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Time periodic solutions of the Navier-Stokes equations under general outflow condition in a two dimensional symmetric channel

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Abstract. In this paper we will prove that there exists a time periodic solution of the Navier-Stokes equations with the inhomogeneous boundary condition for infinite symmetric channels in $\mathbb{R}^2$. In two and three dimensional more generalized infinite channels (than treated in this paper) H. Beirão da Veiga [5] proved that there exists time periodic solutions of the Navier-Stokes equations with the homogeneous boundary condition under a small time periodic flux. G. P. Galdi and A. M. Robertson [11] obtained time-periodic Poiseuille flow in a straight channel with a smooth cross section. C. J. Amick [3] proved that in two and three dimensional unbounded channels there exists solutions of the stationary Navier-Stokes equations with the nonhomogenous boundary condition. H. Morimoto and H. Fujita [19] and H. Morimoto [20] proved that in a two dimensional certain unbounded symmetric channel there exists symmetric solutions of the stationary Navier-Stokes equations with a special symmetric Dirichlet boundary condition. T-P. Kobayashi [15] demonstrated that for two and three dimensional infinite channels time periodic solutions of the Navier-Stokes equations exist under the same condition as C. J. Amick [3]. In this paper using the condition of H. Morimoto and H. Fujita [19] and H. Morimoto [20], we obtain time priodic solutions.

Key words: time periodic solutions of the Navier-Stokes equations, general outflow condition, stationary symmetric Navier-Stokes flow, symmetry, 2-D infinite channels, the poiseuille velocity.

1. Introduction

1.1. Problems

First of all, we define infinite channels in $\mathbb{R}^2$. Let L_1 and L_2 be positive real numbers. Let

$$\omega^i = \{x \in \mathbb{R}^2; -L_i < x_1 < L_i\} \quad (i = 1, 2).$$

We consider the Poiseuille velocities

$$\boldsymbol{P}_i^\alpha(x) = \left(0, \frac{3\alpha}{4L_i^3}(L_i^2 - x_1^2)\right) \quad \text{in } \omega_i \ (i = 1, 2), \tag{1.1}$$

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where α is a flux of $\boldsymbol{P}_i^{\alpha}$ on the cross section of ω^i.

In this paper we suppose that a domain Ω satisfies the following conditions. Let H be a positive constant. We set

$$\omega_H^1 = \{x \in \omega^1; x_2 \geq H\},$$
$$\omega_{-H}^2 = \{x \in \omega^2; x_2 \leq -H\}.$$

Let Ω be a unbounded smooth domain satisfying $\Omega \cap \omega_H^1 = \omega_H^1$ and $\Omega \cap \omega_{-H}^2 = \omega_{-H}^2$. We set

$$\omega_0 = \Omega \backslash \big(\omega_H^1 \cup \omega_{-H}^2\big).$$

Let ω_0 be a bounded domain. The boundary $\partial\omega_0$ has $J+1$ disjoint closed boundary components $\Gamma_1, \ldots, \Gamma_J$, Γ_{J+1}, i.e. $\partial\omega_0 = \cup_{j=1}^{J+1}\Gamma_j$ with $\Gamma_i \cap \Gamma_j = \emptyset$ $(i \neq j)$. Γ_{J+1} is the outer boundary of ω_0 and not smooth and $\Gamma_1, \ldots, \Gamma_J$ are the smooth inner boundaries of ω_0. Let $\partial\Omega$ be the boundary of the unbounded domain Ω. $\partial\Omega$ has $J+2$ disjoint closed boundary components Γ_0^+, Γ_0^-, $\Gamma_1, \ldots, \Gamma_J$ i.e. $\partial\Omega = \cup_{j=1}^{J}\Gamma_j \cup \Gamma_0^+ \cup \Gamma_0^-$ with $\Gamma_i \cap \Gamma_0^+ = \emptyset$ and $\Gamma_i \cap \Gamma_0^- = \emptyset$, where Γ_0^+ and Γ_0^- are the right hand side and the left hand side of the outer boundaries of Ω respectively and $\Gamma_1, \ldots, \Gamma_J$ are inner boundaries of Ω. Futhermore the domain satisfies the following symmetric condition.

Assumption 1.1 *The domain Ω is symmetric with respect to the x_2-axis and the boundary $\partial\Omega$ has connected components Γ_0^+, Γ_0^-, $\Gamma_1, \ldots, \Gamma_J$ and the inner boundaries $\Gamma_j (1 \leq j \leq J)$ intersects the x_2-axis.*

An incompressible viscous fluid filles Ω. Let $\boldsymbol{u} = \boldsymbol{u}(t, x)$, $p = p(t, x)$ be the unknown velocity and the unknown pressure of an incompressible viscous fluid in Ω respectively, while $\nu > 0$ is the kinematic viscosity. We consider the nonstationary Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } (0, T) \times \Omega, \tag{1.2}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } (0, T) \times \Omega \tag{1.3}$$

with the Dirichlet boundary condition

$$\boldsymbol{u} = \boldsymbol{\beta} \quad \text{on } (0, T) \times \partial\Omega, \tag{1.4}$$

where $\boldsymbol{f} = (f_1(t,x), f_2(t,x))$ is the external force and $\boldsymbol{\beta} = (\beta_1(t,x), \beta_2(t,x))$ is the given function on $(0,T) \times \partial\Omega$. The boundary condition $\boldsymbol{\beta}$ must satisfy

$$\int_{\partial\Omega} \boldsymbol{\beta}(t) \cdot \boldsymbol{n} d\sigma = 0 \quad (\forall t \in (0,T)), \tag{1.5}$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$. Let us call the condition (1.5) "*General Outflow Condition*", (GOC). Futhermore if $\boldsymbol{\beta}$ satisfies

$$\int_{\Gamma_0^+} \boldsymbol{\beta}(t) \cdot \boldsymbol{n} d\sigma = \int_{\Gamma_0^-} \boldsymbol{\beta}(t) \cdot \boldsymbol{n} d\sigma = \int_{\Gamma_j} \boldsymbol{\beta}(t) \cdot \boldsymbol{n} d\sigma = 0$$

$$(\forall t \in (0,T),\ j = 1, \ldots, J), \tag{1.6}$$

then we call the condition (1.6) "*Stringent Outflow Condition*", (SOC). We consider at infinity

$$\begin{cases} \boldsymbol{u} \to \boldsymbol{P}_1^\alpha & \text{as } x_2 \to \infty \text{ in } \omega_H^1 \\ \boldsymbol{u} \to \boldsymbol{P}_2^\alpha & \text{as } x_2 \to -\infty \text{ in } \omega_{-H}^2 \end{cases} \tag{1.7}$$

and the time periodic condition

$$\boldsymbol{u}(0) = \boldsymbol{u}(T) \quad \text{in } \Omega. \tag{1.8}$$

It is well known that there exists a smooth vector function $\boldsymbol{P}^\alpha$ which is symmetric with respect to the x_2-axis and satisfies

$$\operatorname{div} \boldsymbol{P}^\alpha = 0 \qquad \text{in } \Omega, \tag{1.9}$$

$$\boldsymbol{P}^\alpha = \boldsymbol{0} \qquad \text{on } \partial\Omega, \tag{1.10}$$

$$\boldsymbol{P}^\alpha = \boldsymbol{P}_1^\alpha \quad \text{in } \omega_H^1, \tag{1.11}$$

$$\boldsymbol{P}^\alpha = \boldsymbol{P}_2^\alpha \quad \text{in } \omega_{-H}^2. \tag{1.12}$$

For the proof, see C. J. Amick [3, Theorem 3.3]. Let us call $\boldsymbol{P}^\alpha$ "the extended Poiseuille velocity".

S. Kaniel and M. Shinbrot [13] proved the uniqueness of the time periodic solution of the Navier-Stokes equations. V. I. Yudovič [26] proved the existence of time periodic solutions of the Navier-Stokes equations with the Dirichlet boundary condition satisfying (SOC) in two and three dimensional bounded domains. J. L. Lions [17] considered time priodic problems for the Navier-Stokes equations with the homogeneous boundary condition. A. Takeshita [23] studied the existence and uniquness of time periodic solutions of the Navier-Stokes equations in two dimensional bounded domains. In a symmetric domain H. Morimoto [21] obtained a time periodic symmetric solution with the time-independent symmetric Dirichlet boundary conditions satisfying (GOC). T-P. Kobayashi [14] found time periodic symmetric solutions of the Navier-Stokes equations with the time-dependent symmetric Dirichlet boundary condition satisfying (GOC) in the similar domain as H. Morimoto [21]. T-P. Kobayashi [15] investigated the relation between stationary solutions and time periodic solutions of the Navier-Stokes equations in two and three dimensional channles.

1.2. Function spaces

In this section we introduce some function spaces. Hereafter we use the following symmetric rules.

