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A removable singularity theorem of J-holomorphic mappings for strongly pseudo-convex manifolds

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Abstract. We investigate a removable singularity theorem and other some basic properties of a J-holomorphic mapping for strongly pseudo-convex manifolds, which are necessary for constructing the moduli space of J-holomorphic mappings.

Key words: pseudo-convex manifold, pseudo-holomorphic.

1. Introduction

The theory of J-holomorphic curves is one of the most developing subjects in the study of symplectic geometry ever since its inception in Gromov's paper [6]. The first step in this theory is based on the construction of the moduli space and many geometric analyses of J-holomorphic curves. The crucial fact about J-holomorphic curves in symplectic geometry is that for A is a homology class of $H_2(M;\mathbb{Z})$ the moduli space $\mathcal{M}(A, J)$ of simple J-holomorphic curves is a compact finite dimensional smooth manifold if J is generic. This theory has a close relation with a lot of geometrical subjects including Floer homology and Seiberg-Witten invariant.

Although many fruitful results have been obtained by this theory, these results are applicable only in even-dimensional geometry. Strongly pseudo-convex manifolds also admit many symplectic-like features, for example, an almost complex strucutre J and a non-degenerate pseudo-Hermitian structure L. It is also well known that these structures give relevant properties to J-holomorphic mappings for strongly pseudo-convex manifolds [11].

On a symplectic manifold, we use two key properties of J-holomorphic curves to prove that the moduli space $\mathcal{M}(A, J)$ is a finite dimensional compact smooth manifold. These properties are the strong ellipticity of the equation which defines J-holomorphy and the removablity of singularity of J-holomorphic curves. In this paper, we consider J-holomorphic mappings

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from a 3-dimensional Sasakian manifold into a strongly pseudo-convex manifold and show a removable singularity property of them.

Removable singularity type theorems have been widely studied when the domain of mappings is a two dimensional disc. In this case the conformal invariance of the energy of J-holomorphic curves plays an important role. For example, we can consider the set of J-holomorphic curves with uniformly bounded energy as the set of L_1^2-bounded harmonic mappings, which are solutions of an elliptic partial differential equation.

A difficulty in study of strongly pseudo-convex CR geometry often comes from the control of the smoothness of the structures with respect to its characteristic direction. Since it is natural in strongly pseudo-convex CR geometry to consider a J-holomorphic mapping as a map whose domain is also an odd dimensional manifold, we have to modify the original harmonic theory for two-dimensional discs. Moreover, since the homothetic change of a contact structure does not imply the conformal change of the natural metric, the conformality of energy does not make sense in our case. To avoid these difficulties, in this paper, we will mainly consider the case of J-holomorphic mappings from regular Sasakian manifolds.

Let Σ^3 be a compact 3-dimensional Sasakian manifold. Then, for each point $p \in \Sigma$ there exists a neighborhood of p of the form $D^2 \times S^1 \subset \Sigma$ when Σ is assumed to be regular, that is its characteristic vector field is regular, where D^2 is a 2-dimensional unit disc. In this neighborhood the orbits generated by the characteristic vector field ξ_Σ are circles S^1 of the second factor. Since the volume of $D^2 \times S^1 \subset \Sigma$ is finite, the energy of every J-holomorphic mapping is also finite (see Section 3 for more details).

In these settings, we will see that a J-holomorphic mapping which has an isolated orbit singularity can be extended continuously.

Theorem *Let $u : (D^2 - \{(0,0)\}) \times S^1 \longrightarrow M^{2n+1}$ be a smooth J-holomorphic mapping into a strongly pseudo-convex manifold. Then u can be continuously extended as $u : D^2 \times S^1 \longrightarrow M^{2n+1}$, when the volume for $D^2 \times S^1$ is finite.*

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2. Preliminaries

We first recall basic notions of a strongly pseudo-convex manifold M^{2n+1}.

A $(2n+1)$-dimensional *strongly pseudo-convex manifold* M is an oriented smooth manifold which carries a structure (P, J, θ), where $P \subset TM$ is a $2n$-dimensional real subbundle of TM with an almost complex structure $J : P \longrightarrow P$ satisfying

$$N_J(X, Y) = 0 \quad \text{for } X, Y \in \Gamma(P),$$

where $N_J(X, Y) = [X, Y] - [JX, JY] - J([JX, Y] + [X, JY])$ is Nijenhuis tensor of J, and $\theta \in \Gamma((TM/P)^*)$ is a contact form whose *Levi-form*

$$L(X, Y) = -d\theta(JX, Y) \quad \text{for } X, Y \in P$$

is positive definite. In what follows, a strongly pseudo-convex manifold is abbreviated as an s.p.c. manifold.

Let (M, θ, J) be an s.p.c. manifold. Consider the complexification of TM and its $\sqrt{-1}$ eigen-subspace $S = \{X - \sqrt{-1}JX \mid X \in P\} \subset \mathbb{C}TM$. It holds $S \cap \overline{S} = \{0\}$ and $[\Gamma(S), \Gamma(S)] \subset \Gamma(S)$, where $\overline{S}$ is the complex conjugation of S.

