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Scaling limit for the Dereziński-Gérard model

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Abstract. We consider a scaling limit for the Dereziński-Gérard model. We derive an effective potential by taking a scaling limit for the total Hamiltonian of the Dereziński-Gérard model. Our method to derive an effective potential is independent of whether or not the quantum field has a nonnegative mass. As an application of our theory developed in the present paper, we derive an effective potential of the Nelson model.

Key words: Fock space, scaling limit, effective potential, the Dereziński-Gérard model, the Nelson model, Weyl relations.

1. Introduction

The Dereziński-Gérard model was introduced by Dereziński and Gérard [5] as an abstract model of particle-field interaction. (they call it the Pauli-Fierz model, but we call it the Dereziński-Gérard model for the sake of clarity.) The Hamiltonian of the Dereziński-Gérard model is given by

$$H := A \otimes I + I \otimes H_{\mathrm{b}} + \tilde{\phi}(v).$$

The first and second terms mean Hamiltonians of a particle and a field, respectively. The third term means the interaction between a particle and a field.

Davies [4] initiated a scaling limit for models of quantum particles coupled to a Bose field and obtained effective potentials. In [1], Arai constructed a theory of abstract scaling limit. He applied it to the Pauli-Fierz model without A^2-term in the dipole approximation and derived the effective potential introduced by Bethe [3] and Welton [14] to explain the Lamb shift. He not only derived the effective potential of Bethe and Welton rigorously but also clarified a mathematical meaning of it. Hiroshima [6] considered a scaling limit for the Pauli-Fierz model with A^2-term in the diploe approximation. In [7], he derived the Yukawa potential by taking a weak coupling limit and removing the ultraviolet cutoff simultaneously for the

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massive Nelson model. In [8], he analyzed the same problem by means of functional integration. Various kinds of scaling limits for the Pauli-Fierz model in the dipole approximation with A^2-term were studied by Hiroshima [9]. Suzuki [12] studied a scaling limit for the generalized spin boson (GSB) model, which was introduced by Arai and Hirokawa in [2], and applied it to the models in nuclear physics. He also studied a scaling limit for a general version of the Nelson model in [13].

The purpose of the present paper is to consider a scaling limit for the Dereziński-Gérard model. As far as the author knows, a scaling limit for the Dereziński-Gérard model has not been investigated. This is one of the motivations of the present paper.

The present paper is organized as follows. Section 2 consists of three subsections. In Subsection 2.1, we describe the model considered in the present paper. We devote Subsection 2.2 to investigating properties of operators $\tilde{a}^\sharp(v)$ and $\tilde{\phi}(v)$ extensively for later use. In Subsection 2.3, we introduce some assumptions and state the main result. In Section 3, we briefly explain an abstract scaling limit theory used in the present paper. Section 4 also consists of three subsections. In Subsection 4.1, we transform a scaled Hamiltonian into an operator handled easily. In Subsection 4.2, we prove two conditions, which are needed to apply abstract scaling limit theorems. In Subsection 4.3, we prove the main theorem. In Section 5, we consider an application of our theory to the Nelson model and derive an effective potential of the Nelson model. In Appendix A, we prove the Weyl relations for $\tilde{\phi}(v)$. As a corollary of the Weyl relations, we obtain a necessary and sufficient condition that $\tilde{\phi}(v)$ and $\tilde{\phi}(w)$ strongly commute. In Appendix B, we show a relative boundedness of $\tilde{\phi}(v)\tilde{\phi}(w)$ with respect to $I \otimes H_{\mathrm{b}}$.

2. Definition of the Model and the Main Result

2.1. Definition of the model of the model

In the present paper, we denote the inner product and the norm of a Hilbert space $\mathcal{X}$ by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\| \cdot \|_{\mathcal{X}}$, respectively. The inner product is antilinear in the first variable. If there is no danger of confusion, then we omit the subscript $\mathcal{X}$ in $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\| \cdot \|_{\mathcal{X}}$.