Let X be a function space on the symmetric domain Ω. X^S is a set of all symmetric X functions with recpect to the x_2-axis. For a vector function $\boldsymbol{v}(x) = (v_1(x), v_2(x))$, $\boldsymbol{v}(x)$ is symmetric with recpect to the x_2-axis if and only if v_1 is an odd function with recpect to the x_2-axis and v_2 is an even function with respect to the x_2-axis, that is to say, v_1 and v_2 satisfy

$$-v_1(-x_1, x_2) = v_1(x_1, x_2) \quad ((x_1, x_2) \in \Omega),$$
$$v_2(-x_1, x_2) = v_2(x_1, x_2) \quad ((x_1, x_2) \in \Omega).$$

$\mathbb{C}_0^\infty(\Omega)$ is the set of all smooth functions with compact support contained in Ω. $\mathbb{C}_{0,\sigma}^\infty(\Omega)$ is all $\mathbb{C}_0^\infty(\Omega)$ function $\boldsymbol{\varphi}$ with $\operatorname{div} \boldsymbol{\varphi} = 0$ in Ω. $\mathcal{H}(\Omega)$ is the closure of $\mathbb{C}_{0,\sigma}^\infty(\Omega)$ for the usual $\mathbb{L}^2(\Omega)$ norm. The $\mathbb{L}^2$ inner product and norm on Ω are denoted as $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_{2,\Omega}$ respectively. We often omite Ω. $\mathbb{H}_0^1(\Omega)$ and $\mathcal{V}(\Omega)$ are the closure of $\mathbb{C}_0^\infty(\Omega)$ and $\mathbb{C}_{0,\sigma}^\infty(\Omega)$ for the usual Dirichlet norm $\|\nabla\cdot\|_{2,\Omega}$ respectively. $\mathbb{H}_0^1(\Omega)$ and $\mathcal{V}(\Omega)$ are the Hilbert spaces with respect to the inner product $((\boldsymbol{u}, \boldsymbol{v}))_\Omega = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_\Omega$. We often omite Ω. $\mathbb{H}_\sigma^1(\Omega)$ is all $\mathbb{H}^1(\Omega)$ functions $\boldsymbol{u}$ with $\operatorname{div} \boldsymbol{u} = 0$ in Ω. Let $(\mathcal{H}(\Omega))'$, $(\mathcal{V}(\Omega))'$, $(\mathcal{H}^S(\Omega))'$ and $(\mathcal{V}^S(\Omega))'$ be the dual spaces of $\mathcal{H}(\Omega)$, $\mathcal{V}(\Omega)$, $\mathcal{H}^S(\Omega)$ and $\mathcal{V}^S(\Omega)$ respectively.

Let $\gamma \in \mathcal{L}(\mathbb{H}^1(\Omega), \mathbb{L}^2(\partial\Omega))$ be the trace operator. The space $\mathbb{H}^{\frac{1}{2}}(\partial\Omega)$

denotes $\gamma(\mathbb{H}^1(\Omega))$. $\mathbb{H}^{-\frac{1}{2}}(\partial\Omega)$ is the dual space of $\mathbb{H}^{\frac{1}{2}}(\partial\Omega)$.

Let X be a Banach space and X' be the dual space of X. The spaces $C([0,T];X)$, $C^1([0,T];X)$, $L^2((0,T);X)$ and $L^\infty((0,T);X)$ are the usual Banach spaces. If $\boldsymbol{u}$ belongs to $C_\pi([0,T];X)$, $\boldsymbol{u} \in C([0,T];X)$ satisfies the time periodic condition $\boldsymbol{u}(0) = \boldsymbol{u}(T)$ in X. $C^1_\pi([0,T];X)$ is similar to $C_\pi([0,T];X)$. If $\boldsymbol{u}$ belongs to $H^1((0,T);X)$, $\boldsymbol{u}$ belongs to $L^2((0,T);X)$ and its weak derivative $\boldsymbol{u}'$ belongs to $L^2((0,T);X)$. $H^1_\pi((0,T);X)$ is the set of all $H^1((0,T);X)$ functions $\boldsymbol{\varphi}$ satisfying the time periodic condition $\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}(T)$ in X. Since it is well known that $C([0,T];X)$ contains $H^1((0,T);X)$, the time periodic condition in $H^1((0,T);X)$ is meaningful.

1.3. Definition of time periodic solutions

Our definition of a time periodic weak solution of the Navier-Stokes equations is as follows.

Definition 1.1 Suppose that a domain Ω satisfies Assumption 1.1.

Then a measurable function $\boldsymbol{u} = \boldsymbol{u}(t,x)$ on $(0,T)\times\Omega$ is called a weak solution of the Navier-Stokes equations (1.2), (1.3), (1.4), (1.7), (1.8) if and only if $\boldsymbol{u} - \boldsymbol{P}^\alpha \in L^2((0,T);\mathbb{H}^{1,S}_\sigma(\Omega)) \cap L^\infty((0,T);\mathbb{L}^{2,S}(\Omega))$ such that $\boldsymbol{u}$ satisfies

$$
\begin{aligned}
&-\int_0^T (\boldsymbol{u},\boldsymbol{\varphi})\psi' dt + \int_0^T \{\nu(\nabla\boldsymbol{u},\nabla\boldsymbol{\varphi}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{\varphi})\}\psi dt \\
&= \int_0^T {}_{(\mathcal{V}^S(\Omega))'}\langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_{\mathcal{V}^S(\Omega)}\psi dt \quad \big(\boldsymbol{\varphi}\in\mathcal{V}^S(\Omega),\ \psi\in C_0^\infty(0,T)\big) \qquad (1.13)
\end{aligned}
$$

and

$$
\boldsymbol{u} = \boldsymbol{\beta} \quad \text{on } (0,T)\times\partial\Omega \qquad (1.14)
$$

in the trace sence. Moreover the weak solution $\boldsymbol{u}$ is a time periodic solution if $\boldsymbol{u} - \boldsymbol{P}^\alpha$ belongs to $C_\pi([0,T];\mathbb{L}^{2,S}(\Omega))$.

We call $\boldsymbol{u}$ "a time periodic weak solution of the Navie-Stokes equations".

Hereafter $\langle\cdot,\cdot\rangle$ represent ${}_{(\mathcal{V}^S(\Omega))'}\langle\cdot,\cdot\rangle_{\mathcal{V}^S(\Omega)}$.

Remark 1.1 Definition 1.1 is meaningful even if Ω does not satisfies Assumption 1.1. In this case all symmetric conditions may be disregarded.

1.4. Results

Before stating our results in the channels ω_i we define a constant concerned with the Poiseuille velocity.

Definition 1.2 We set

$$\sigma_i(\alpha) = \sup_{\boldsymbol{\varphi} \in \mathcal{V}(\omega^i)} \frac{((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{P}_i^\alpha)_{\omega^i}}{\|\nabla \boldsymbol{\varphi}\|_{2,\omega^i}^2} \qquad (i = 1, 2), \tag{1.15}$$

$$\sigma_i^S(\alpha) = \sup_{\boldsymbol{\varphi} \in \mathcal{V}^S(\omega^i)} \frac{((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{P}_i^\alpha)_{\omega^i}}{\|\nabla \boldsymbol{\varphi}\|_{2,\omega^i}^2} \qquad (i = 1, 2). \tag{1.16}$$

Remark 1.2 The $\mathbb{L}^2$ inner products and norms in (1.15) and (1.16) are denoted in the channels ω^i, that is to say, the constant $\sigma_i(\alpha)$ and $\sigma_i^S(\alpha)$ do not depend on Ω. In the paper of C. J. Amick [3], the constant $\sigma_i(\alpha)$ and $\sigma_i^S(\alpha)$ are defined in two and three dimensional channels.

The constants $\sigma_i(\alpha)$ and $\sigma_i^S(\alpha)$ have the following properties.

Proposition 1.1 (C. J. Amick [3, Remarks, p. 494 and p. 499]) *The equalities*

$$\sigma_1(\alpha) = \sigma_2(\alpha),$$
$$\sigma_1^S(\alpha) = \sigma_2^S(\alpha)$$

hold true.

As for the proof of Proposition 1.1, see H. Morimoto and H. Fujita [19]. Hereafter we set

$$\sigma(\alpha) = \sigma_1(\alpha),$$
$$\sigma^S(\alpha) = \sigma_1^S(\alpha).$$

Futhermore we obtain the following result.

Proposition 1.2 (H. Morimoto and H. Fujita [19, Lemma 2 and Lemma 4]) *The following inequalities*

$$\sigma(\alpha) \geq \sigma^S(\alpha) > 0$$

and equalities

$$\sigma(\alpha) = |\alpha|\sigma(1),$$
$$\sigma^S(\alpha) = |\alpha|\sigma^S(1)$$

hold true.

Remark 1.3 The explicit value of the constants $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$ are shown in C. J. Amick [4].

But in the three dimensional channels Proposition 1.1 does not hold true. For the detail, see T-P. Kobayashi [15].

Our main reslut is as follows.

