{n_1}y_2^{n_2}, y_3^{n_3}\right] &\equiv \left[y_1^{n_1}, y_3^{n_3}\right]\left[y_2^{n_2}, y_3^{n_3}\right] \pmod{L_{k_1+k_2+1}(P)} \qquad \text{(by L6)}\\
&\equiv [y_1, y_3]^{n_1n_3}[y_2, y_3]^{n_2n_3} \pmod{L_{k_1+k_2+1}(P)}\\
&\qquad \text{(by Lemma 1)} \quad \square
\end{aligned}$$

Lemma 5 *If a finite nilpotent group* G *is generated by two elements* a, b*, then for each* $i \ge 2$ $L_i(G)/L_{i+1}(G)$ *is generated by* $\{[x_1, x_2, \dots, x_i]L_{i+1}(G) \mid (x_j) \in \{a, b\}^i,\ x_1 \ne x_2\}$*. And a element of* $L_i(G)/L_{i+1}(G)$ *is reprsented by*

$$\prod_i [x_1, x_2, \dots, x_i]^{n_x} L_{i+1}(G),$$

where the product $\prod_i$ *runs over* $x = (x_j) \in \{a, b\}^i$ *such that* $x_1 \neq x_2$.

Proof. When $i = 2$, we shall show that $xL_3(G) = [a^{\xi_{11}}b^{\xi_{12}}w_1, a^{\xi_{21}}b^{\xi_{22}}w_2]L_3(G)$ $(w_1, w_2 \in L_2(G))$ is reprsented by $[a, b]L_3(G)$.

$$\begin{aligned} xL_3(G) &= \left[a^{\xi_{11}}b^{\xi_{12}}, a^{\xi_{21}}b^{\xi_{22}}\right]L_3(G) \\ &= \left[a^{\xi_{11}}, a^{\xi_{21}}b^{\xi_{22}}\right]\left[b^{\xi_{12}}, a^{\xi_{21}}b^{\xi_{22}}\right]L_3(G) \\ &= \left[a^{\xi_{11}}, b^{\xi_{22}}\right]\left[b^{\xi_{12}}, a^{\xi_{21}}\right]L_3(G) \\ &= [a, b]^{\xi_{11}\xi_{22}-\xi_{21}\xi_{12}}L_3(G) \end{aligned}$$

When $i - 1$, let us suppose that the claim is true. It is enough that for any $y \in L_{i-1}(G)$ and any $z \in G$, $[y, z]L_{i+1}(G)$ is reprsented in the form

$$\prod_i [x_1, x_2, \ldots, x_i]^{l_x} L_{i+1}(G).$$

By induction, one has $y = \left(\prod_{i-1}[x_1, x_2, \ldots, x_{i-1}]^{n_x}\right)v$ for some $v \in L_i(G)$ and $z = a^{\xi_{i1}}b^{\xi_{i2}}w_i$ for some $w_i \in L_2(G)$. Hence, from Lemma 1,

$$\begin{aligned} [y, z]L_{i+1}(G) &= \left[\left(\prod_{i-1}[x_1, x_2, \ldots, x_{i-1}]^{n_x}\right)v, a^{\xi_{i1}}b^{\xi_{i2}}w_i\right]L_{i+1}(G) \\ &= \left[\prod_{i-1}[x_1, x_2, \ldots, x_{i-1}]^{n_x}, a^{\xi_{i1}}\right] \\ &\quad \cdot \left[\prod_{i-1}[x_1, x_2, \ldots, x_{i-1}]^{n_x}, b^{\xi_{i2}}\right]L_{i+1}(G) \\ &= \prod_{i-1}\left[[x_1, x_2, \ldots, x_{i-1}]^{n_x}, a^{\xi_{i1}}\right] \\ &\quad \cdot \prod_{i-1}\left[[x_1, x_2, \ldots, x_{i-1}]^{n_x}, b^{\xi_{i2}}\right]L_{i+1}(G) \\ &= \prod_i [x_1, x_2, \ldots, x_i]^{l_x} L_{i+1}(G). \end{aligned}$$ □

We shall need the following lemmas.