For a linear operator T on a Hilbert space, we denote its domain by $D(T)$. If T is densely defined, the adjoint of T is denoted by T^*. For linear operators S and T on a Hilbert space, $D(S + T) := D(S) \cap D(T)$ unless otherwise stated.

For a self-adjoint operator S on a Hilbert space, we denote its spectum and its resolvent set by $\sigma(S)$ and $\rho(S)$, respectively. The spectral measure associated with S is denoted by $E_S(\cdot)$. If S is bounded below, then we set

$$E_0(S) := \inf \sigma(S)$$

and call it the ground state energy of S.

To describe the Bose field, one uses the Boson Fock space over a separable complex Hilbert space $\mathcal{X}$:

$$\begin{aligned}\mathcal{F}_{\mathrm{b}}(\mathcal{X}) &:= \bigoplus_{n=0}^{\infty} \bigotimes_{s}^{n} \mathcal{X} \\ &= \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \,\middle|\, n \geq 0,\ \psi^{(n)} \in \bigotimes_{s}^{n} \mathcal{X},\ \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2 < \infty \right\},\end{aligned}$$

where $\bigotimes_s^n \mathcal{X}$ denotes the n -fold symmetric tensor product of $\mathcal{X}$ with $\bigotimes_s^0 \mathcal{X} := \mathbb{C}$ (the space of complex numbers). The vector $\Omega := \{1, 0, \ldots\}$ is called the Fock vacuum in $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$.

One of the main objects on $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$ is the annihilation operator $a(f)$ which is a densely defined closed linear operator on $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$ such that for all $\eta = \{\eta^{(n)}\}_{n=0}^{\infty} \in D(a(f)^*)$, $(a(f)^*\eta)^{(0)} = 0$ and $(a(f)^*\eta)^{(n)} = \sqrt{n} S_n(f \otimes \eta^{(n-1)})$, $n \geq 1$, where S_n is the symmetrization operator on $\bigotimes^n \mathcal{X}$. The adjoint $a(f)^*$, which is called the creation operator, and the annihilation operator $a(g)$ obey the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0$$

for all $f, g \in \mathcal{X}$ on the dense subspace

$$\begin{aligned}\mathcal{F}_0(\mathcal{X}) := \{\eta \in \mathcal{F}_{\mathrm{b}}(\mathcal{X}) \mid &\text{ there exists a number } n_0 \\ &\text{such that } \eta^{(n)} = 0 \text{ for all } n \geq n_0\},\end{aligned}$$

where $[\cdot, \cdot]$ means the commutator.

For every self-adjoint operator S on $\mathcal{X}$, one can define a self-adjoint operator $d\Gamma(S)$, called the second quantization of S, by

$$d\Gamma(S) := \bigoplus_{n=0}^{\infty} S^{(n)},$$

with $S^{(0)} := 0$ and $S^{(n)}$ is the closure of

$$\left(\sum_{j=1}^{n} \underbrace{I \otimes \cdots \otimes \overbrace{S}^{j} \otimes \dots I}_{n} \right) \Bigg|_{\hat{\bigotimes}^n D(S)},$$

where I denotes the identity and $\hat{\bigotimes}^n D(S)$ the algebraic tensor product of $D(S)$. If S is nonnegative, then so is $d\Gamma(S)$. The second quantization of the identity, $N_{\mathrm{b}} := d\Gamma(I)$, is called the number operator.

As the state space of the Dereziński-Gérard model, we take the tensor product Hilbert space

$$\mathcal{F} := L^2(\mathbb{R}^N) \otimes \mathcal{F}_{\mathrm{b}}\big(L^2(\mathbb{R}^d)\big).$$

The Hilbert space $\mathcal{F}$ is identified with the space

$$\bigoplus_{n=0}^{\infty} \left[L^2(\mathbb{R}^N) \otimes \bigotimes_{s}^{n} L^2(\mathbb{R}^d) \right],$$

and we use this identification freely in what follows.