Theorem 1.1 *Suppose that a domain* Ω *satisfies Assumption* 1.1*,* $\boldsymbol{\beta} \in H^1_\pi((0,T);\mathbb{H}^{\frac{1}{2},S}(\partial\Omega))$ *satisfies* (GOC) *and vanishes on* $\Gamma_0^+ \cup \Gamma_0^-$*,* $\boldsymbol{f}$ *belongs to* $L^2((0,T);(\mathcal{V}^S(\Omega))')$ *and* $\sigma^S(\alpha) < \nu$*.*

Then, there exists at least one time periodic weak solution $\boldsymbol{u}$ *of the Navier-Stokes equation* (1.2)*,* (1.3)*,* (1.4)*,* (1.7)*,* (1.8)*.*

Remark 1.4 Supposing that $\sigma(\alpha) < \nu$, T-P. Kobayashi [15] proved that in two and three dimensional channels there exists time periodic weak solutions of the Navier-Stokes equations with the homogeneous boundary condition. It is not necessary that the boundary condition $\boldsymbol{\beta}$ vanishes on $\Gamma_0^+ \cup \Gamma_0^-$. We suppose that the support of $\boldsymbol{\beta}$ is compact and

$$\int_{\Gamma_0^+} \boldsymbol{\beta}\cdot\boldsymbol{n}d\sigma = \int_{\Gamma_0^-} \boldsymbol{\beta}\cdot\boldsymbol{n}d\sigma = 0 \quad \text{on } [0,T].$$

1.5. Results for stationary solutions

Supposing that $\sigma(\alpha) < \nu$, then C. J. Amick [3] proved that in two and three dimensional channels there exists a weak solution of the stationary Navier-Stokes equations with the homogeneous boundary condition. Supposing that $\sigma^S(\alpha) < \nu$, in a two dimensional symmetric semi infinite channel H. Morimoto and H. Fujita [19] proved that there exists a symmetric weak solution of the stationary Navier-Stokes equations with a cetain symmetric inhomogeneous boundary condition satisfying (GOC). H. Morimoto [20] demonstrated that if $\sigma^S(\alpha) < \nu$ holds and the smooth and symmetric boundary condition $\boldsymbol{\beta}$ satisfies (GOC) and vanishes on $\Gamma_0^+ \cup \Gamma_0^-$, there exists a symmetric weak solution of the stationary Navier-Stokes equations.

Remark 1.5 For the results in this paper, we use the similar conditions to H. Morimoto [20] for the flux condition and the boundary condition.

1.6. Leray's Inequality

If Ω satisfies Assumption 1.1, the following Proposition 1.3 holds true.

Proposition 1.3 *Suppose that a domain Ω satisfies Assumption 1.1 and $\boldsymbol{\beta} \in H^1_\pi((0,T);\mathbb{H}^{\frac{1}{2},S}(\partial\Omega))$ satisfies (GOC) and vanishes on $\Gamma_0^+ \cup \Gamma_0^-$.*

Then for any $\varepsilon > 0$ there exists an extension $\boldsymbol{b}_\varepsilon \in H^1_\pi((0,T);\mathbb{H}^{1,S}_\sigma(\Omega))$ of $\boldsymbol{\beta}$ such that $\boldsymbol{b}_\varepsilon$ has a compact support and the inequality

$$\big|((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{b}_\varepsilon(t))\big| < \varepsilon\|\nabla \boldsymbol{v}\|_2^2 \quad (\boldsymbol{v} \in \mathcal{V}^S(\Omega), t \in [0,T]) \tag{1.17}$$

holds true.

Proposition 1.3 is the time periodic style of Lemma 4.3 of H. Morimoto [20]. The main course of the proof of Proposition 1.3 is similar to Theorem 1 of H. Fujita [9]. For the outline of the proof, see Section 2. We call the estimate (1.17) "Leray's Inequality". The condition "$\boldsymbol{\beta}$ vanishes on $\Gamma_0^+ \cup \Gamma_0^-$" is strong for "Leray's Inequality". If we suppose that the support of $\boldsymbol{\beta}$ is compact and

$$\int_{\Gamma_0^+} \boldsymbol{\beta}\cdot\boldsymbol{n}d\sigma = \int_{\Gamma_0^-} \boldsymbol{\beta}\cdot\boldsymbol{n}d\sigma = 0 \quad \text{on } [0,T],$$

we obtain "Leray's Inequality" (1.17).

Remark 1.6 "Leray's Inequality" need not hold true for a given domain and a given function satisfying (GOC). See A. Takeshita [22]. If the boundary data satisfy (SOC), then we obtain an extension of the boundary data satisfying "Leray's Inequality". For example, see R. Finn [7], H. Fujita [8].

1.7. Lemma

Lemma 1.1 (the Poincaré inequality) *Suppose a domain Ω satisfies Assumption 1.1.*

Then there exists a constant $C(\Omega)$ depending on Ω such that inequality

$$\|\boldsymbol{u}\|_2 \le C(\Omega)\|\nabla \boldsymbol{u}\|_2 \quad (\boldsymbol{u} \in \mathbb{H}^1_0(\Omega))$$

holds true.

Lemma 1.2 (R. Temam [24]) *The following inequalities*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\mathbb{L}^4(\Omega)}^2 \le 2^{\frac{1}{2}}\|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}\|_2 && (\boldsymbol{u}\in\mathbb{H}_0^1(\Omega)),\\ &\big|((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w})\big| \le C\|\nabla\boldsymbol{u}\|_2\|\nabla\boldsymbol{v}\|_2\|\nabla\boldsymbol{w}\|_2 && (\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w}\in\mathbb{H}_0^1(\Omega)) \end{aligned} \tag{1.18}$$

and equalities

$$\begin{aligned} &((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}) = -((\boldsymbol{u}\cdot\nabla)\boldsymbol{w},\boldsymbol{v}) && \big(\boldsymbol{u}\in\mathbb{H}_\sigma^1(\Omega),\ \boldsymbol{v},\ \boldsymbol{w}\in\mathbb{H}_0^1(\Omega)\big),\\ &((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{v}) = 0 && \big(\boldsymbol{u}\in\mathbb{H}_\sigma^1(\Omega),\ \boldsymbol{v}\in\mathbb{H}_0^1(\Omega)\big) \end{aligned}$$

hold true, where the constant $C = 2^{\frac{1}{2}}C(\Omega)$.

Lemma 1.3 (K. Masuda [18, Lemma 2.2]) *Let* X_0 *be a dense subset of a Banach space* X *and* $\boldsymbol{\varphi}$ *belongs to* $L^2((0,T);X)$.

Then for all $\varepsilon > 0$ *there exists finite sequences* $\{\boldsymbol{\psi}_j\}\subset X_0$ *and* $\{q_j\}\subset C[0,T]$ *such that*

$$\Big\|\boldsymbol{\varphi} - \sum_{j=1}^{L} q_j\boldsymbol{\psi}_j\Big\|_{L^2((0,T);X)} < \varepsilon$$

holds true.

Lemma 1.4 (K. Masuda [18]) *Suppose a domain* Ω *satisfies Assumption* 1.1.

Then for any $\varepsilon > 0$ *and* $\boldsymbol{w}_3 \in C([0,T];\mathbb{L}^{2,S}(\Omega))$, *there exists a constant* M, *an integer* N *and functions* $\boldsymbol{\psi}_j\in\mathbb{L}^{2,S}(\Omega)$ $(j=1,\dots,N)$ *such that the inequality*

$$\begin{aligned} &\int_0^T \big|((\boldsymbol{w}_1\cdot\nabla)\boldsymbol{w}_2,\boldsymbol{w}_3)\big|dt\\ &\quad\le \varepsilon\int_0^T \big(\|\nabla\boldsymbol{w}_1\|_2^2 + \|\nabla\boldsymbol{w}_2\|_2^2 + \|\boldsymbol{w}_1\|_2\|\nabla\boldsymbol{w}_2\|_2\big)dt\\ &\qquad + M\sum_{j=1}^{N}\int_0^T \big|(\boldsymbol{w}_1,\boldsymbol{\psi}_j)\big|^2 dt \quad \big(\boldsymbol{w}_1,\boldsymbol{w}_2\in L^2((0,T);\mathcal{V}^S(\Omega))\big) \end{aligned}$$

holds true.

This kind of inequality appears in K. Masuda [18, p. 632, Lemma 2.5]. This inequality is its two dimensional and symmetric version.

Lemma 1.5 (R. Finn [7]) *Let Ω be any domain in $\mathbb{R}^2$ and*

$$\rho(x) = \mathrm{dist}(x, \partial\Omega) \quad (x \in \Omega).$$

Then the inequality

$$\int_\Omega \left|\frac{\boldsymbol{v}}{\rho}\right|^2 dx \le C_H \int_\Omega |\nabla \boldsymbol{v}|^2 dx \quad \big(\boldsymbol{v} \in \mathbb{H}_0^1(\Omega)\big)$$

holds true.

This inequality is called Hardy's inequality.

We define a functional $\boldsymbol{r}$ from

$$\boldsymbol{\varphi} \in \mathcal{V}^S(\Omega) \mapsto \nu(\nabla \boldsymbol{P}^\alpha, \nabla \boldsymbol{\varphi}) + ((\boldsymbol{P}^\alpha \cdot \nabla)\boldsymbol{P}^\alpha, \boldsymbol{\varphi}), \tag{1.19}$$

where $\boldsymbol{P}^\alpha$ is defined in (1.9), (1.10), (1.11), (1.12).