Since M is a contact manifold, there exists a unique nonvanishing vector field ξ which satisfies $\theta(\xi) = 1$ and $i(\xi)\, d\theta = 0$. We call this vector field the characteristic vector field. By definition we have $TM = P \oplus \mathbb{R}\xi$, and hereafter we consider the Levi-form as a tensor on TM by extending J on TM as $J\xi = 0$. Then, we have a canonical Riemannian metric $g = g_{(\theta,J)}$ on M defined by

$$g(X, Y) = L(X, Y) + \theta(X)\theta(Y).$$

Since the 1-form θ and the vector field ξ satisfy $L_\xi\theta = 0$, we see that a necessary and sufficient condition of $L_\xi J = 0$ is that ξ is Killing with respect to the metric $g = g_{(\theta,J)}$. We call an almost complex structure J *normal* or *K-contact*, when $L_\xi\theta = 0$. We call an s.p.c. manifold with a normal almost complex structure *Sasakian manifold*.

Let $(\Sigma^3, \theta_\Sigma, j)$ be a connected 3-dimensional Sasakian manifold and (M, θ, J) be an s.p.c. manifold. Consider a smooth mapping $u : \Sigma \longrightarrow M$

which obeys the conditions

1. $du \circ j = J \circ du$ on P_Σ, (JH-1)
2. $du(\xi_\Sigma) = \lambda\xi$, for a positive smooth function $\lambda > 0$ on Σ, (JH-2)
3. $u^*\theta = \lambda\theta_\Sigma$, (JH-3)

where, $P_\Sigma = \ker\theta_\Sigma$ and ξ_Σ denotes the characteristic vector field on Σ.

We note that the equation (JH-1) is just an analogue of J-holomorphic curve in symplectic geometry, which M. Gromov initially defined [6]. So, we call such a map *J-holomorphic mapping for an s.p.c. manifold M.*

By the conditions (JH-1) and (JH-3), J-holomorphic mappings preserve the holomorphic structures, i.e. we have $u_*(S_\Sigma) \subset S$ for a J-holomorphic mapping u. This condition is often referred to CR-holomorphy of a mapping between CR manifolds.

Remark If the function $\lambda \in C^\infty_{>0}(\Sigma)$ is identically 1 over Σ, then the J-holomorphic mapping $u : \Sigma \longrightarrow M$ is an isometric immersion.

Examples
i) *Linear subspace sections*: Let $f(z_1, \ldots, z_{n+1})$ be a weighted homogeneous polynomial with an isolated singular point $0 \in \mathbb{C}^{n+1}$. It is well known that the link of the zero locus $M' = f^{-1}(0)$ has an s.p.c. structure.

Let $M = S^{2n+1} \cap M'$ be the link and $u : \Sigma \longrightarrow M$ be a natural inclusion map, where Σ is the link of the zero locus of the weighted homogeneous polynomial $f_0(z_1, z_2, z_3) = f(z_1, z_2, z_3, 0, \ldots, 0)$, the restriction of f to a four dimensional subspace V. So Σ is a section of M by V. Then it is observed that u is a J-holomorphic mapping into M.

ii) *Veronese mapping*: Let $\Sigma = S^3 \subset \mathbb{C}^2$ and $M = S^5 \subset \mathbb{C}^3$ be the unit spheres. Then we can see that the Veronese mapping $v : S^3 \longrightarrow S^5$

$$v(z_1, z_2) = \left(z_1^2, \sqrt{2}z_1 z_2, z_2^2\right),$$

is a J-holomorphic mapping.

More generally, the lift of J-holomorphic curves can be considered as an example of J-holomorphic mappings. Let $\Sigma^3 \longrightarrow \underline{\Sigma}^2$ and $M^{2n+1} \longrightarrow \underline{M}^{2n}$ be negative S^1-bundles over a Riemann surface and a $2n$-dimensional Kähler manifold, respectively. Let $\underline{u} : \underline{\Sigma} \longrightarrow \underline{M}$ be a holomorphic curve between base spaces and we assume that there exists a bundle isomorphism from the pull-back bundle $\underline{u}^*M$ to the bundle Σ which preserves connections. Then we can see that the bundle isomorphism induces a lift $u : \Sigma \longrightarrow \underline{u}^* M \subset M$ over the holomorphic curve $\underline{u}$, which is J-holomorphic in our sense.

The geometrical meaning of the positive function $\lambda \in C^\infty(\Sigma)$ defined in the condition (JH-2) is, roughly speaking, how the image $u(\Sigma)$ is covered by u. In the above, we have stated the definition in a weak form. However we may exploit a stronger condition than (JH-2). Namely, with an easy consideration, we can show that the function λ must be a constant.

Proposition 2.1 *For every mapping $u : \Sigma \longrightarrow M$ satisfying* (JH-2) *and* (JH-3)*, the function λ defined above is constant.*

Proof. For $X \in P_\Sigma$ we have $X\lambda = 0$, since

$$\begin{aligned} 0 &= d\theta(du(\xi_\Sigma), duX) = u^* d\theta(\xi_\Sigma, X) \\ &= (d\lambda \wedge \theta_\Sigma + \lambda d\theta_\Sigma)(\xi_\Sigma, X) = X\lambda. \end{aligned}$$

Therefore, from the strong pseudo-convexity of Σ we deduce $\xi_\Sigma \lambda = 0$, and hence $d\lambda = 0$. □

3. Energy and harmonicity

Let $u : \Sigma \longrightarrow M$ be a smooth mapping. We define the energy functional $E(u)$ of u by

$$E(u) = \frac{1}{2} \int_\Sigma \mathrm{Tr}_{g_\Sigma}(u^* g)\, dv_\Sigma.$$

Here, $g = g_{(\theta,J)}$ is the canonical metric on M, and $g_\Sigma = g_{(\theta_\Sigma, j)}$, $dv_\Sigma = dv_{(\theta_\Sigma, j)}$ are the canonical metric and the volume form defined by (θ_Σ, j) on Σ. The integrand is called the energy density and has an expression

$$\mathrm{Tr}_{g_\Sigma}(u^* g) = |du(e)|^2_{(\theta,J)} + |du(je)|^2_{(\theta,J)} + |du(\xi_\Sigma)|^2_{(\theta,J)},$$

where $e \in P_\Sigma$ is a horizontal unit vector.