Lemma 6 *Let p be an odd prime and let i $(2 \le i \le 5)$ be natural number. If a nilpotent group P is generated by two elements a, b both of which have order p, then for all $x_1, \dots, x_i \in \{a, b\}$, and for all natural numbers $n_1, \dots, n_i$, a commutator $[x_1^{n_1}, x_2^{n_2}, \dots, x_i^{n_i}]$ of weight i have order p or 1 on $L_{i+1}(P)$.*

Proof.

$$\begin{aligned}\left[x_1^{n_1}, x_2^{n_2}, \dots, x_i^{n_i}\right]^p &\equiv \left[x_1^{n_1}, x_2^{n_2}, \dots, x_i^{pn_i}\right] \pmod{L_{i+1}(P)} \qquad \text{(by L7)}\\ &\equiv \left[x_1^{n_1}, x_2^{n_2}, \dots, x_{i-1}^{n_{i-1}}, 1\right] \pmod{L_{i+1}(P)}\\ &\equiv 1 \pmod{L_{i+1}(P)}. \end{aligned}$$

□

Lemma 7 *Let p be an odd prime number. If a group P is generated by two elements a, b both of which has order p, then $L_4(P)/L_5(P)$ is generated by three elements $[a, b, a, a]L_5(P)$, $[a, b, b, a]L_5(P)$, and $[a, b, b, b]L_5(P)$.*

Proof. Since Lemma 4, it is enough to prove that $[a, b, a, b]L_5(P)$ is represented by $[a, b, b, a]L_5(P)$.

$$\begin{aligned}&[a, b, a, b]\\ &\equiv [a, b, a, b]^{a^{-1}} \pmod{L_5(P)}\\ &\equiv \left[b, [a, b]^{-1}, a^{-1}\right]^{-[a,b]} \left[a^{-1}, b^{-1}, [a, b]\right]^{-b} \pmod{L_5(P)}\\ &\equiv \left[[b, [a, b]]^{-[a,b]^{-1}}, a^{-1}\right]^{-1} \left[[a, b^{-1}]^{-a^{-1}}, [a, b]\right]^{-1} \pmod{L_5(P)}\\ &\equiv \left[[a, b, b][a, b, b, [a, b]^{-1}], a^{-1}\right]^{-1} \left[[a, b]^{b^{-1}a^{-1}}, [a, b]\right]^{-1} \pmod{L_5(P)}\\ &\equiv \left[[a, b, b], a^{-1}\right]^{-1} \left[[a, b][a, b, b^{-1}a^{-1}], [a, b]\right]^{-1} \pmod{L_5(P)}\\ &\equiv [a, b, b, a]^{a^{-1}} \left[[a, b], [a, b]\right]^{-1} \pmod{L_5(P)}\\ &\equiv [a, b, b, a] \pmod{L_5(P)}. \end{aligned}$$

□

3. Examples

3.1. *p*-group of class 5

We will get the following equations for p-groups which are generated by two elements of order p and which have nilpotent class five. We study the following group:

$$P_5 := \left\langle a, b \mid a^p = b^p = 1,\ [x_1, x_2, x_3, x_4, x_5, x_6] = 1 \text{ for all } (x_i) \in \{a, b\}^6 \right\rangle.$$

We have the following equations.

Q1 $\left[C_{123}^i, C_{45}^j\right] = C_{12345}^{ij} C_{12354}^{-ij}$

Q2 $\left[x_1, x_2^i\right] = C_{12}^i C_{122}^{\binom{i}{2}} C_{1222}^{\binom{i}{3}} C_{12222}^{\binom{i}{4}} C_{12212}^{\binom{i}{3}} C_{12221}^{-\binom{i}{3}}$

Q3 $\left[x_1^i, x_2\right] = C_{12}^i C_{121}^{\binom{i}{2}} C_{1211}^{\binom{i}{3}} C_{12111}^{\binom{i}{4}} C_{12112}^{(i-1)i(2i-1)/6} C_{12121}^{-(i-1)i(2i-1)/6}$

Q4 $C_{2134} = C_{1234}^{-1}$

Q5 $C_{21345} = C_{12345}^{-1}$

Q6 $C_{213} = C_{123}^{-1} C_{12312} C_{12321}^{-1}$

Q7 $C_{12212} = C_{12221}$

Q8 $C_{12112} = C_{12121}$

Proposition 2 *$L_5(P_5)$ is generated by four elements $[a, b, a, a, a]$, $[a, b, b, a, a]$, $[a, b, b, b, a]$ and $[a, b, b, b, b]$.*