Lemma 1.6 (C. J. Amick [3, pp. 490–491]) *The map $\boldsymbol{r}$ is a linear and continuous functional on $\mathcal{V}^S(\Omega)$.*

Therefore $\boldsymbol{r} \in (\mathcal{V}^S(\Omega))'$ satisfies

$${}_{(\mathcal{V}^S(\Omega))'}\langle \boldsymbol{r}, \boldsymbol{\varphi}\rangle_{\mathcal{V}^S(\Omega)} = \nu(\nabla \boldsymbol{P}^\alpha, \nabla \boldsymbol{\varphi}) + ((\boldsymbol{P}^\alpha \cdot \nabla)\boldsymbol{P}^\alpha, \boldsymbol{\varphi}) \quad (\boldsymbol{\varphi} \in \mathcal{V}^S(\Omega)).$$

2. Proof of Proposition 1.3

In this subsection we prove the following Lemmas for the proof of Proposition 1.3. These are stationary results except Lemma 2.6. In this subsection we suppose that a domain Ω satisfies Assumption 1.1. But even if a domain Ω does not satisfy Assumption 1.1 or Ω is a three dimensional domain, the following Lemmas hold true.

Lemma 2.1 *Suppose that a domain Ω satisfies Assumption* 1.1*, $\boldsymbol{\beta} \in \mathbb{H}^{\frac{1}{2}}(\partial\Omega)$ and the support of $\boldsymbol{\beta}$ is compact.*

Then there exists a bounded subdomain $\widetilde{\Omega} \subset \Omega$ such that the following *conditions hold true.*

(1) *The support of $\boldsymbol{\beta}$ is contained in $\partial\widetilde{\Omega}$ (the boundary of $\widetilde{\Omega}$).*

(2) *If we set*

$$\widetilde{\boldsymbol{\beta}} = \begin{cases} \boldsymbol{\beta} & \text{on } \partial\Omega \cap \partial\widetilde{\Omega} \\ \mathbf{0} & \text{on } \partial\widetilde{\Omega} \backslash \partial\Omega, \end{cases}$$

then

$$\|\widetilde{\boldsymbol{\beta}}\|_{\mathbb{H}^{\frac{1}{2}}(\partial\widetilde{\Omega})} = \|\boldsymbol{\beta}\|_{\mathbb{H}^{\frac{1}{2}}(\partial\Omega)}$$

holds true.

Lemma 2.2 *Suppose that a domain Ω satisfies Assumption* 1.1*, $\boldsymbol{\beta} \in \mathbb{H}^{\frac{1}{2}}(\partial\Omega)$ satisfies (GOC) and the support of $\boldsymbol{\beta}$ is compact.*

Then there exists one and only one $\boldsymbol{\psi} \in \mathbb{H}^1_\sigma(\Omega)$ such that the support of $\boldsymbol{\psi}$ is compact and

$$\boldsymbol{\psi} = \boldsymbol{\beta} \quad \text{on} \quad \partial\Omega,$$
$$\|\boldsymbol{\psi}\|_{\mathbb{H}^1(\Omega)} \leq C\|\boldsymbol{\beta}\|_{\mathbb{H}^{\frac{1}{2}}(\partial\Omega)}$$

hold true, where the constant C depends only on Ω and the support of $\boldsymbol{\beta}$.

Proof. We may use the bounded smooth domain $\widetilde{\Omega}$ and $\widetilde{\boldsymbol{\beta}}$ in Lemma 2.1. In the domain $\widetilde{\Omega}$ we consider the stationary Stokes equations

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \nabla p = \mathbf{0} \quad & \text{in } \widetilde{\Omega}, \\ \operatorname{div}\boldsymbol{u} = 0 \quad & \text{in } \widetilde{\Omega}, \\ \boldsymbol{u} = \widetilde{\boldsymbol{\beta}} \quad & \text{on } \partial\widetilde{\Omega}. \end{aligned}$$

See Galdi [10, Section IV.1, Theorem 1.1]. □

Lemma 2.3 *Suppose that a domain Ω satisfies Assumption* 1.1*, $\boldsymbol{\beta} \in \mathbb{H}^{\frac{1}{2}}(\partial\Omega)$ satisfies (GOC) and the support of $\boldsymbol{\beta}$ is compact. Let $\boldsymbol{\psi} \in \mathbb{H}^1_\sigma(\Omega)$ be the extension of $\boldsymbol{\beta}$ obtained in Lemma* 2.2*.*

Then for all $\boldsymbol{\xi} \in (\mathbb{H}^1_\sigma(\Omega))'$ there exists a $\boldsymbol{\zeta} \in \mathbb{H}^{-\frac{1}{2}}(\partial\Omega)$ such that

$$
{}_{\mathbb{H}^1_\sigma}\langle \boldsymbol{\psi}, \boldsymbol{\xi} \rangle_{(\mathbb{H}^1_\sigma)'} = {}_{\mathbb{H}^{\frac{1}{2}}}\langle \boldsymbol{\beta}, \boldsymbol{\zeta} \rangle_{\mathbb{H}^{-\frac{1}{2}}} \tag{2.1}
$$

holds true.

Lemma 2.4 *Suppose that a domain Ω satisfies Assumption* 1.1*, $\boldsymbol{\beta} \in \mathbb{H}^{\frac{1}{2}}(\partial\Omega)$ satisfies (SOC) and the support of $\boldsymbol{\beta}$ is compact. Let $\boldsymbol{\psi} \in \mathbb{H}^1_\sigma(\Omega)$ be the extension of $\boldsymbol{\beta}$ obtained in Lemma* 2.2*.*

Then there exists a $\varphi \in H^2(\Omega)$ such that the support of φ is compact and

$$
\mathrm{rot}\varphi = \boldsymbol{\psi} \quad \text{in } \Omega,
$$

$$
\|\varphi\|_{H^2(\Omega)} \le C_1 \|\boldsymbol{\psi}\|_{\mathbb{H}^1(\Omega)} \le C_2 \|\boldsymbol{\beta}\|_{\mathbb{H}^{\frac{1}{2}}(\partial\Omega)}
$$

hold true where the constant C_1 and C_2 depend on Ω and the support of $\boldsymbol{\beta}$.

For the proof of Lemma 2.4, see Galdi [10, Section VIII, Lemma 4.1].

Lemma 2.5 *Suppose that a domain Ω satisfies Assumption* 1.1*, $\boldsymbol{\beta} \in \mathbb{H}^{\frac{1}{2}}(\partial\Omega)$ satisfies (SOC) and the support of $\boldsymbol{\beta}$ is compact. Let $\varphi \in H^2(\Omega)$ be the extension of $\boldsymbol{\beta}$ obtained in Lemma* 2.4*.*

Then for all $u \in (H^2(\Omega))'$ there exists a $\boldsymbol{\Psi} \in \mathbb{H}^{-\frac{1}{2}}(\partial\Omega)$ such that

$$
{}_{H^2}\langle \varphi, u \rangle_{(H^2)'} = {}_{\mathbb{H}^{\frac{1}{2}}}\langle \boldsymbol{\beta}, \boldsymbol{\Psi} \rangle_{\mathbb{H}^{-\frac{1}{2}}} \tag{2.2}
$$

holds true.

Lemma 2.6 *Suppose that a domain Ω satisfies Assumption* 1.1*, $\boldsymbol{\beta} \in H^1_\pi((0,T); \mathbb{H}^{\frac{1}{2}}(\partial\Omega))$ satisfies (SOC) and the support of $\boldsymbol{\beta}$ is compact.*

Then for all $\varepsilon > 0$ there exists an extension $\boldsymbol{g}_\varepsilon \in H^1_\pi((0,T); \mathbb{H}^1_\sigma(\Omega))$ of $\boldsymbol{\beta}$ such that the support of $\boldsymbol{g}_\varepsilon$ is compact and

$$
\left|((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{g}_\varepsilon(t))\right| < \varepsilon \|\boldsymbol{v}\|_2^2 \quad (\boldsymbol{v} \in \mathcal{V}(\Omega), t \in [0,T]) \tag{2.3}
$$

hold true.

Remark 2.1 The inequality (2.3) is Leray's Inequality. It is the time periodic version. The proof is similar to H. Fujita [8].

Using Lemma 2.6, we can prove Proposition 1.3. Since the detail of the proof is similar to H. Fujita [9], we omit the proof.