By definition of the energy functional, we have $E(u) = \frac{1}{2}(2\lambda + \lambda^2)\mathrm{vol}_{(\theta_\Sigma, j)}$ for any J-holomorphic mapping u. In particular, if we use a Sasakian structure of Σ having finite volume $\mathrm{vol}_{(\theta_\Sigma, j)} < \infty$, then the energy $E(u)$ of J-holomorphic mapping u is also finite. On the other hand, the volume of the image $u(\Sigma)$ of Σ by the mapping u is defined by $V(u) =$

$\int_\Sigma \det_{g_{(\theta_\Sigma,j)}}(u^*g_{(\theta,J)})^{1/2}\,dv_{(\theta_\Sigma,j)}$, and we have $V(u) = \lambda^2 \mathrm{vol}_{(\theta_\Sigma,j)}$ for any J-holomorphic mapping.

Remark Although in the case of symplectic geometry we have "energy-area equality" which asserts that the energy of a symplectic J-holomorphic curve is equal to the area, in our case we do not expect to get a similar identity. On the other hand, we can observe an analogy with symplectic geometry using terms of harmonicity with respect to the energy functional E of J-holomorphic mappings for an s.p.c. manifold.

We call a smooth mapping harmonic when it is a critical point of the energy functional E (c.f. [11]).

Lemma *For an arbitrary connections D, D^M on Σ and M, and its induced connection D^u by u, respectively, we have,*

$$D^u_X(duY) - D^u_Y(duX) - du([X,Y]) = T(duX, duY),$$
$$D^u_X(JduY) - J\big(D^u_X duY\big) = \big(D^M_{duX}J\big)duY,$$

where T is the torsion tensor of D and $X, Y \in \mathfrak{X}(\Sigma)$.

Proof of lemma. It is obviously noticed that the left hand side of each of the above formula is tensor. Take local coordinates (x^1, x^2, x^3) and $(y^1, \ldots, y^{2n+1})$ on Σ and M, respectively. Then, for $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$

$$D^u_X(duY) = \sum \frac{\partial^2 u^k}{\partial x^i \partial x^j}\frac{\partial}{\partial y^k} + \sum \frac{\partial u^k}{\partial x^j}\frac{\partial u^l}{\partial x^i} D_{\frac{\partial}{\partial y^l}} \frac{\partial}{\partial y^k} \tag{3.1}$$

and thus

$$D^u_X(duY) - D^u_Y(duX) = \sum \frac{\partial u^k}{\partial x^j}\frac{\partial u^l}{\partial x^i} T\left(\frac{\partial}{\partial y^l}, \frac{\partial}{\partial y^k}\right) = T(duX, duY).$$

Similarly, since

$$JduY = J\left(\sum \frac{\partial u^k}{\partial x^j}\frac{\partial}{\partial y^k}\right) = \sum \frac{\partial u^k}{\partial x^j} J \frac{\partial}{\partial y^k},$$

the identity (3.1) induces,

$$D_X^u(JduY) - J\big(D_X^u duY\big) = \sum \frac{\partial u^k}{\partial x^j}\frac{\partial u^l}{\partial x^i}\,\big(D_{\frac{\partial}{\partial y^l}} J\big)\frac{\partial}{\partial y^k} = (D_{duX}J)duY.$$

These complete the proof. □

Proposition 3.1 *Every J-holomorphic mapping $u : \Sigma^3 \longrightarrow M^{2n+1}$ between s.p.c. manifolds is harmonic.*

Proof. It is well known that the condition for a mapping u to be harmonic is characterized as follows: Let ∇, ∇^M be the Levi-Civita connection on Σ, M respectively, and ∇^u be the connections of u^*TM canonically induced from ∇^M by u. We define the tension field $\tau(u)$ of u by

$$\tau(u) = \sum_i \nabla^u_{e_i} du(e_i) - du(\nabla_{e_i} e_i),$$

where $\{e_i\}$ is an orthonormal basis of $T\Sigma$. Note that this definition is independent of the choice of $\{e_i\}$. Then, a mapping u is harmonic if and only if $\tau(u) \equiv 0$.

We assume that u is J-holomorphic. First we examine the second term of the tension field $\tau(u)$. Take a unit vector $e \in P_\Sigma$, and hereafter, we will use the orthonormal basis $\{e, je, \xi_\Sigma\}$ of $T\Sigma$. Then by the condition (JH-1) of the definition, we have,

$$\begin{aligned} du(\nabla_{je} je) &= du(j\nabla_{je} e) + du((\nabla_{je} j)e) \\ &= Jdu(\nabla_e je) - du(j[e, je]) + du((\nabla_{je} j)e) \\ &= -\,du(\nabla_e e) + \theta(du(\nabla_e e))\xi + du(j(\nabla_e j)e) \\ &\quad - du(j[e, je]) + du((\nabla_{je} j)e) \end{aligned}$$