Proof. Since $L_4(P_5)/L_5(P_5)$ is generated by three elements $\overline{[a, b, a, a]}$, $\overline{[a, b, a, b]}$ and $\overline{[a, b, b, b]}$, we deduce that $L_5(P_5)/L_6(P_5) \cong L_5(P_5)$ is generated by six elements $\overline{[a, b, a, a, a]}$, $\overline{[a, b, a, a, b]}$, $\overline{[a, b, a, b, a]}$, $\overline{[a, b, a, b, b]}$, $\overline{[a, b, b, b, a]}$, and $\overline{[a, b, b, b, b]}$. From the equations above, we get

$$\begin{aligned} C_{abaab} &= C_{ababa} && \text{(by Q8)} \\ &= C_{abbaa} && \text{(by Lemma 7)}, \\ C_{abbab} &= C_{abbba}. && \text{(by Q7)} \end{aligned}$$

Therefore $L_5(P_5)/L_6(P_5)$ is generated by four elements $\overline{[a, b, a, a, a]}$, $\overline{[a, b, b, a, a]}$, $\overline{[a, b, b, b, a]}$, and $\overline{[a, b, b, b, b]}$. □

Proof. Equations (Q1)–(Q3) follow from induction and (C1)–(C6).

(Q4)

$$\begin{aligned} C_{2134} &= \left[C_{12}^{-1}, x_3, x_4\right] \\ &= \left[C_{123}^{-C_{12}^{-1}}, x_4\right] && \text{(by C5)} \\ &= C_{1234}^{-C_{123}^{-C_{12}^{-1}}} && \text{(by C5)} \\ &= C_{1234}^{-1}. \end{aligned}$$

(Q5)
$$\begin{aligned} C_{21345} &= \left[C_{1234}^{-1}, x_5\right] && \text{(by Q4)}\\ &= C_{12345}^{-C_{1234}^{-1}} && \text{(by C5)}\\ &= C_{12345}^{-1}. \end{aligned}$$

(Q6)
$$\begin{aligned} C_{213} &= \left[C_{12}^{-1}, x_3\right]\\ &= C_{123}^{-C_{12}^{-1}} && \text{(by C5)}\\ &= C_{123}^{-1}\left[C_{123}^{-1}, C_{12}^{-1}\right]\\ &= C_{123}^{-1}C_{12312}C_{12321}^{-1}. && \text{(by Q1)} \end{aligned}$$

(Q7) Indeed

$$\begin{aligned} \left[x_1, x_2^i\right] &= \left[x_2^i, x_1\right]^{-1}\\ &= \left(C_{21}^{i}C_{212}^{\binom{i}{2}}C_{2122}^{\binom{i}{3}}C_{21222}^{\binom{i}{4}}C_{21221}^{(i-1)i(2i-1)/6}C_{21212}^{-(i-1)i(2i-1)/6}\right)^{-1}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Q3)}\\ &= C_{21212}^{(i-1)i(2i-1)/6}C_{21221}^{-(i-1)i(2i-1)/6}C_{21222}^{-\binom{i}{4}}C_{2122}^{-\binom{i}{3}}C_{212}^{-\binom{i}{2}}C_{21}^{-i}\\ &= \underbrace{C_{12212}^{-(i-1)i(2i-1)/6}C_{12221}^{(i-1)i(2i-1)/6}C_{12222}^{\binom{i}{4}}}_{\text{(by Q5)}}\ \underbrace{C_{1222}^{\binom{i}{3}}}_{\text{(by Q4)}}\\ &\quad\cdot \underbrace{C_{122}^{\binom{i}{2}}C_{12212}^{-\binom{i}{2}}C_{12221}^{\binom{i}{2}}}_{\text{(by Q6)}}C_{12}^{i}\\ &= C_{12}^{i}C_{122}^{\binom{i}{2}}C_{12212}^{-\binom{i}{2}}C_{12221}^{\binom{i}{2}}\underbrace{C_{12212}^{i\binom{i}{2}}C_{12221}^{-i\binom{i}{2}}}_{\text{(by Q1)}}C_{1222}^{\binom{i}{3}}C_{12222}^{\binom{i}{4}}\\ &\quad\cdot C_{12212}^{(i-1)i(2i-1)/6}C_{12221}^{-(i-1)i(2i-1)/6}\\ &= C_{12}^{i}C_{122}^{\binom{i}{2}}C_{1222}^{\binom{i}{3}}C_{12222}^{\binom{i}{4}}C_{12212}^{(i-1)i(5i-4)/6}C_{12221}^{-(i-1)i(5i-4)/6} \end{aligned}$$