3. Proof of Theorem 1.1

3.1. Time periodic solution in a bounded symmetric domain

In this section $\boldsymbol{b}_\varepsilon$ is the extension for Proposition 1.3. We suppose that $\{\Omega^n\}$ is a symmetric and bounded domain of Ω and satisfies $\Omega^n \subset \Omega^{n+1}$ and $\cup_{n\in\mathbb{N}}\Omega^n = \Omega$, where Ω^1 and $\partial\Omega^1$ (the boundary of Ω^1) containe the support of $\boldsymbol{b}_\varepsilon$ and $\boldsymbol{\beta}$ respectively. In the bounded domains Ω^n we consider a time periodic problem of the Navier-Stokes equations

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad & \text{in } (0,T)\times\Omega^n,\\
\operatorname{div}\boldsymbol{u} = 0 \qquad & \text{in } (0,T)\times\Omega^n,\\
\boldsymbol{u} = \boldsymbol{\beta}^n + \boldsymbol{P}^\alpha \qquad & \text{on } (0,T)\times\partial\Omega^n,\\
\boldsymbol{u}(0) = \boldsymbol{u}(T) \qquad & \text{in } \Omega^n,
\end{aligned}$$

where

$$\boldsymbol{\beta}^n = \begin{cases} \boldsymbol{\beta} & \text{on } (0,T)\times\partial\Omega\cap\partial\Omega^n \\ \boldsymbol{0} & \text{on } (0,T)\times\partial\Omega\backslash\partial\Omega^n. \end{cases}$$

Since the connected components of $\partial\Omega^n$ are symmetric and intersect the x_2-axis, $\boldsymbol{\beta}^n + \boldsymbol{P}^\alpha$ satisfies (GOC) on $\partial\Omega^n$ and is symmetric with respect to the x_2-axis, therefore there exists a $\boldsymbol{u}_n$ satisfying

$$\begin{aligned}
&\boldsymbol{u}_n \in L^2\big((0,T);\mathbb{H}^{1,S}_\sigma(\Omega^n)\big) \cap L^\infty\big((0,T);\mathbb{L}^{2,S}(\Omega^n)\big),\\
&\boldsymbol{u}_n' \in L^2((0,T);(\mathcal{V}^S(\Omega^n))') \quad \text{(weak derivative of } \boldsymbol{u}_n),\\
&\boldsymbol{u}_n \in C_\pi([0,T];\mathbb{L}^{2,S}(\Omega^n))
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}(\boldsymbol{u}_n,\boldsymbol{\varphi})_{\Omega^n} + \nu(\nabla\boldsymbol{u}_n,\nabla\boldsymbol{\varphi})_{\Omega^n} + ((\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n,\boldsymbol{\varphi})_{\Omega^n} &= \langle\boldsymbol{f},\boldsymbol{\varphi}\rangle_n \\
&\qquad (\boldsymbol{\varphi}\in\mathcal{V}^S(\Omega^n)),\\
\boldsymbol{u}_n = \boldsymbol{\beta}^n + \boldsymbol{P}^\alpha \quad (0,T)\times \text{on } \partial\Omega^n,&
\end{aligned}$$

where $\langle\cdot,\cdot\rangle_n$ denotes the duality pair of $(\mathcal{V}^S(\Omega^n))'$ and $\mathcal{V}^S(\Omega^n)$. For the proof, see T-P. Kobayashi [14, Theorem 1.1].

Remark 3.1 In the paper of T-P. Kobayashi [14], a given function $\boldsymbol{\beta}^n$ on $\partial\Omega^n$ belongs to $C^1_\pi([0,T];\mathbb{H}^{\frac{1}{2},S}(\partial\Omega^n))$. But we obtain "Leray's Inequality", even if a given function on $\partial\Omega^n$ belongs to $H^1_\pi((0,T);\mathbb{H}^{\frac{1}{2},S}(\partial\Omega^n))$. Therefore in the symmetric bounded domain Ω^n there exists the time periodic solution as above.

We set

$$\boldsymbol{v}_n = \begin{cases} \boldsymbol{u}_n - \boldsymbol{b}_\varepsilon - \boldsymbol{P}^\alpha & \text{in } \Omega_n \\ \boldsymbol{0} & \text{in } \Omega\backslash\Omega_n. \end{cases}$$

Then we obtain

$$\begin{aligned} &\frac{d}{dt}(\boldsymbol{v}_n,\boldsymbol{\varphi}) + \nu((\boldsymbol{v}_n,\boldsymbol{\varphi})) + ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi}) + ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi}) \\ &\quad + ((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi}) + ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{P}^\alpha,\boldsymbol{\varphi}) + ((\boldsymbol{P}^\alpha\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi}) \\ &= \langle \boldsymbol{F},\boldsymbol{\varphi}\rangle \quad (\boldsymbol{\varphi}\in\mathcal{V}^S(\Omega^n)), \end{aligned} \tag{3.1}$$

where $\boldsymbol{\varphi}\in\mathcal{V}^S(\Omega^n)$ is extended as a $\boldsymbol{0}$ function to the outside of Ω^n and

$$\langle \boldsymbol{F},\boldsymbol{\varphi}\rangle = \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle - (\boldsymbol{b}_{\varepsilon,t},\boldsymbol{\varphi}) - (\nabla\boldsymbol{b}_\varepsilon,\nabla\boldsymbol{\varphi}) - ((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi}) - \langle \boldsymbol{r},\boldsymbol{\varphi}\rangle \qquad (\boldsymbol{\varphi}\in\mathcal{V}^S(\Omega)).$$

Now we obtain $\boldsymbol{F}\in L^2((0,T);(\mathcal{V}^S(\Omega))')$ because the estimate

$$|\langle \boldsymbol{F},\boldsymbol{\varphi}\rangle| \le \big(\|\boldsymbol{f}\|_{(\mathcal{V}^S)'} + \|\boldsymbol{b}_{t,\varepsilon}\|_{\mathbb{H}^1} + \|\boldsymbol{b}_\varepsilon\|_{\mathbb{H}^1} + C_s\|\boldsymbol{b}_\varepsilon\|^2_{\mathbb{H}^1} + \|\boldsymbol{r}\|_{(\mathcal{V}^S)'}\big)\|\nabla\boldsymbol{\varphi}\|_2 \qquad (\boldsymbol{\varphi}\in\mathcal{V}^S(\Omega))$$

holds true and $\boldsymbol{b}_\varepsilon\in C([0,T];\mathbb{H}^{1,S}_\sigma(\Omega))$, where C_s is the constant of Sobolev's Imbedding Theorem $\mathbb{H}^1(\Omega)\hookrightarrow\mathbb{L}^4(\Omega)$.

We will prove that $\|\boldsymbol{v}_n(0)\|_2$ is a bounded sequence with respect to n. We use the above two Propositions.

Proposition 3.1 (H. Morimoto [20, Lemma 4.4]) *Suppose that* $\theta\in C^\infty(\mathbb{R})$ *satisfies* $0\le\theta\le 1$, $\theta(t)=1$ $(t>1)$ *and* $\theta=0$ $(t\le 0)$. *For all* $\delta>0$, *we set*

$$\theta_\delta(x) = \begin{cases} \theta(\delta(x_2 - H - 1)) & (x \in \omega_H^1) \\ \theta(-\delta(x_2 + H + 1)) & (x' \in \omega_{-H}^2) \\ 0 & \text{otherwise.} \end{cases}$$

Then for all $\varepsilon > 0$, there exists an $\boldsymbol{s} \in \mathbb{C}_{0,\sigma}^{\infty,S}(\Omega)$ and a constant C_0 such that

$$((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{P}^\alpha) \le ((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{s}) + \big((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{P}^\alpha\theta_\delta^2\big)_{\omega_H^1\cup\omega_{-H}^2} + (\varepsilon + c_0\delta)\|\nabla\boldsymbol{\varphi}\|_2^2 \quad (\boldsymbol{\varphi} \in \mathcal{V}^S(\Omega)) \tag{3.2}$$

holds true.

Proposition 3.1 is the special case of Lemma 4.4 in H. Morimoto [20]. We may assume that the support of $\boldsymbol{s} \in \mathbb{C}_{0,\sigma}^{\infty,S}(\Omega)$ is contained in Ω^1.

Proposition 3.2 (C. J. Amick [3, Theorem 4.3 and Corollary 4.4])

$$\lim_{\delta\to 0}\sup_{\boldsymbol{\varphi}\in\mathcal{V}^S(\Omega)} \frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{P}^\alpha\theta_\delta^2)_{\omega_H^1\cup\omega_{-H}^2}}{\|\nabla\boldsymbol{\varphi}\|_{2,\Omega}^2} = \sigma^S(\alpha) \tag{3.3}$$

holds true.

Now we set $\boldsymbol{\varphi} = \boldsymbol{v}_n$ in the Equation (3.1). We obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + \nu\|\nabla\boldsymbol{v}_n\|_2^2 + ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{b}_\varepsilon, \boldsymbol{v}_n) = ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n, \boldsymbol{P}^\alpha) + \langle \boldsymbol{F}, \boldsymbol{v}_n\rangle. \tag{3.4}$$

Using Proposition 3.1 for the first term of the right hand side of (3.4), then we have

$$((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n, \boldsymbol{P}^\alpha) \le ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n, \boldsymbol{s}) + \big((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n, \boldsymbol{P}^\alpha\theta_\delta^2\big)_{\omega_H^1\cup\omega_{-H}^2} + (\varepsilon + C_0\delta)\|\nabla\boldsymbol{v}_n\|_2^2$$

We obtain

$$\big((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n, \boldsymbol{P}^\alpha\theta_\delta^2\big)_{\omega_H^1\cup\omega_{-H}^2} \le (\sigma^S(\alpha) + \varepsilon)\|\nabla\boldsymbol{v}_n\|_2^2$$

for enough small $\delta > 0$ by Proposition 3.2. Using Leray's inequality for the third term of the left hand side of (3.4), we have