We apply to this the formulae

$$(\nabla_X j)Y = g_\Sigma(X, Y)\xi_\Sigma - \theta_\Sigma(Y)X, \qquad \nabla_X \xi_\Sigma = -jX,$$

which are valid for the Levi-Civita connection on a normal s.p.c. manifold [2]. Then, we see that the third term of the above is

$$du(j(\nabla_e j)e) = g_\Sigma(e, e)\, du(j\xi_\Sigma) - \theta_\Sigma(e)\, du(je),$$

and hence it vanishes because $j\xi_\Sigma = 0$ and $\theta_\Sigma(e) = 0$. Similarly, the fifth term vanishes. Moreover the second term also vanishes, since

$$\theta_\Sigma(\nabla_e e) = g_\Sigma(\nabla_e e, \xi_\Sigma) = -g_\Sigma(e, \nabla_e \xi_\Sigma) = g_\Sigma(e, je) = 0.$$

Therefore, we obtain, via (JH-1),

$$du(\nabla_{je} je) + du(\nabla_e e) = -J du([e, je]).$$

For the first term of the tension field we will have a similar computation by applying the formulae

$$\big(\nabla^M_X J\big)Y = g((1+h)X, Y)\xi - \theta(Y)\,(1+h)X, \qquad \nabla^M_X \xi = -J(1+h)X,$$

where h is a $(1,1)$-tensor defined by $h = \frac{1}{2}L_\xi J$ on a general s.p.c. manifold [2]. In fact we have

$$\begin{aligned}\nabla^u_{je} du(je) = &-\nabla^u_e du(e) + \theta\big(\nabla^u_e du(e)\big)\xi - J du([e, je]) \\ &+ g(hJdu(e), du(e))\xi + J(\nabla_{du(e)} J) du(e).\end{aligned}$$

For the second term of the right hand side of the above,

$$\begin{aligned}\theta\big(\nabla^u_e du(e)\big) &= \lambda^{-1} g\big(\nabla^u_e du(e), \lambda\xi\big) \\ &= \lambda^{-1} e g(du(e), du(\xi_\Sigma)) - \lambda^{-1} g\big(du(e), \nabla^u_e du(\xi_\Sigma)\big) \\ &= -\lambda^{-1} g\big(du(e), \nabla^u_e du(\xi_\Sigma)\big) \\ &= -g(du(e), \nabla_{du(e)}\xi) = g(du(e), J(1+h)du(e)) \\ &= g(du(e), Jh du(e)).\end{aligned}$$

On the other hand for any vector $X \in P$,

$$\begin{aligned}2JhX &= J(L_\xi J)X = JL_\xi JX - JJL_\xi X \\ &= JL_\xi JX + L_\xi X - \theta(L_\xi X)\xi \\ &= JL_\xi JX - L_\xi JJX - L_\xi \theta(X)\xi = -2hJX.\end{aligned}$$

So the second term and the fourth term together reduce to zero. We easily see that the last term vanishes. Thus we have also

$$\nabla^u_e du(e) + \nabla^u_{je} du(je) = -J du([e, je]),$$

so that $\tau(u)$ reduces

$$\nabla^u_{\xi_\Sigma} du(\xi_\Sigma) - du(\nabla_{\xi_\Sigma}\xi_\Sigma) = \lambda^{-1}(\xi\lambda)\xi.$$

It vanishes, since λ is constant from Proposition 2.1. □

4. Proof of the removable singularity

To prove the removable singularity theorem, we first show a key lemma which asserts the monotonicity of the volume density function of J-holomorphic mappings. To show the key lemma, we will use two basic properties of the Levi-Civita connection on a Sasakian manifold below.

Proposition 4.1 *On an arbitrary compact s.p.c. manifold Σ^{2m+1}, we have, for any $X \in \mathfrak{X}(\Sigma)$,*

$$\int_\Sigma \sum_{i=1}^{2m} g_\Sigma(\nabla_{e_i} X, e_i)\, dv_\Sigma = 0,$$

where $\{e_i\}$ is an orthonormal frame of P_Σ.

Proof. First we notice that the integral curves of ξ_Σ are geodesic on a contact metric manifold. Since ξ_Σ is perpendicular to P_Σ, we have $\mathrm{div}X = \sum_{i=1}^{2m} g_\Sigma(\nabla_{e_i} X, e_i) + g_\Sigma(\nabla_{\xi_\Sigma} X, \xi_\Sigma)$, and

$$g_\Sigma(\nabla_{\xi_\Sigma} X, \xi_\Sigma) = \xi_\Sigma \theta_\Sigma(X) - g_\Sigma(X, \nabla_{\xi_\Sigma}\xi_\Sigma) = \xi_\Sigma\theta_\Sigma(X).$$

Since the integration of the term including ξ_Σ vanishes (Proposition 3.6. of [13]), we obtain the desired formula. □

Proposition 4.2 *On a compact Sasakian manifold Σ^{2m+1}, we have*

$$\theta_\Sigma(\nabla_X X) = 0.$$

for $X \in \Gamma(P_\Sigma)$.