and (Q2) so that

$$C_{12212}^{(i-1)i(4i-2)/6} = C_{12221}^{(i-1)i(4i-2)/6}.$$

Since there exists an integer i such that $1 \leq i \leq p-1$ and $f(i) = (i-1)i(2i-1)/3 \neq 0$, one has

$$C_{12212} = C_{12221}.$$

(Q8) Interchanging the index 1 and 2 in (Q7), we obtain

$$C_{21121} = C_{21112}.$$

From (Q5),

$$C_{12121} = C_{12112}. \qquad \square$$

3.2. *p*-group of class 6

We study the following group:

$$P_6 := \big\langle a, b \mid a^p = b^p = 1,$$
$$[x_1, x_2, x_3, x_4, x_5, x_6, x_7] = 1 \text{ for all } (x_i) \in \{a, b\}^7 \big\rangle.$$

We obtain the following equations.

R1 $\left[C_{123}^{i}, C_{45}^{j}\right] = C_{12345}^{ij} C_{12354}^{-ij} C_{123445}^{-ij} C_{123454}^{ij} C_{123554}^{ij} C_{123545}^{-ij}$

R2 $\left[C_{1234}^{i}, C_{56}^{j}\right] = C_{123456}^{ij} C_{123465}^{-ij}$

R3 $\left[C_{12345}^{i}, x_6^{j}\right] = C_{123456}^{ij}$

R4 $\left[C_{1234}^{i}, x_5^{j}\right] = C_{12345}^{ij} C_{123455}^{i\binom{j}{2}}$

R5 $\left[C_{123}, C_{456}\right] = C_{123645}^{-1} C_{123654} C_{123456} C_{123546}^{-1}$

R6 $C_{121112} = C_{121121}$

R7 $C_{122212} = C_{122221}$

R8 $\left[C_{123}^{i}, x_4^{j}\right] = C_{1234}^{ij} C_{12344}^{i\binom{j}{2}} C_{123444}^{i\binom{j}{3}}$

R9 $C_{121212} = C_{121221}$

Proposition 3 *$L_6(P_6)$ is generated by five elements $[a, b, a, a, a, a]$, $[a, b, b, a, a, a]$, $[a, b, b, b, a, a]$, $[a, b, b, b, b, a]$ and $[a, b, b, b, b, b]$.*

Proof. Since $L_5(P_6)/L_6(P_6)$ is generated by $\overline{[a, b, a, a, a]}$, $\overline{[a, b, b, a, a]}$, $\overline{[a, b, b, b, a]}$, and $\overline{[a, b, b, b, b]}$ by Proposition 2, $L_6(P_6)/L_7(P_6)$ is generated by $\overline{[a,b,a,a,a,a]}$, $\overline{[a,b,a,a,a,b]}$, $\overline{[a,b,b,a,a,a]}$, $\overline{[a,b,b,a,a,b]}$, $\overline{[a,b,b,b,a,a]}$, $\overline{[a,b,b,b,a,b]}$, $\overline{[a,b,b,b,b,a]}$, and $\overline{[a,b,b,b,b,b]}$. And from the above equations, we obtain

$$\begin{aligned} C_{abaaab} &= C_{abaaba} && \text{(by R6)} \\ &= C_{ababaa} && \text{(by Q8)} \\ &= C_{abbaaa} && \text{(by Lemma 7)}, \\ C_{abbaab} &= C_{abbaba} && \text{(by R9)} \\ &= C_{abbbaa} && \text{(by Q7)}, \\ C_{abbbab} &= C_{abbbba} && \text{(by R7)}. \end{aligned}$$

This completes the proof. □

Proof. Equations (R1)–(R4) and (R8) follow from induction and (C1)–(C6).