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + (\nu - \sigma^S(\alpha) - 4\varepsilon - C_0\delta)\|\nabla \boldsymbol{v}_n\|_2^2 \\ \leq ((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{v}_n, \boldsymbol{s}) + C\|\boldsymbol{F}\|_{(\mathcal{V}^S)'}^2,$$

where we choose ε and δ such that $\nu - \sigma^S(\alpha) - 5\varepsilon - C_0\delta$ is greater than 0. It is easy to follow from (3.1) that

$$\begin{aligned} &((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{v}_n, \boldsymbol{s}) \\ &\quad = -\frac{d}{dt}(\boldsymbol{v}_n, \boldsymbol{s}) - \nu((\boldsymbol{v}_n, \boldsymbol{s})) - ((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{b}_\varepsilon, \boldsymbol{s}) - ((\boldsymbol{b}_\varepsilon \cdot \nabla)\boldsymbol{v}_n, \boldsymbol{s}) \\ &\qquad - ((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{P}^\alpha, \boldsymbol{s}) - ((\boldsymbol{P}^\alpha \cdot \nabla)\boldsymbol{v}_n, \boldsymbol{s}) + \langle \boldsymbol{F}, \boldsymbol{s}\rangle \\ &\quad \leq -\frac{d}{dt}(\boldsymbol{v}_n, \boldsymbol{s}) + \varepsilon\|\nabla \boldsymbol{v}_n\|_2^2 + C\big(\|\nabla \boldsymbol{s}\|_2^2 + C_s\|\nabla \boldsymbol{s}\|_2^2\|\boldsymbol{b}_\varepsilon\|_{\mathbb{H}^1}^2 + \|\boldsymbol{F}\|_{(\mathcal{V}^S)'}^2\big), \end{aligned}$$

where the constant C is depndent of $\boldsymbol{P}^\alpha$, ε and the Poincaré inequality and C_s is the constant of Sobolev's Imbedding Theorem $\mathbb{H}^1(\Omega) \hookrightarrow \mathbb{L}^4(\Omega)$. We set

$$K_1(t) = C\big(\|\nabla \boldsymbol{s}\|_2^2 + C_s\|\nabla \boldsymbol{s}\|_2^2\|\boldsymbol{b}_\varepsilon\|_{\mathbb{H}^1}^2 + \|\boldsymbol{F}\|_{(\mathcal{V}^S)'}^2\big).$$

Using the Poincaré inequality, we have

$$\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + \mu\|\boldsymbol{v}_n\|_2^2 \leq -2\frac{d}{dt}(\boldsymbol{v}_n, \boldsymbol{s}) + 2K_1(t),$$

where

$$\mu = 2\frac{\nu - \sigma^S(\alpha) - 5\varepsilon - C_0\delta}{C(\Omega)^2}.$$

For $\xi > 0$ (smaller than μ) multiplying by $e^{(\mu-\xi)t}$, then we obtain

$$\begin{aligned}
&e^{(\mu-\xi)t}\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + \mu e^{(\mu-\xi)t}\|\boldsymbol{v}_n\|_2^2 \\
&\quad \le -2e^{(\mu-\xi)t}\frac{d}{dt}(\boldsymbol{v}_n, \boldsymbol{s}) + 2K_1(t)e^{(\mu-\xi)t} \\
&\quad = -2\frac{d}{dt}\{(\boldsymbol{v}_n, \boldsymbol{s})e^{(\mu-\xi)t}\} + 2(\mu-\xi)e^{(\mu-\xi)t}(\boldsymbol{v}_n, \boldsymbol{s}) + 2K_1(t)e^{(\mu-\xi)t} \\
&\quad \le -2\frac{d}{dt}\{(\boldsymbol{v}_n, \boldsymbol{s})e^{(\mu-\xi)t}\} + \xi e^{(\mu-\xi)t}\|\boldsymbol{v}_n\|_2^2 + \big(C\|\boldsymbol{s}\|_2^2 + 2K_1(t)\big)e^{(\mu-\xi)t},
\end{aligned} \tag{3.5}$$

where the constant C depends only on μ, ξ and ε. If we set

$$K_2(t) = \big(C\|\boldsymbol{s}\|_2^2 + 2K_1(t)\big)e^{(\mu-\xi)t},$$

it follows from (3.5) that

$$\frac{d}{dt}\big(e^{(\mu-\xi)t}\|\boldsymbol{v}_n\|_2^2\big) \le -2\frac{d}{dt}\{(\boldsymbol{v}_n, \boldsymbol{s})e^{(\mu-\xi)t}\} + K_2(t). \tag{3.6}$$

Integrating (3.6) on $[0, T]$, then we have

$$\|\boldsymbol{v}_n(T)\|_2^2 e^{(\mu-\xi)T} \le \|\boldsymbol{v}_n(0)\|_2^2 - 2(\boldsymbol{v}_n(T), \boldsymbol{s})e^{(\mu-\xi)T} + 2(\boldsymbol{v}_n(0), \boldsymbol{s}) + K,$$

where

$$K = \int_0^T K_2(t)dt.$$

Since $\boldsymbol{v}_n$ is time periodic in $\mathbb{L}^2(\Omega)$, for all $\eta > 0$ the inequality

$$\begin{aligned}
\|\boldsymbol{v}_n(0)\|_2^2 e^{(\mu-\xi)T} \le \|\boldsymbol{v}_n(0)\|_2^2 &+ \big(\eta\|\boldsymbol{v}_n(0)\|_2^2 + C\|\boldsymbol{s}\|_2^2\big)e^{(\mu-\xi)T} \\
&+ \eta\|\boldsymbol{v}_n(0)\|_2^2 + C\|\boldsymbol{s}\|_2^2 + K
\end{aligned}$$

holds true, where the constant C is dependent of η. We set

$$\begin{aligned}
&H = Ke^{-(\mu-\xi)T} + C\|\boldsymbol{s}\|_2^2(e^{-(\mu-\xi)T} + 1), \\
&\gamma = 1 - \eta - (1+\eta)e^{-(\mu-\xi)T}.
\end{aligned}$$

We choose $\eta > 0$ such that γ is greater than 0. Then

$$\|\boldsymbol{v}_n(0)\|_2^2 \leq \frac{H}{\gamma} := M_1 \tag{3.7}$$

holds true. Consequetly we see that $\{\boldsymbol{v}_n(0)\}$ is a bounded sequence in $\mathbb{L}^2(\Omega)$ with respect to n.

3.2. Weak limit

Using Leray's inequality for the equation (3.4), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_n\|_2^2 + (\nu - 2\varepsilon)\|\nabla \boldsymbol{v}_n\|_2^2 \leq \left|((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{P}^\alpha, \boldsymbol{v}_n)\right| + C\|\boldsymbol{F}\|^2_{(\mathcal{V}^S)'}, \tag{3.8}$$

where the constant C depends only on ε. The inequality

$$\left|((\boldsymbol{v}_n \cdot \nabla)\boldsymbol{P}^\alpha, \boldsymbol{v}_n)\right| \leq C(P^\alpha)\|\boldsymbol{v}_n\|_2^2 \tag{3.9}$$

holds true, where the constant $C(P^\alpha)$ depends only on the extended Poiseuille velocity $\boldsymbol{P}^\alpha$. Using (3.9), the Gronwall inequality and integrating (3.8) from 0 to t ($\leq T$), then we obtain

$$\|\boldsymbol{v}_n(t)\|_2^2 \leq M_1 e^{C(P^\alpha)T} + C\int_0^T e^{C(P^\alpha)t}\|\boldsymbol{F}\|^2_{(\mathcal{V}^S)'}dt =: M_2. \tag{3.10}$$

Integrating (3.8) on $[0, T]$, we see that

$$\int_0^T \|\nabla \boldsymbol{v}_n\|_2^2 dt \leq \frac{1}{(\nu - 2\varepsilon)}\left(C(P^\alpha)TM_2 + C\int_0^T \|\boldsymbol{F}\|^2_{(\mathcal{V}^S)'}dt\right) =: M_3 \tag{3.11}$$

holds true.

For any $\boldsymbol{\varphi} \in \mathbb{C}^{\infty,S}_{0,\sigma}(\Omega)$, there exists an $N \in \mathbb{N}$ such that Ω_N contains the support of $\boldsymbol{\varphi}$. We will prove that $\{(\boldsymbol{v}_n(t), \boldsymbol{\varphi})\}_{n\geq N}$ is uniformly bounded and equicontinuous on $[0, T]$ with respect to n. A calculations

$$|(\boldsymbol{v}_n(t), \boldsymbol{\varphi})| \leq \|\boldsymbol{v}_n(t)\|_2\|\boldsymbol{\varphi}\|_2 \leq M_2\|\boldsymbol{\varphi}\|_2$$

and

$$
\begin{aligned}
&|(\boldsymbol{v}_n(t),\boldsymbol{\varphi})-(\boldsymbol{v}_n(s),\boldsymbol{\varphi})| \\
&\quad=\left|\int_s^t \frac{d}{d\tau}(\boldsymbol{v}_n(\tau),\boldsymbol{\varphi})d\tau\right| \\
&\quad\le \int_s^t \nu|((\boldsymbol{v}_n,\boldsymbol{\varphi}))|+\big|((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi})\big|+\big|((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi})\big|+\big|((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi})\big| \\
&\qquad+\big|((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{P}^\alpha,\boldsymbol{\varphi})\big|+\big|((\boldsymbol{P}^\alpha\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi})\big|+|\langle\boldsymbol{F},\boldsymbol{\varphi}\rangle|d\tau \\
&\quad\le \int_s^t \big(\nu\|\nabla\boldsymbol{v}_n\|_2+2^{\frac{1}{2}}\|\boldsymbol{v}_n\|_2\|\nabla\boldsymbol{v}_n\|_2+2C(\Omega)C_s\|\nabla\boldsymbol{v}_n\|_2\|\boldsymbol{b}_\varepsilon\|_{\mathbb{H}^1} \\
&\qquad+2C(P^\alpha)C(\Omega)\|\nabla\boldsymbol{v}_n\|_2+\|\boldsymbol{F}\|_{(\mathcal{V}^S)'}\big)\|\nabla\boldsymbol{\varphi}\|_2 d\tau \\
&\quad\le M_4|t-s|^{\frac{1}{2}}\|\nabla\boldsymbol{\varphi}\|_2
\end{aligned}
$$

yield, where the constant C_s depends only on Sobolev's Imbedding Theorem $\mathbb{H}^1(\Omega)\hookrightarrow\mathbb{L}^4(\Omega)$ and the constant M_4 does not depend on n.