Proof. We only show $\theta_\Sigma(\nabla_{jX} jX) = 0$.

$$\begin{aligned}\nabla_{jX} jX &= (\nabla_{jX} j)X + j\nabla_{jX} X = g(jX, X)\xi_\Sigma - \theta_\Sigma(X) jX + j\nabla_{jX} X \\ &= j\nabla_{jX} X.\end{aligned}$$

Hence, $\theta_\Sigma(\nabla_{jX} jX) = \theta_\Sigma(j\nabla_{jX} X) = 0$. □

Proposition 4.3 (monotonicity) *For any J-holomorphic mapping $u : \Sigma^{2m+1} \longrightarrow M$, we have for a sufficiently small r and for any point $p \in u(\Sigma) \subset M$,*

$$V(r) \geq \mathrm{const} \cdot r^{2m+1},$$

where $V(r) = \mathrm{vol}(B(p; r) \cap u(\Sigma))$.

Proof. The argument used here is an analogy of classical one obtained in the case of harmonic maps from a unit 2-disc (c.f. [8]).

We take an isometric embedding $M \subset \mathbb{R}^N$ for a sufficiently large N and let B be the second fundamental form of this embedding. Since the embedding is isometry, we have $\langle X, Y \rangle = g(X, Y)$ for $X, Y \in TM$, where $\langle , \rangle$ is the canonical inner product of $\mathbb{R}^N$. On a small neighborhood of p, fix an orthonormal frame $\{e_i\}$ of P_Σ.

First, we consider a decomposition $X = X^\top + X^\perp$ for a vector field $X \in \mathfrak{X}(\mathbb{R}^N)$ with respect to $T\mathbb{R}^N|_M = TM \oplus TM^\perp$. Then, by definition of the second fundamental form B, we have

$$\begin{aligned}\sum_{i=1}^{2m} \langle \nabla^u_{e_i} X, du_i \rangle &= \sum_{i=1}^{2m} \langle \nabla^u_{e_i} X^\perp, du_i \rangle + \sum_{i=1}^{2m} \langle \nabla^u_{e_i} X^\top, du_i \rangle \\ &= -\sum_{i=1}^{2m} \langle X^\perp, \nabla^u_{e_i} du_i \rangle + \sum_{i=1}^{2m} \langle \nabla^u_{e_i} X^\top, du_i \rangle \\ &= -\sum_{i=1}^{2m} \langle X, B(du_i, du_i) \rangle + \sum_{i=1}^{2m} \langle \nabla^u_{e_i} X^\top, du_i \rangle.\end{aligned}$$

Here, $du_i = du(e_i), i = 1, \ldots, 2m$.

Since $u : \Sigma \longrightarrow M$ is an immersion, we can decompose the second term of the right hand side with respect to $T_u M = u_* T\Sigma \oplus (u_* T\Sigma)^\perp$ as

$$X^\top = \tan_u X + \mathrm{nor}_u X, \quad \tan_u X = du(\tilde{X} + a\xi_\Sigma).$$

Here, $\tilde{X} \in P_\Sigma, a = \lambda^{-1}\theta(X^\top) \circ u \in C^\infty(\Sigma)$. By the harmonicity of u we can calculate

$$\begin{aligned}\sum_{i=1}^{2m} \big\langle \nabla^u_{e_i} \mathrm{nor}_u X, du_i \big\rangle &= -\sum_{i=1}^{2m} \big\langle \mathrm{nor}_u X, \nabla^u_{e_i} du_i \big\rangle \\ &= -\sum_{i=1}^{2m} \big\langle \mathrm{nor}_u X, du(\nabla_{e_i} e_i) \big\rangle = 0.\end{aligned}$$

From Proposition 4.2, we have $\nabla_{e_i} e_i \in P_\Sigma$ and hence $sum_{i=1}^{2m} du(\nabla_{e_i} e_i) \in P$. Therefore,

$$\begin{aligned}\sum_{i=1}^{2m} \big\langle \nabla^u_{e_i} du\tilde{X}, du_i \big\rangle &= \sum_{i=1}^{2m} du_i\, g\big(du\tilde{X}, du_i\big) - g\big(du\tilde{X}, \nabla^u_{e_i} du_i\big) \\ &= \sum_{i=1}^{2m} \lambda g_\Sigma\big(\nabla_{e_i} \tilde{X}, e_i\big),\end{aligned}$$

$$\sum_{i=1}^{2m} \big\langle \nabla^u_{e_i} du(a\xi_\Sigma), du_i \big\rangle = -\sum_{i=1}^{2m} g\big(du(a\xi_\Sigma), \nabla^u_{e_i} du_i\big) = 0.$$

So the P_Σ-component of $\sum_{i=1}^{2m} \langle \nabla^u_{e_i} X, du_i \rangle$ is,

$$\sum_{i=1}^{2m} \big\langle \nabla^u_{e_i} X, du_i \big\rangle = \sum_{i=1}^{2m} -\langle X, B(du_i, du_i) \rangle + \lambda g_\Sigma\big(\nabla_{e_i} \tilde{X}, e_i\big).$$

On the other hand, for the $du_0 = du(\xi_\Sigma)$ component, we have

$$\begin{aligned}\big\langle \nabla^u_{\xi_\Sigma} X, du_0 \big\rangle &= \big\langle \nabla^u_{\xi_\Sigma} X^\perp, du_0 \big\rangle + \big\langle \nabla^u_{\xi_\Sigma} X^\top, du_0 \big\rangle \\ &= -\langle X, B(du_0, du_0) \rangle + g\big(\nabla^u_{\xi_\Sigma} X^\top, du_0\big),\end{aligned}$$

of which the second term further is

$$\begin{aligned}g\big(\nabla^u_{\xi_\Sigma} X^\top, du_0\big) &= du_0\, g\big(X^\top, du_0\big) - g\big(X^\top, \nabla^u_{\xi_\Sigma} du_0\big) \\ &= du_0\, g(\tan_u X, du_0) \\ &= du_0\, g\big(du(\tilde{X} + a\xi_\Sigma), du_0\big) = \lambda^2 \xi_\Sigma\, a.\end{aligned}$$

Thus $\int_\Sigma \langle \nabla^u_{\xi_\Sigma} X, du_0\rangle\, dv_\Sigma = -\int_\Sigma \langle X, B(du_0, du_0)\rangle\, dv_\Sigma$.