The subspace $\mathcal{D}_0$ of $\mathcal{F}$ is defined as follows:

$$\mathcal{D}_0 := \{\psi \in \mathcal{F} \mid \text{there exists an } n_0 \text{ such that, for all } n \geq n_0,\ \psi^{(n)} = 0\}.$$

Let A be a self-adjoint operator on $L^2(\mathbb{R}^N)$. Let ω be a nonnegative Borel measurable function such that $0 < \omega(k) < \infty$ a.e. $k \in \mathbb{R}^d$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Then ω defines a multiplication operator on $L^2(\mathbb{R}^d)$, which is nonnegative, injective and self-adjoint. We denote it by the same symbol. The function ω represents a dispersion relation of one free boson associated with the Bose field under consideration. A typical example of ω is $\omega(k) = \sqrt{m^2 + |k|^2}$. Here $m \geq 0$ is the mass of the boson. As indicated in this example, we define the mass of the boson by

$$m := \operatorname*{ess.inf}_{k \in \mathbb{R}^d} \omega(k).$$

If $m > 0$ (resp. $m = 0$), we call the associated Bose field massive (resp. massless). Let

$$H_{\mathrm{b}} := d\Gamma(\omega)$$

which acts on $\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$. The free Hamiltonian of the Dereziński-Gérard model is given by

$$H_0 := A \otimes I + I \otimes H_{\mathrm{b}}.$$

To define the interaction part of the Dereziński-Gérard model, we introduce an analogue $\tilde{\phi}(v)$ of the Segal field operator for a bounded operator v from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N) \otimes L^2(\mathbb{R}^d)$. To do this, we first define the operator $\tilde{a}^*(v)$, which is an analogue of the usual creation operator. The domain and the operation of $\tilde{a}^*(v)$ are given by

$$\begin{aligned} D(\tilde{a}^*(v)) := \Bigg\{ \psi = (\psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \,\Bigg|& \\ &\sum_{n=0}^{\infty} n \big\| \big(I_{L^2(\mathbb{R}^N)} \otimes S_n\big)\big(v \otimes I_{\otimes_s^{n-1} L^2(\mathbb{R}^d)}\big)\psi^{(n-1)} \big\|^2 < \infty \Bigg\}, \end{aligned}$$

$$(\tilde{a}^*(v)\psi)^{(0)} := 0,$$

$$(\tilde{a}^*(v)\psi)^{(n)} := \sqrt{n}\big(I_{L^2(\mathbb{R}^N)} \otimes S_n\big)\big(v \otimes I_{\otimes_s^{n-1} L^2(\mathbb{R}^d)}\big)\psi^{(n-1)}, \quad n \geq 1.$$

It is easy to see that $D(\tilde{a}^*(v)) \supset \mathcal{D}_0$. Thus the operator $\tilde{a}^*(v)$ is densely defined. We set

$$\tilde{a}(v) := (\tilde{a}^*(v))^*.$$

The domain and the operation of $\tilde{a}(v)$ is as follows:

$$\begin{aligned} D(\tilde{a}(v)) := \Bigg\{ \psi = (\psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \,\Bigg|& \\ &\sum_{n=0}^{\infty} (n+1) \big\| \big(I_{L^2(\mathbb{R}^N)} \otimes S_n\big)\big(v^* \otimes I_{\otimes_s^{n} L^2(\mathbb{R}^d)}\big)\psi^{(n+1)} \big\|^2 < \infty \Bigg\}, \end{aligned}$$

$$(\tilde{a}(v)\psi)^{(n)} := \sqrt{n+1}\big(I_{L^2(\mathbb{R}^N)} \otimes S_n\big)\big(v^* \otimes I_{\otimes_s^{n} L^2(\mathbb{R}^d)}\big)\psi^{(n+1)}.$$

It is easily verified that $\tilde{a}^\sharp(v)(\tilde{a}^\sharp = \tilde{a}$ or $\tilde{a}^*)$ is closed and $\mathcal{D}_0$ is a core for $\tilde{a}^\sharp(v)$.