Since the time periodic solution $\{\boldsymbol{v}_n\}$ is a bounded sequence in $L^2((0,T);\mathcal{V}^S(\Omega))\cap L^\infty((0,T);\mathcal{H}^S(\Omega))$, therefore there exists a subsequence $\{\boldsymbol{v}_{nk}\}_k$ of $\{\boldsymbol{v}_n\}$ and an element $\boldsymbol{v}$ of $L^2((0,T);\mathcal{V}^S(\Omega))\cap L^\infty((0,T);\mathcal{H}^S(\Omega))$ such that

$$
\boldsymbol{v}_{nk}\to\boldsymbol{v}\quad\text{in}\quad\begin{cases}L^\infty((0,T);\mathcal{H}^S(\Omega)) & \text{weak star}\\ L^2((0,T);\mathcal{V}^S(\Omega)) & \text{weakly}\end{cases}\qquad(k\to\infty)\tag{3.12}
$$

holds true. For any $\boldsymbol{\varphi}\in\mathbb{C}_{0,\sigma}^{\infty,S}(\Omega)$, there exists a subsequence $\{\boldsymbol{v}_{nki}\}$ of $\{\boldsymbol{v}_{nk}\}$ such that

$$
\lim_{i\to\infty}(\boldsymbol{v}_{nki},\boldsymbol{\varphi})=(\boldsymbol{v},\boldsymbol{\varphi})\tag{3.13}
$$

holds true using *the Ascoli-Arzelà* Theorem. We will prove the convergence (3.13) for any $\boldsymbol{\varphi}\in\mathbb{L}^{2,S}(\Omega)$. Since we know $\mathbb{L}^{2,S}(\Omega)=\mathcal{H}^S(\Omega)\oplus(\mathcal{H}^S(\Omega))^\perp$, we have $\boldsymbol{\varphi}=\boldsymbol{\varphi}_\sigma+\boldsymbol{\varphi}_p$ ($\boldsymbol{\varphi}_\sigma\in\mathcal{H}^S(\Omega),\boldsymbol{\varphi}_p\in(\mathcal{H}^S(\Omega))^\perp$). Since $\mathbb{C}_{0,\sigma}^{\infty,S}(\Omega)$ is dence in $\mathcal{H}^S(\Omega)$, for any $\delta>0$ there exists a $\boldsymbol{\varphi}_\sigma^\delta\in\mathbb{C}_{0,\sigma}^{\infty,S}(\Omega)$ such that

$$
\left\|\boldsymbol{\varphi}_\sigma^\delta-\boldsymbol{\varphi}_\sigma\right\|_2<\delta
$$

holds true. We have

$$\begin{aligned}|(\boldsymbol{v}-\boldsymbol{v}_n,\boldsymbol{\varphi})| &\le \left|(\boldsymbol{v}-\boldsymbol{v}_n,\boldsymbol{\varphi}_\sigma-\boldsymbol{\varphi}_\sigma^\delta)\right| + \left|(\boldsymbol{v}-\boldsymbol{v}_n,\boldsymbol{\varphi}_\sigma^\delta)\right| \\ &\le 2M_2\delta + \left|(\boldsymbol{v}-\boldsymbol{v}_n,\boldsymbol{\varphi}_\sigma^\delta)\right| \end{aligned} \tag{3.14}$$

because $\boldsymbol{v}_n$ is bounded in $L^\infty((0,T);\mathcal{H}^S(\Omega))$. We can choose a subsequence $\{\boldsymbol{v}_{nk}\}_k$ of $\{\boldsymbol{v}_n\}_n$ such that the second term of the right hand side of (3.14) goes to 0. Therefore for any $\boldsymbol{\varphi} \in \mathbb{L}^{2,S}(\Omega)$ there exists a subsequence $\{\boldsymbol{v}_{nk}\}$ such that $(\boldsymbol{v}_{nk},\boldsymbol{\varphi})$ converges to $(\boldsymbol{v},\boldsymbol{\varphi})$ uniformly on $[0,T]$.

3.3. Time periodic solution

We mulitiply (3.1) by $\psi \in C_0^\infty(0,T)$ and integrate on $[0,T]$. For any $\boldsymbol{\varphi} \in \mathbb{C}_{0,\sigma}^{\infty,S}(\Omega)$, there exists an $N \in \mathbb{N}$ (choosing an N such that Ω_N contains the support of $\boldsymbol{\varphi}$) such that for all $n \ge N$

$$\begin{aligned} -\int_0^T (\boldsymbol{v}_n,\boldsymbol{\varphi})\psi' + \big\{\nu((\boldsymbol{v}_n,\boldsymbol{\varphi})) + ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi}) + ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi}) \\ + ((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi}) + ((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{P}^\alpha),\boldsymbol{\varphi}) + ((\boldsymbol{P}^\alpha\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi})\big\}\psi dt \\ = \int_0^T \langle \boldsymbol{F},\boldsymbol{\varphi}\rangle \psi dt \end{aligned} \tag{3.15}$$

holds true. We can choose a subsequence $\{\boldsymbol{v}_{nk}\}_k$ such that the left hand side of (3.15) except the nonliner term converges to

$$\begin{aligned} -\int_0^T (\boldsymbol{v},\boldsymbol{\varphi})\psi' + \big\{\nu((\boldsymbol{v},\boldsymbol{\varphi})) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi}) \\ + ((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi}) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{P}^\alpha),\boldsymbol{\varphi}) + ((\boldsymbol{P}^\alpha\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi})\big\}\psi dt \end{aligned}$$

from (3.12). We prove that there exists a subsequence $\{\boldsymbol{v}_{nki}\}$ such that

$$\int_0^T ((\boldsymbol{v}_{nki}\cdot\nabla)\boldsymbol{v}_{nki},\boldsymbol{\varphi})\psi dt \to \int_0^T ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi})\psi dt \quad (i\to\infty) \tag{3.16}$$

holds true. We have

$$\int_0^T ((\boldsymbol{v}_{nk}\cdot\nabla)\boldsymbol{v}_{nk},\boldsymbol{\varphi})\psi dt - \int_0^T ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi})\psi dt$$
$$= \int_0^T ((\boldsymbol{v}_{nk}-\boldsymbol{v})\cdot\nabla\boldsymbol{v}_{nk},\boldsymbol{\varphi})\psi dt - \int_0^T (\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi},\boldsymbol{v}_{nk}-\boldsymbol{v})\psi dt. \tag{3.17}$$

Firstly, we consider the first term of the right hand side of (3.17). By Lemma 1.4, for any $\delta > 0$ there exists a constant M, an integer N and $\boldsymbol{\psi}_l \in \mathbb{L}^{2,S}(\Omega)$ $(l = 1,\dots,N)$ such that

$$\left|\int_0^T ((\boldsymbol{v}_{nk}-\boldsymbol{v})\cdot\nabla\boldsymbol{v}_{nk},\psi\boldsymbol{\varphi})dt\right|$$
$$\le \delta\int_0^T \big(\|\nabla\boldsymbol{v}_{nk}-\nabla\boldsymbol{v}\|_2^2 + \|\nabla\boldsymbol{v}_{nk}\|_2^2 + \|\boldsymbol{v}_{nk}-\boldsymbol{v}\|_2\|\nabla\boldsymbol{v}_{nk}\|_2\big)dt$$
$$+ M\sum_{l=1}^N \int_0^T |(\boldsymbol{v}_{nk}-\boldsymbol{v},\boldsymbol{\psi}_l)|^2 dt. \tag{3.18}$$

holds. Since it follows that the time periodic solution $\{\boldsymbol{v}_{nk}\}$ is a bounded sequence in $L^2((0,T);\mathcal{V}^S(\Omega)) \cap L^\infty((0,T);\mathcal{H}^S(\Omega))$ with respect to n, there exists a constant M_5 (independent of n) such that

$$\delta\int_0^T \big(\|\nabla\boldsymbol{v}_{nk}-\nabla\boldsymbol{v}\|_2^2 + \|\nabla\boldsymbol{v}_{nk}\|_2^2 + \|\boldsymbol{v}_{nk}-\boldsymbol{v}\|_2\|\nabla\boldsymbol{v}_{nk}\|_2\big)dt \le M_5\delta$$

holds true. Secondly, we consider the second term of the right hand side of (3.17). Since we know that $\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi}$ belongs to $L^2((0,T);\mathbb{L}^{2,S}(\Omega))$, there exists a $\boldsymbol{\Phi}_\delta \in \mathbb{C}_0^\infty((0,T)\times\Omega)$ such that $\boldsymbol{\Phi}_\delta$ is symmetric with respect to the x_2-axis and

$$\|\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi} - \boldsymbol{\Phi}_\delta\|_{L^2((0,T);\mathbb{L}^2(\Omega))} < \delta$$

holds true. For any $t \in [0,T]$ we obtain the decomposition

$$\boldsymbol{\Phi}_\delta(t) = \boldsymbol{\Phi}_{\delta,\sigma}(t) + \boldsymbol{\Phi}_{\delta,p}(t) \quad \big(\boldsymbol{\Phi}_{\delta,\sigma}(t) \in \mathcal{H}^S(\Omega), \boldsymbol{\Phi}_{\delta,p}(t) \in (\mathcal{H}^S(\Omega))^\perp\big).$$