Let $\mathbf{x}$ be the position vector in $\mathbb{R}^N$, and apply the argument above to a vector field of the form $X = f(\|\mathbf{x}\|)\mathbf{x} \in \mathfrak{X}(\mathbb{R}^N)$ for a compactly supported function f on $\mathbb{R}^N$. Then, for each $i = 1, \ldots, 2m$, we have

$$\begin{aligned}\big\langle \nabla^u_{e_i} X, du_i\big\rangle &= f'(\|\mathbf{x}\|)(du_i\|\mathbf{x}\|)\langle \mathbf{x}, du_i\rangle + f(\|\mathbf{x}\|)\big\langle \nabla^u_{e_i}\mathbf{x}, du_i\big\rangle \\ &= f'(\|\mathbf{x}\|)\,\frac{\langle \mathbf{x}, du_i\rangle^2}{\|\mathbf{x}\|} + f(\|\mathbf{x}\|)\|du_i\|^2.\end{aligned}$$

So if we write $E_i = \frac{du_i}{\|du_i\|} = \lambda^{-\frac{1}{2}} du_i$, $E_0 = \xi$, then since $\langle E_i, E_j\rangle = \delta_{ij}$, $\tan_u \mathbf{x} = \sum_{i=0}^{2m}\langle \mathbf{x},\, E_i\rangle E_i$, we obtain

$$2m\lambda f(\|\mathbf{x}\|) = \sum_{i=1}^{2m} \lambda g_\Sigma\big(\nabla_{e_i}\tilde{X}, e_i\big) - \langle X, \mathrm{Tr}_{P_\Sigma} B\rangle - \sum_{i=1}^{2m} f'(\|\mathbf{x}\|)\,\frac{\langle \mathbf{x}, du_i\rangle^2}{\|\mathbf{x}\|},$$

and thus

$$2m f(\|\mathbf{x}\|) = \sum_{i=1}^{2m} g_\Sigma\big(\nabla_{e_i}\tilde{X}, e_i\big) - \frac{1}{\lambda}\langle X, \mathrm{Tr}_{P_\Sigma} B\rangle - \sum_{i=1}^{2m} f'(\|\mathbf{x}\|)\,\frac{\langle \mathbf{x}, E_i\rangle^2}{\|\mathbf{x}\|},$$

For the direction of ξ_Σ we have

$$f(\|\mathbf{x}\|) = \xi_\Sigma\, a - \lambda^{-2}\langle X, B(du_0, du_0)\rangle - f'(\|\mathbf{x}\|)\,\frac{\langle \mathbf{x}, \xi\rangle^2}{\|\mathbf{x}\|}.$$

Here, $\mathrm{Tr}_{P_\Sigma} B = \sum_{i=1}^{2m} B(du_i, du_i)$.

Now, let f be a cut-off function whose support is in $B(p; r+\epsilon)$ such that $f(x) = 1$ and $f'(x) \le 1/\epsilon$ for $x \in B(p; r)$. Then, sum up the above for a cut-off f over $i = 0, \ldots, 2m$ and integrate over Σ,

$$\begin{aligned}&(2m+1)\int_\Sigma f(\|\mathbf{x}\|)\, dv_\Sigma \\ &\quad = -\int_\Sigma f'(\|\mathbf{x}\|)\,\frac{\|\tan_u \mathbf{x}\|^2}{\|\mathbf{x}\|} + \left\langle X, \frac{1}{\lambda}\mathrm{Tr}_{P_\Sigma} B + \frac{1}{\lambda^2} B(du_0, du_0)\right\rangle dv_\Sigma,\end{aligned}$$

so that

$$(2m+1)\int_{B(p;r+\epsilon)} f(\|\mathbf{x}\|)\,dv_\Sigma$$
$$\leq -\int_\Sigma f'(\|\mathbf{x}\|)\,\|\tan_u \mathbf{x}\|\,dv_\Sigma + \int_\Sigma \langle f(\|\mathbf{x}\|)\mathbf{x}, \mathrm{trace}_u B\rangle\,dv_\Sigma,$$

and hence

$$(2m+1)\int_{B(p;r+\epsilon)} f(\|\mathbf{x}\|)\,dv_\Sigma$$
$$\leq \frac{r+\epsilon}{\epsilon}\int_{B(p;r+\epsilon)-B(p;r)} dv_\Sigma + (r+\epsilon)\max\|B\|\int_{B(p;r+\epsilon)} f(\|\mathbf{x}\|)\,dv_\Sigma,$$

where $\mathrm{trace}_u B = \sum_{i=0}^{2m} B(E_i,E_i) = \frac{1}{\lambda}\mathrm{Tr}_{P_\Sigma} B + \frac{1}{\lambda^2}B(du_0,du_0)$.