We define an analogue of the Segal field operator by

$$\tilde{\phi}(v) := \frac{1}{\sqrt{2}}(\tilde{a}(v) + \tilde{a}^*(v)).$$

The total Hamiltonian of the Dereziński-Gérard model is defined by

$$H := H_0 + \tilde{\phi}(v).$$

2.2. Properties of $\tilde{a}^\sharp(v)$ and $\tilde{\phi}(v)$

We devote this subsection to investigating properties of $\tilde{a}^\sharp(v)$ and $\tilde{\phi}(v)$ such as commutation relations, relative boundedness and so on. These properties are analogous to those of the usual annihilation/creation operator and the Segal field operator on a Boson Fock space.

We introduce a function space $L^{\infty,2} = L^{\infty,2}(\mathbb{R}^N \times \mathbb{R}^d)$ by

$$L^{\infty,2} := \left\{ u \,\middle|\, \operatorname{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} |u(x,k)|^2 dk < \infty \right\}.$$

$L^{\infty,2}$ is a Banach space with the norm

$$\|u\|_{L^{\infty,2}} := \left(\operatorname{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} |u(x,k)|^2 dk \right)^{1/2}.$$

For a function $\tilde{v} \in L^{\infty,2}$, we define a bounded operator v from $L^2(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N) \otimes L^2(\mathbb{R}^d)$ by

$$(vf)(x,k) = \tilde{v}(x,k) f(x), \quad f \in L^2(\mathbb{R}^N). \tag{2.1}$$

It is easily verified that the operator norm of v is given by

$$\|v\| = \|\tilde{v}\|_{L^{\infty,2}}$$

(cf. [10]). In what follows, we identify an operator v defined by (2.1) with its corresponding function $\tilde{v}$ and denote both objects simply by v.

For $v \in L^{\infty,2}$, the domain and the operation of $\tilde{a}^*(v)$ are given by

$$D(\tilde{a}^*(v)) = \left\{ \psi \in \mathcal{F} \,\middle|\, \sum_{n=1}^{\infty} \frac{1}{n} \int_{\mathbb{R}^{N+dn}} \left| \sum_{j=1}^{n} v(x,k_j)\psi^{(n-1)}\big(x,k_1,\ldots,\hat{k_j},\ldots,k_n\big) \right|^2 \cdot dxdk_1\cdots dk_n < \infty \right\}$$

and

$$\begin{aligned} &(\tilde{a}^*(v)\psi)^{(n)}(x,k_1,\ldots,k_n) \\ &\qquad\qquad = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} v(x,k_j)\psi^{(n-1)}\big(x,k_1,\ldots,\hat{k_j},\ldots,k_n\big) \quad n \geq 1, \\ &(\tilde{a}^*(v)\psi)^{(0)}(x) = 0, \end{aligned}$$

where $\hat{}$ denotes the omission of the object wearing the hat.

Similarly, for $v \in L^{\infty,2}$, the domain and the operation of $\tilde{a}(v)$ are given by

$$D(\tilde{a}(v)) = \left\{ \psi \in \mathcal{F} \,\middle|\, \sum_{n=1}^{\infty} (n+1) \int_{\mathbb{R}^{N+dn}} \left| \int_{\mathbb{R}^d} \overline{v(x,k)}\psi^{(n+1)}(x,k,k_1,\ldots,k_n)dk \right|^2 \cdot dxdk_1\cdots dk_n < \infty \right\}$$

and

$$(\tilde{a}(v))^{(n)}(x,k_1,\ldots,k_n) = \sqrt{n+1} \int_{\mathbb{R}^d} \overline{v(x,k)}\psi^{(n+1)}(x,k,k_1,\ldots,k_n)dk,$$

where $\overline{v(x,k)}$ denotes the complex conjugate of $v(x,k)$.