It is easy to follow that $\boldsymbol{\Phi}_{\delta,\sigma}$ belongs to $L^2((0,T);\mathcal{H}^S(\Omega))$. According to Lemma 1.3, there exists finite sequences $\{\boldsymbol{\xi}_j\}_{j=1,\dots,L} \subset \mathbb{C}_{0,\sigma}^{\infty,S}(\Omega)$ and

$\{q_j\}_{j=1,\dots,L} \subset C[0,T]$ such that

$$\Big\| \boldsymbol{\Phi}_{\delta,\sigma} - \sum_{j=1}^{L} q_j \boldsymbol{\xi}_j \Big\|_{L^2((0,T);H(\Omega))} < \delta$$

holds true. Now we divide the second term of the right hand side of (3.17) into

$$-\int_0^T (\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi} - \boldsymbol{\Phi}_\delta, \boldsymbol{v}_{nk} - \boldsymbol{v})\psi dt - \int_0^T \Big(\boldsymbol{\Phi}_{\delta,\sigma} - \sum_{j=1}^{L} q_j\boldsymbol{\xi}_j, \boldsymbol{v}_{nk} - \boldsymbol{v}\Big)\psi dt$$
$$-\sum_{j=1}^{L}\int_0^T q_j(\boldsymbol{\xi}_j, \boldsymbol{v}_{nk} - \boldsymbol{v})\psi dt.$$

There exists a constant M_6 (independent of n) such that the estimates

$$\Big|\int_0^T (\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi} - \boldsymbol{\Phi}_\delta, \boldsymbol{v}_{nk} - \boldsymbol{v})\psi dt\Big|$$
$$< \Big(\sup_{[0,T]}|\psi|\Big) C(\Omega)\delta\Big(\int_0^T \|\nabla\boldsymbol{v}_{nk} - \nabla\boldsymbol{v}\|_2^2 dt\Big) \le M_6\delta,$$
$$\Big|\int_0^T \Big(\boldsymbol{\Phi}_{\delta,\sigma} - \sum_{j=1}^{L} q_j\boldsymbol{\xi}_j, \boldsymbol{v}_{nk} - \boldsymbol{v}\Big)\psi dt\Big|$$
$$< \Big(\sup_{[0,T]}|\psi|\Big) C(\Omega)\delta\Big(\int_0^T \|\nabla\boldsymbol{v}_{nk} - \nabla\boldsymbol{v}\|_2^2 dt\Big) \le M_6\delta$$

hold true. Consequently we obtain

$$\Big|\int_0^T ((\boldsymbol{v}_{nk}\cdot\nabla)\boldsymbol{v}_{nk}, \boldsymbol{\varphi})\psi dt - \int_0^T ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{\varphi})\psi dt\Big|$$
$$< \delta(M_5 + 2M_6) + M\sum_{l=1}^{N}\int_0^T |(\boldsymbol{v}_{nk} - \boldsymbol{v}, \boldsymbol{\psi}_l)|^2 dt$$
$$+ M_7\sum_{j=1}^{L}\int_0^T |(\boldsymbol{\xi}_j, \boldsymbol{v}_{nk} - \boldsymbol{v})| dt, \tag{3.19}$$

where

$$M_7 = \sup_{[0,T]} |\psi| \max_{1\le j\le J} \sup_{[0,T]} |q_j|.$$

We can choose a subsequence $\{\boldsymbol{v}_{nki}\}_{i\in\mathbb{N}}$ of $\{\boldsymbol{v}_{nk}\}_{k\in\mathbb{N}}$ such that the second and third terms of the right hand side of (3.19) converge to zero by (3.13). Therefore (3.16) holds true. We obtain (3.15) for the subsequence $\{\boldsymbol{v}_{nki}\}$. As i goes to infinity, $\boldsymbol{v}$ satisfies

$$\begin{aligned}
&-\int_0^T (\boldsymbol{v},\boldsymbol{\varphi})\psi' + \big\{\nu((\boldsymbol{v},\boldsymbol{\varphi})) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi}) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi})\\
&\qquad + ((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi}) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{P}^\alpha,\boldsymbol{\varphi}) + ((\boldsymbol{P}^\alpha\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi})\big\}\psi dt\\
&= \int_0^T \langle \boldsymbol{F},\boldsymbol{\varphi}\rangle\psi dt.
\end{aligned}$$

Since the inclusion $\mathbb{C}^{\infty,S}_{0,\sigma}(\Omega)\subset\mathcal{V}^S(\Omega)$ is dense, consequently we have

$$\begin{aligned}
&-\int_0^T (\boldsymbol{v},\boldsymbol{\varphi})\psi' + \big\{\nu((\boldsymbol{v},\boldsymbol{\varphi})) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi}) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi})\\
&\qquad + ((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi}) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{P}^\alpha,\boldsymbol{\varphi}) + ((\boldsymbol{P}^\alpha\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi})\big\}\psi dt\\
&= \int_0^T \langle \boldsymbol{F},\boldsymbol{\varphi}\rangle\psi dt \quad (\boldsymbol{\varphi}\in\mathcal{V}^S(\Omega), \psi\in C_0^\infty(0,T)).
\end{aligned} \tag{3.20}$$

Then $\boldsymbol{v}$ satisfies

$$\begin{aligned}
&\frac{d}{dt}(\boldsymbol{v},\boldsymbol{\varphi}) + \nu((\boldsymbol{v},\boldsymbol{\varphi})) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi}) + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi}) + ((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi})\\
&\quad + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{P}^\alpha,\boldsymbol{\varphi}) + ((\boldsymbol{P}^\alpha\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi}) = \langle \boldsymbol{F},\boldsymbol{\varphi}\rangle \quad (\boldsymbol{\varphi}\in\mathcal{V}^S(\Omega)).
\end{aligned} \tag{3.21}$$

in the distribution sence on $(0,T)$. Let $\boldsymbol{w}\in L^2((0,T);\mathcal{V}^S(\Omega))$. A map

$$\begin{aligned}
\boldsymbol{w}\mapsto \int_0^T &\big\{-\nu((\boldsymbol{w},\boldsymbol{\varphi})) - ((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi}) - ((\boldsymbol{w}\cdot\nabla)\boldsymbol{b}_\varepsilon,\boldsymbol{\varphi}) - ((\boldsymbol{b}_\varepsilon\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi})\\
&- ((\boldsymbol{w}\cdot\nabla)\boldsymbol{P}^\alpha,\boldsymbol{\varphi}) - ((\boldsymbol{P}^\alpha\cdot\nabla)\boldsymbol{w},\boldsymbol{\varphi}) + \langle \boldsymbol{F},\boldsymbol{\varphi}\rangle\big\}\psi dt\\
&\qquad\qquad (\boldsymbol{\varphi}\in\mathcal{V}^S(\Omega), \psi\in C_0^\infty(0,T))
\end{aligned} \tag{3.22}$$

is cotinuous functional on $L^2((0,T);\mathcal{V}^S(\Omega))$. Therefore $\boldsymbol{v}$ has a weak derivative $\boldsymbol{v}' \in L^2((0,T);(\mathcal{V}^S(\Omega))')$ and $\boldsymbol{v}$ belongs to $C([0,T];\mathcal{H}^S(\Omega))$. For any $\boldsymbol{\varphi} \in \mathbb{L}^{2,S}(\Omega)$, it is evident that there exists a subsequence $\{\boldsymbol{v}_{nk}\}$ such that the limit (3.13) holds true with respect to $\boldsymbol{\varphi}$. Therefore it follows that

$$(\boldsymbol{v}(0) - \boldsymbol{v}(T), \boldsymbol{\varphi}) = (\boldsymbol{v}(0) - \boldsymbol{v}_{nk}(0), \boldsymbol{\varphi}) + (\boldsymbol{v}_{nk}(T) - \boldsymbol{v}(T), \boldsymbol{\varphi}) \to 0 \quad (k \to \infty)$$

holds true. Consequently $\boldsymbol{v} \in C_\pi([0,T];\mathcal{H}^S(\Omega))$. We set $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{b}_\varepsilon + \boldsymbol{P}^\alpha$. Then $\boldsymbol{u}$ is a time periodic weak solution. □

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