(R5) $\left[C_{123}, C_{456}\right]$

$$\begin{aligned} &= \left[C_{123}, C_{45}^{-1} x_6^{-1} C_{45} x_6\right] \\ &= \left[C_{123}, x_6^{-1} C_{45} x_6\right]\left[C_{123}, C_{45}^{-1}\right] \underbrace{\left[C_{123}, C_{45}^{-1}, x_6^{-1} C_{45} x_6\right]}_{\in L_7(P_6)} \\ &= \left[C_{123}, C_{45} x_6\right]\left[C_{123}, x_6^{-1}\right]\left[C_{123}, x_6^{-1}, C_{45} x_6\right] \\ &\quad \left(C_{12354} C_{12345}^{-1} C_{123554}^{-1} C_{123545} C_{123445} C_{123454}^{-1}\right) && \text{(by R1)} \\ &= [C_{123}, x_6][C_{123}, C_{45}][C_{123}, C_{45}, x_6] \underbrace{\left(C_{1236}^{-1} C_{12366}^{\binom{-1}{2}} C_{123666}^{\binom{-1}{3}}\right)}_{\text{(by R8)}} \\ &\quad \left[C_{123}, x_6^{-1}, x_6\right]\left[C_{123}, x_6^{-1}, C_{45}\right]\left[C_{123}, x_6^{-1}, C_{45}, x_6\right] \\ &\quad \left(C_{12354} C_{12345}^{-1} C_{123554}^{-1} C_{123545} C_{123445} C_{123454}^{-1}\right) \\ &= C_{1236}\left(C_{12345} C_{12354}^{-1} C_{123445}^{-1} C_{123454} C_{123554} C_{123545}^{-1}\right) && \text{(by R1)} \\ &\quad \left[\left(C_{12345} C_{12354}^{-1} C_{123445}^{-1} C_{123454} C_{123554} C_{123545}^{-1}\right), x_6\right] && \text{(by R1)} \\ &\quad \left(C_{1236}^{-1} C_{12366}^{\binom{-1}{2}} C_{123666}^{\binom{-1}{3}}\right) \end{aligned}$$

$$\left[\left(C_{1236}^{-1} C_{12366}^{\binom{-1}{2}} C_{123666}^{\binom{-1}{3}}\right), x_6\right] \quad \text{(by R8)}$$

$$\left[\left(C_{1236}^{-1} C_{12366}^{\binom{-1}{2}} C_{123666}^{\binom{-1}{3}}\right), C_{45}\right] \quad \text{(by R8)}$$

$$\left(C_{12354} C_{12345}^{-1} C_{123554}^{-1} C_{123545} C_{123445} C_{123454}^{-1}\right)$$

$$= C_{1236}\left(C_{12345} C_{12354}^{-1} C_{123445}^{-1} C_{123454} C_{123554} C_{123545}^{-1}\right)$$

$$[C_{12345}, x_6]^{C_{12354}^{-1}} \left[C_{12354}^{-1}, x_6\right]$$

$$\left(C_{1236}^{-1} C_{12366}^{\binom{-1}{2}} C_{123666}^{\binom{-1}{3}}\right)$$

$$\left[C_{1236}^{-1}, x_6\right]^{C_{12366}^{\binom{-1}{2}}} \left[C_{12366}^{\binom{-1}{2}}, x_6\right]$$

$$\left[C_{1236}^{-1}, C_{45}\right]$$

$$\left(C_{12354} C_{12345}^{-1} C_{123554}^{-1} C_{123545} C_{123445} C_{123454}^{-1}\right)$$

$$= C_{1236}\left(C_{12345} C_{12354}^{-1} C_{123445}^{-1} C_{123454} C_{123554} C_{123545}^{-1}\right)$$

$$C_{123456} C_{123546}^{-1} \quad \text{(by R3)}$$

$$\left(C_{1236}^{-1} C_{12366}^{\binom{-1}{2}} C_{123666}^{\binom{-1}{3}}\right) \quad \text{(by R4)}$$

$$C_{12366}^{-1} C_{123666}^{\binom{-1}{2}} \quad \text{(by R3)}$$

$$C_{123645}^{-1} C_{123654} \quad \text{(by R2)}$$

$$\left(C_{12354} C_{12345}^{-1} C_{123554}^{-1} C_{123545} C_{123445} C_{123454}^{-1}\right)$$

$$= C_{123456} C_{123546}^{-1} C_{123645}^{-1} C_{123654}.$$

(R6) By substituting $(x_1, x_2, x_1, x_1, x_2, x_1)$ for $(x_1, x_2, x_3, x_4, x_5, x_6)$ in (R5),

$$1 = [C_{121}, C_{121}] = C_{121112}^{-1} C_{121121} C_{121121} C_{121211}^{-1}.$$

Using (Q8), we have $C_{121121} = C_{121211}$ so that we get

$$C_{121112} = C_{121121}.$$

(R7) By substituting $(x_1, x_2, x_2, x_1, x_2, x_2)$ for $(x_1, x_2, x_3, x_4, x_5, x_6)$ in (R5),