We define an operator $\langle v(x,\cdot), w(x,\cdot)\rangle_{L^2(\mathbb{R}^d_k)}$ on $\mathcal{F}$ as follows:

$$\begin{aligned} &\big(\langle v(x,\cdot), w(x,\cdot)\rangle_{L^2(\mathbb{R}^d_k)}\psi\big)^{(n)}(x,k_1,\ldots,k_n) \\ &\quad := \langle v(x,k), w(x,k)\rangle_{L^2(\mathbb{R}^d_k)}\psi^{(n)}(x,k_1,\ldots,k_n). \end{aligned}$$

The operator $\langle v(x,\cdot), w(x,\cdot)\rangle_{L^2(\mathbb{R}^d_k)}$ is a bounded operaor on $\mathcal{F}$. In what follows, if there is no danger of confusion, we abbreviate $\langle v(x,\cdot), w(x,\cdot)\rangle$ as $\langle\!\langle v, w\rangle\!\rangle$.

It is easy to see that

$$\|\langle\!\langle v, w\rangle\!\rangle\| \le \|v\|\|w\|. \tag{2.2}$$

(2.2) is frequently used throughtout the present paper.

The following commutation relations are analogues of the usual canonical commutation relations (CCRs). They are easily proven by direct calculations. So we leave the proof to the reader.

Proposition 2.1 *For all $v, w \in L^{\infty,2}$, the following commutation relations hold on $\mathcal{D}_0$:*

$$[\tilde{a}(v), \tilde{a}^*(w)] = \langle\!\langle v, w\rangle\!\rangle, \quad [\tilde{a}(v), \tilde{a}(w)] = 0, \quad [\tilde{a}^*(v), \tilde{a}^*(w)] = 0. \tag{2.3}$$

We next prove a relative boundedness for $\tilde{a}^\sharp(v)$ and $\tilde{\phi}(v)$ with respect to $I \otimes N_{\mathrm{b}}^{1/2}$. (cf. [10, Lemma 4.8, 4.10]).

Proposition 2.2

(1) *For $v \in L^{\infty,2}$, $D(I \otimes N_{\mathrm{b}}^{1/2}) \subset D(\tilde{a}^\sharp(v))$ and for $\psi \in D(I \otimes N_{\mathrm{b}}^{1/2})$*

$$\|\tilde{a}(v)\psi\| \le \|v\|\big\|I \otimes N_{\mathrm{b}}^{1/2}\psi\big\|, \tag{2.4}$$

$$\|\tilde{a}^*(v)\psi\| \le \|v\|\big\|I \otimes (N_{\mathrm{b}}+1)^{1/2}\psi\big\| \tag{2.5}$$

hold.

(2) *For $v \in L^{\infty,2}$, $D(I \otimes N_{\mathrm{b}}^{1/2}) \subset D(\tilde{\phi}(v))$ and for $\psi \in D(I \otimes N_{\mathrm{b}}^{1/2})$*

$$\big\|\tilde{\phi}(v)\psi\big\| \le \sqrt{2}\|v\|\big\|I \otimes N_{\mathrm{b}}^{1/2}\psi\big\| + \frac{1}{\sqrt{2}}\|v\|\|\psi\| \tag{2.6}$$

holds.

Proof. Direct calculations. □

By using Proposition 2.2, we can prove the following theorem.

Theorem 2.3 *For $v \in L^{\infty,2}$, any vectors belonging to $\mathcal{D}_0$ are entire vectors for $\tilde{\phi}(v)$ and $\tilde{a}^\sharp(v)$.*

Theorem 2.3 and the Nelson's analytic vector theorem [11, Theorem X.39] imply the following theorem.