This leads at the limit of $\epsilon \to 0$, $(2m+1)V(r) \leq r\frac{\mathrm{d}}{\mathrm{d}r}V(r) + rV(r)\max\|B\|$, and $\frac{\mathrm{d}}{\mathrm{d}r}\log V(r) - \frac{(2m+1)}{r} + \max\|B\| \geq 0$. Or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}r}\log V(r) - \frac{\mathrm{d}}{\mathrm{d}r}\log r^{2m+1} + \frac{\mathrm{d}}{\mathrm{d}r}\max\|B\|r \geq 0.$$

Thus $\frac{\mathrm{d}}{\mathrm{d}r}\log\frac{V(r)}{r^{2m+1}}e^{\max\|B\|r} \geq 0$.

This means that for a sufficiently small r the function $r \mapsto \frac{V(r)}{r^{2m+1}}e^{\max\|B\|r}$ is non-increasing. So, if we set $r \ll R$, we get the desired inequality by setting

$$\mathrm{const.} = \left(\lim_{r\to 0}\frac{V(r)}{r^{2m+1}}\right)e^{-\max\|B\|R}. \qquad \square$$

Now, we are in a position to prove Theorem.

Proof of Theorem. Let us write $\Sigma = D^2 \times S^1$, and consider S^1 as the unit interval with the end points identified. When we deal with a regular Sasakian 3-fold, we can take a neighborhood of this form. In this case we can consider each S^1 as an orbit of the characteristic vector field. For the proof of Theorem, it is sufficient to prove that for any sequence $a_n = (x_n, y_n, 0) \to (0,0,0)$ $(n \to \infty)$ on the plane D^2 there exists a unique limit point $p_0 = \lim_{n\to\infty} u(a_n) \in M$. In fact, if this is done, we define

$$u(0,0,a) = \psi_{\lambda a}(p_0),$$

where $\{\psi_a \mid a \in \mathbb{R}\}$ is the 1-parameter transformation group generated by ξ on M. Then this implies the required continuous extension in Theorem. Indeed, obviously $du(\xi_\Sigma) = \lambda\xi$ and it suffices to see the continuity at $(0,0,0)$. For any sequence $(x_n, y_n, z_n) \to (0,0,0)$,

$$\begin{aligned} \lim_{t\to\infty} u(x_n, y_n, z_n) &= \lim_{n\to\infty} u \circ \psi^{\Sigma}_{z_n}(x_n, y_n, 0) \\ &= \lim_{n\to\infty} \psi_{\lambda z_n} \circ u(x_n, y_n, 0). \end{aligned}$$

This reduces to $\lim_{n\to\infty} u(x_n, y_n, 0)$, by the continuity of $\psi : \mathbb{R} \times M \longrightarrow M$.

Lemma *Let $a_n = (x_n, y_n, 0) \to (0,0,0)$ $(n \to \infty)$ be a sequence on the plane satisfying $\lim_{n\to\infty} u(a_n) = p \in M$. Then for any positive number $\delta > 0$ and for each such components $A \subset u^{-1}(B(p;\delta))$ such that $u(A) \cap B(p;\delta/2) \neq \emptyset$ we have a uniform estimate of the volume of A from below as follows:*

$$\mathrm{vol}(u(A)) \geq \mathrm{const} \cdot \delta^3.$$

In particular, the number of components of $u^{-1}(B(p;\delta))$ is finite.

Proof of lemma. This follows from the monotonicity of the restricted J-holomorphic mapping $u : A \longrightarrow M$ (Proposition 4.3). Indeed we can take a point p' for which we have $u(A) \cap B(p';\delta/2) \subset B(p;\delta)$, by the assumption $u(A) \cap B(p;\delta/2) \neq \emptyset$. So we can get the result applying the monotonicity we have seen above for u and $p' \in u(A)$.

The finiteness of the number of such components can be proved by the finiteness of the volume of Σ. □

This lemma implies that there exists a connected component A of $u^{-1}(B(p;\delta))$ which includes an infinite subsequence of a_n. That is, there exists a component A such that the closure $\overline{A}$ intersects with the origin $(0,0,0)$ and thus we can take a (continuous) path $\alpha(t)$ which passes through the each point of the subsequence of a_n.

Next we show that the image by u of a circle $\gamma_r = \{(x,y,0) \mid x^2 + y^2 = r^2\}$ passes through a neighborhood of p when r is sufficiently small. Fix r and choose n such that $d(0, a_{n+1}) \leq r \leq d(0, a_n)$. Let us denote the intersection point on $\alpha(t)$ with $\gamma_r \times S^1 = \{(x,y,z) \mid x^2 + y^2 = r^2\}$ by α_r. (More explicitly, we define $\alpha_r = \alpha(t_r)$, $t_r = \max\{t \mid \alpha(t) \in \gamma_r \times S^1\}$ and we now consider the correspondence $n \mapsto d(0, a_n)$ is nonincreasing.)

Define a curve $\beta(t)$ as the projection of $\alpha(t)$ to the plane, $\beta(t) = \pi_{D^2} \circ \alpha(t)$. Let $s(t)$ be the line (with unit speed) on D^2 from a_n to $\beta_r = \pi_{D^2} \circ \alpha_r$. Since the speed $\mathrm{Tr}_{g_{(\theta_\Sigma, j)}}(u^* g_{(\theta,J)})$ of u is bounded by $m_\lambda = \max\{\lambda, \lambda^2\}$, the distance

$$d(u(\beta_r), u(a_n)) \leq \int_{a_n}^{\beta_r} \|du(s'(t))\| \, dv_\Sigma \leq m_\lambda d(\beta_r, a_n)$$

is arbitraly small when n is large so that $d(a_n, a_{n+1})$ and r are sufficiently small. We also have $d(u(\alpha_r), u(a_n)) \leq 2\delta$, since $\alpha_r, a_n \in B(p; \delta)$.