$$1 = [C_{122}, C_{122}] = C_{122212}^{-1} C_{122221} C_{122122} C_{122212}^{-1}.$$

Using (Q7), we have $C_{122212} = C_{122122}$ so that we get

$$C_{122212} = C_{122221}.$$

(R9) By (R5), we have

$$\begin{aligned} [C_{123}, C_{456}] &= [C_{456}, C_{123}]^{-1}, \\ 1 &= C_{123645}^{-1} C_{123654} C_{123456} C_{123546}^{-1} \\ &\quad C_{456312}^{-1} C_{456321} C_{456123} C_{456213}^{-1}. \end{aligned}$$

By substituting $(x_1, x_2, x_1, x_1, x_2, x_2)$ for $(x_1, x_2, x_3, x_4, x_5, x_6)$ in the above equation, one gets

$$\begin{aligned} 1 &= C_{121221} C_{121212}^{-1} \\ &\quad C_{122112}^{-1} C_{122121} \underbrace{C_{122121} C_{122211}^{-1}}_{=1 \quad \text{(by Q7)}} \\ &\quad \underbrace{C_{121212}^{-1} C_{121122}}_{=1 \quad \text{(by Q8)}} \\ &= C_{121221} C_{121212}^{-1} \\ &\quad C_{122112}^{-1} C_{122121} \\ &= C_{121221}^{2} C_{121212}^{-2}. \qquad \text{(by Lemma 7)} \end{aligned}$$

□

4. Proof of the theorem

Proof. Since P is generated by two elements of order p, we have $\Phi(P) = L_2(P)$. If P is an abelian group, then we have the desired conclusion. By Proposition 1, $p \equiv 1 \pmod 7$. There exists a generator system $\{a, b\}$ of P such that a and b are of order p and such that $a^\alpha = a^u w_1$ and $b^\alpha = b^v w_2$ for some u, v integers and some $w_1, w_2 \in \Phi(P)$. Thus it is enough that we shall prove the remainder of the theorem. Our calculation in Examples allows us to compare the dimension of the vector space $L_i(P)/L_{i+1}(P)$ over the finite field $\mathbb{F}_p$ with that of the subspace which is generated by the elements fixed by α.

If $v \equiv u^6 \pmod p$, we get

$$\begin{aligned}[a,b]^\alpha &= \left[a^u w_1, b^{u^6} w_2\right] \\ &\equiv [a,b]^{u^7} \pmod{L_3(P)} && \text{(by Lemma 3)} \\ &\equiv [a,b] \pmod{L_3(P)}. && \text{(by Lemma 5)}\end{aligned}$$

Since α induces a f.p.f. automorphism on $L_2(P)/L_3(P)$, we have $[a,b] \in L_3(P)$. Lemma 5 states that $L_2(P)/L_3(P)$ is generated by $[a,b]L_3(P)$. Hence, we get $L_2(P)/L_3(P) = 1$. Since P is nilpotent, we get $L_2(P) = 1$. Therefore $[a,b] = 1$.

If $v \equiv u^3 \pmod p$, we get

$$\begin{aligned}[a,b,b]^\alpha &= \left[a^u w_1, b^{u^3} w_2, b^{u^3} w_2\right] \\ &\equiv [a,b,b]^{u^7} \pmod{L_4(P)} && \text{(by Lemma 3)} \\ &\equiv [a,b,b] \pmod{L_4(P)}.\end{aligned}$$

Since α induces a f.p.f. automorphism on $L_3(P)/L_4(P)$, we have $[a,b,b] \in L_4(P)$ and $[a,b,b,a]$, $[a,b,b,b] \in L_5(P)$. Then an elementary but tedious calculation shows that

$$[a,b,b]^\alpha = [a,b,b]w_5 \qquad \text{for some } w_5 \in L_5(P). \tag{1}$$

$$\begin{aligned}[a,b,a,a,a]^\alpha &= \left[a^u w_1, b^{u^3} w_2, a^u w_1, a^u w_1, a^u w_1\right] \\ &\equiv [a,b,a,a,a]^{u^7} \pmod{L_6(P)} && \text{(by Lemma 3)} \\ &\equiv [a,b,a,a,a] \pmod{L_6(P)}.\end{aligned}$$