Theorem 2.4 *For $v \in L^{\infty,2}$, $\tilde{\phi}(v)$ is essentially self-adjoint on $\mathcal{D}_0$.*

Note that the domain and the operation of $I \otimes H_{\mathrm{b}}$ are as follows:

$$D(I \otimes H_{\mathrm{b}}) = \left\{ \psi \in \mathcal{F} \,\middle|\, \int_{\mathbb{R}^{N+dn}} \left| \sum_{i=1}^{n} \omega(k_i) \psi^{(n)}(x, k_1, \dots, k_n) \right|^2 dx dk_1 \cdots dk_n < \infty \right\}$$

and

$$(I \otimes H_{\mathrm{b}} \psi)^{(n)}(x, k_1, \dots, k_n) = \sum_{i=1}^{n} \omega(k_i) \psi^{(n)}(x, k_1, \dots, k_n).$$

Proposition 2.5 *Let v be an element of $L^{\infty,2}$ such that v/ω is also an element of $L^{\infty,2}$. Then the following commutation relations hold on $\mathcal{D}_0 \cap D(I \otimes H_{\mathrm{b}})$:*

$$[\tilde{a}^*(v/\omega), I \otimes H_{\mathrm{b}}] = -\tilde{a}^*(v), \tag{2.7}$$

$$[\tilde{a}(v/\omega), I \otimes H_{\mathrm{b}}] = \tilde{a}(v). \tag{2.8}$$

Proof. Direct calculations. □

We need a relative boundedness of $\tilde{a}^\sharp(v)$ and $\tilde{\phi}(v)$ with respect to the operator $I \otimes H_{\mathrm{b}}^{1/2}$ to prove Theorem 2.7 below. The proof of Proposition 2.6 below is similar to that of Proposition 2.2. So we omit the proof.

Proposition 2.6

(1) *For $v \in L^{\infty,2}$ with $v/\sqrt{\omega} \in L^{\infty,2}$, $D(I \otimes H_{\mathrm{b}}^{1/2}) \subset D(\tilde{a}^\sharp(v))$ and for $\psi \in D(I \otimes H_{\mathrm{b}}^{1/2})$*

$$\|\tilde{a}(v)\psi\| \le \|v/\sqrt{\omega}\| \|I \otimes H_{\mathrm{b}}^{1/2} \psi\|,$$

$$\|\tilde{a}^*(v)\psi\| \le \|v/\sqrt{\omega}\| \|I \otimes H_{\mathrm{b}}^{1/2} \psi\| + \|v\| \|\psi\|$$

hold.

(2) *For* $v \in L^{\infty,2}$ *with* $v/\sqrt{\omega} \in L^{\infty,2}$, $D(I \otimes H_{\mathrm{b}}^{1/2}) \subset D(\tilde{\phi}(v))$ *and for* $\psi \in D(I \otimes H_{\mathrm{b}}^{1/2})$

$$\|\tilde{\phi}(v)\psi\| \leq \sqrt{2}\|v/\sqrt{\omega}\|\|I \otimes H_{\mathrm{b}}^{1/2}\psi\| + \frac{1}{\sqrt{2}}\|v\|\|\psi\|$$

holds.

Proposition 2.6 implies the following theorem.

Theorem 2.7 (Relative boundedness) *If* v *and* $v/\sqrt{\omega}$ *belong to* $L^{\infty,2}$*, then the interaction* $\tilde{\phi}(v)$ *is infinitesimally small with respect to the free Hamiltonian* H_0*.*

Self-adjointness of the total Hamiltonian H follows from Theorem 2.7 and the Kato-Rellich theorem [11, Theorem X.12].

Theorem 2.8 (Self-adjointness of the Hamiltonian) *If* v *and* $v/\sqrt{\omega}$ *belong to* $L^{\infty,2}$*, then the total Hamiltonian* H *is self-adjoint, bounded below and any core for* H_0 *is core for* H*.*

2.3. Main Result

In this subsection, we state the main result of the present paper. To do so, we first introduce some assumptions used throughtout.

(A.1) The operator A is of the form

$$A = -\triangle + V,$$

where $\triangle$ denote the Laplacian on $L^2(\mathbb{R}^N)$ and $V : \mathbb{R}^N \to \mathbb{R}$ is a potential defined on $\mathbb{R}^N$.