Thus the z-coordinate $d(\alpha_r, \beta_r)$ of α_r is bounded above by $\lambda^{-1}(2\delta + m_\lambda d(0, a_n))$. Therefore $u(\beta_r)$ is in a neighborhood of p, for example we can consider as $u(\beta_r) \in B(p; 4\delta)$ when $m_\lambda d(0, a_n) < \delta$.

From now on, we will use an argument inducing a contradiction to prove the theorem. To this end we assume that the plane curves $\alpha_1(t) = (x_1(t), y_1(t), 0)$ and $\alpha_2(t) = (x_2(t), y_2(t), 0)$ would have different limit points; $\lim_{t\to\infty} u(\alpha(t)) = p$ and $\lim_{t\to\infty} u(\alpha_2(t)) = q \ (\neq p)$. Take a positive number $\delta > 0$ such that $d(p, q) > 9\delta$. Then, as we have mentioned above, there exists a number $R > 0$ such that for any $r < R$ the image $u(\gamma_r)$ passes through either near p and q. Hence, we have $l(u(\gamma_r)) = \int_{S^1\times\{0\}} |\partial u/\partial\varphi| \, d\varphi > \delta$, where $l(u(\gamma_r))$ means the length of the image of a circle γ_r.

Using a cylindrical coordinate $(w = x + \sqrt{-1}y = re^{\sqrt{-1}\varphi}, z) \in \mathbb{C} \times S^1$ in Σ, we define a horizontal frame field by $X = \partial/\partial x - \theta_\Sigma(\partial/\partial x)\xi_\Sigma$, $Y = jX = \partial/\partial y - \theta_\Sigma(\partial/\partial y)\xi_\Sigma$ and set $e = \frac{1}{2}(X - \sqrt{-1}Y) \in S$.

First we note that $\|du\|^2 = \|du(X)\|^2 + \|du(Y)\|^2 + \|du(\xi_\Sigma)\|^2$ and $\|du(\xi_\Sigma)\|^2 = \lambda^2$. Since the vector field $du(X)$ is horizontal, its norm can be calculated in the decomposition $TM = P \oplus \mathbb{R}\xi$ as $\|du(\partial/\partial x)\|^2 = \|du(X)\|^2 + \|\theta(\partial/\partial x)du(\xi_\Sigma)\|^2$. Therefore

$$\begin{aligned}
\|du\|^2 &= \|du(X)\|^2 + \|du(Y)\|^2 + \|du(\xi_\Sigma)\|^2 \\
&= \left\|\frac{\partial u}{\partial x}\right\|^2 + \left\|\frac{\partial u}{\partial y}\right\|^2 + \lambda^2\left\{1 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial x}\right)\right\|^2 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial y}\right)\right\|^2\right\} \\
&\geq \frac{1}{r^2}\left\|\frac{\partial u}{\partial \varphi}\right\|^2 + \lambda^2\left\{1 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial x}\right)\right\|^2 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial y}\right)\right\|^2\right\}.
\end{aligned}$$

Here we used the fact $\|\partial u/\partial x\|^2 + \|\partial u/\partial y\|^2 = \|\partial u/\partial r\|^2 + \frac{1}{r^2}\|\partial u/\partial\varphi\|^2 \geq \frac{1}{r^2}\|\partial u/\partial\varphi\|^2$. So integrating this we have,

$$E(u) \geq \int_\Sigma \left(\frac{1}{r^2}\left\|\frac{\partial u}{\partial\varphi}\right\|^2 + \lambda^2\left\{1 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial x}\right)\right\|^2 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial y}\right)\right\|^2\right\}\right) dv_\Sigma$$
$$= \int_0^{2\pi}\int_0^1\int_0^{2\pi} \frac{1}{r^2}\left|\frac{\partial u}{\partial\varphi}\right|^2 r d\varphi dr dz$$
$$+ \lambda^2 \int_\Sigma \left\{1 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial x}\right)\right\|^2 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial y}\right)\right\|^2\right\} dv_\Sigma.$$

Using Schwarz inequality and the fact $l(u(\gamma_r)) \geq \delta$,

$$E(u) \geq \int_0^{2\pi}\int_0^1 \frac{1}{2\pi r}\left(\int_0^{2\pi}\left|\frac{\partial u}{\partial\varphi}\right| d\varphi\right)^2 dr dz$$
$$+ \lambda^2 \int_\Sigma \left\{1 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial x}\right)\right\|^2 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial y}\right)\right\|^2\right\} dv_\Sigma$$
$$\geq \int_0^{2\pi}\int_0^1 \frac{\delta^2}{2\pi r} dr dz + \lambda^2 \int_\Sigma \left\{1 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial x}\right)\right\|^2 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial y}\right)\right\|^2\right\} dv_\Sigma.$$

The right hand side is equal to

$$\delta^2 \int_0^1 \frac{dr}{r} + \lambda^2 \int_\Sigma \left\{1 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial x}\right)\right\|^2 - \left\|\theta_\Sigma\left(\frac{\partial}{\partial y}\right)\right\|^2\right\} dv_\Sigma,$$

whose second term is the integral of a function which does not depend on u, and therefore it can only take a finite value. Since the first term diverges, this is a contradiction to finiteness of the energy E as desired. □

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