Since α induces a f.p.f. automorphsim on $L_5(P)/L_6(P)$, we have $[a,b,a,a,a] \in L_6(P)$. From Proposition 2, we deduce that $L_5(P)/L_6(P) = 1$. This means that $L_5(P) = 1$. Hence one obtains $[a,b,a,a,a] = 1$, $[a,b,b,a,a] = 1$, $[a,b,b,b,a] = 1$, $[a,b,b,b,b] = 1$, $[a,b,b,b] = 1$ and $[a,b,b,a] = 1$. From equation (1) and $L_5(P) = 1$ we have $[a,b,b]^\alpha = [a,b,b]$.

Therefore we conclude that $[a, b, b] = 1$.

If $v \equiv u^2 \pmod{p}$, we get

$$\begin{aligned}[a,b,b,b]^\alpha &= \left[a^u w_1, b^{u^2} w_2, b^{u^2} w_2, b^{u^2} w_2\right] \\ &\equiv [a,b,b,b]^{u^7} \pmod{L_5(P)} \qquad \text{(by Lemma 3)} \\ &\equiv [a,b,b,b] \pmod{L_5(P)}.\end{aligned}$$

Since α induce a f.p.f. automorphism on $L_5(P)/L_6(P)$, $[a,b,b,b] \in L_5(P)$. And $[a,b,b,b,a], [a,b,b,b,b] \in L_6(P)$. Then an commutator calculation show that

$$[a,b,b,b]^\alpha = [a,b,b,b]w_6 \qquad \text{for some } w_6 \in L_6(P). \tag{2}$$

$$\begin{aligned}[a,b,b,a,a]^\alpha &= \left[a^u w_1, b^{u^2} w_2, b^{u^2} w_2, a^u w_1, a^u w_1\right] \\ &\equiv [a,b,b,a,a]^{u^7} \pmod{L_6(P)} \qquad \text{(by Lemma 3)} \\ &\equiv [a,b,b,a,a] \pmod{L_6(P)}\end{aligned}$$

Since α induces a f.p.f. automorphsim on $L_5(P)/L_6(P)$, $[a,b,b,a,a] \in L_6(P)$.

$$\begin{aligned}[a,b,a,a,a,a]^\alpha &= \left[a^u w_1, b^{u^2} w_2, a^u w_1, a^u w_1, a^u w_1, a^u w_1\right] \\ &\equiv [a,b,a,a,a,a]^{u^7} \pmod{L_7(P)} \qquad \text{(by Lemma 3)} \\ &\equiv [a,b,a,a,a,a] \pmod{L_7(P)}\end{aligned}$$

Since α induces a f.p.f. automorphsim on $L_6(P)/L_7(P)$, we have $[a,b,a,a,a,a] \in L_7(P)$. From Proposition 3, we deduce that $L_6(P)/L_7(P) = 1$. We get $L_6(P) = 1$. $[a,b,b,b,a] = [a,b,b,b,b] = 1$. From equation (2) and $L_6(P) = 1$ and $[a,b,b,b] \in L_5(P)$, we have $[a,b,b,b]^\alpha = [a,b,b,b]$. Therefore $[a,b,b,b] = 1$.

If $v \equiv u \pmod{p}$, then for all $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \{a, b\}^7$,

$$\begin{aligned}&[x_1, x_2, x_3, x_4, x_5, x_6, x_7]^\alpha \\ &\quad = \left[x_1^u w_{s_1}, x_2^u w_{s_2}, x_3^u w_{s_3}, x_4^u w_{s_4}, x_5^u w_{s_5}, x_6^u w_{s_6}, x_7^u w_{s_7}\right] \\ &\quad \equiv [x_1, x_2, x_3, x_4, x_5, x_6, x_7]^{u^7} \pmod{L_8(P)} \qquad \text{(by Lemma 3)}\end{aligned}$$

$$\equiv [x_1, x_2, x_3, x_4, x_5, x_6, x_7] \pmod{L_8(P)}$$

Since α induces a f.p.f. automorphism on $L_7(P)/L_8(P)$, for all $(x_i) \in \{a, b\}^7 [x_1, x_2, x_3, x_4, x_5, x_6, x_7] \in L_8(P)$. Hence $L_7(P)/L_8(P) = 1$. Therefore $L_7(P) = 1$.

This completes the proof of the Theorem. □

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