(A.2) A is self-adjoint and bounded below.

(A.3) $D(V) \supset C_0^\infty(\mathbb{R}^N)$ and $C_0^\infty(\mathbb{R}^N)$ is a core for A.

(A.4) Let D_j be the generalized partial differential operator with respect to the variable x_j and $\tilde{A} := A - E_0(A)$. For $j = 1, \ldots, N$, $D(\tilde{A}^{1/2}) \subset D(D_j)$ and there exist constants $a_j, b_j \geq 0$ such that for all $u \in D(\tilde{A}^{1/2})$

$$\|D_j u\| \leq a_j \|\tilde{A}^{1/2}u\| + b_j\|u\|.$$

For the operator v, we impose the following assumptions.

(v.1) The function $v = v(x, k)$ is twice differentiable with respect to $x \in \mathbb{R}^N$ and satisfies

$$v, v/\omega \in L^{\infty,2},$$

$$\partial_j v/\sqrt{\omega}, \partial_j v/\omega^{3/2} \in L^{\infty,2},$$

$$\triangle_x v/\omega, \triangle_x v/\omega^{3/2} \in L^{\infty,2}.$$

Here $\partial_j v$ and $\triangle_x$ denote the partial derivative of the function v with respect to the variable $x_j \in \mathbb{R}$ and the Laplacian with respect to the variable $x \in \mathbb{R}^N$, respectively.

(v.2) The function $x \mapsto \langle v(x, \cdot)/\omega, \triangle_x v(x, \cdot)/\omega\rangle_{L^2(\mathbb{R}^d_k)}$ is real-valued for a.e. $x \in \mathbb{R}^N$.

Remark 1 The condition (v.2) is equivalent to the fact that $\tilde{\phi}(iv/\omega)$ and $\tilde{\phi}(i\triangle_x v/\omega)$ strongly commute. See Corollary A.5.

For $\Lambda > 0$, we introduce a scaled Hamiltonian $H(\Lambda)$ by

$$H(\Lambda) := A \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} + \Lambda H_{\mathrm{I}},$$

where

$$H_{\mathrm{I}} := \tilde{\phi}(v).$$

We denote the orthogonal projection onto $\ker H_{\mathrm{b}}$ by P_0. Note that $\ker H_{\mathrm{b}} = \mathcal{L}\{\Omega\}$, where $\mathcal{L}\{\Omega\}$ denotes the subspace spanned by the set $\{\cdots\}$.

For $v, w \in L^{\infty,2}$, we define a bounded operator $\langle\!\langle v, w\rangle\!\rangle_0$ on $L^2(\mathbb{R}^N)$ as follows:

$$(\langle\!\langle v, w\rangle\!\rangle_0 f)(x) := \langle v(x, k), w(x, k)\rangle_{L^2(\mathbb{R}^d_k)} f(x), \quad f \in L^2(\mathbb{R}^d).$$

The following theorem is the main theorem of the present papar.

Theorem 2.9 *Suppose that* (A.1)–(A.4) *and* (v.1)–(v.2) *hold. Then, for* $z \in \mathbb{C} \setminus \mathbb{R}$ *or for* $z < 0$ *with* $|z|$ *sufficiently large,*

$$s\text{-}\lim_{\Lambda\to\infty} (H(\Lambda) - z)^{-1} = (A + V_{\mathrm{eff}} - z)^{-1} \otimes P_0, \tag{2.9}$$

where

$$V_{\mathrm{eff}} := -\frac{1}{2}\langle\!\langle v/\omega, v\rangle\!\rangle_0.$$

3. Abstract Scaling Limit

In this section, we first explain an abstract scaling limit theorem. Secondary, we define the notion of the partial expectation of operators on a tensor product Hilbert space. The notion of partial expectation enables us to express the limit operator $(K_\infty - z)^{-1} I \otimes P_B$ appearing in Theorem 3.3 below in more explicit way.

To begin with, we introduce the following notions which are useful for describing a scaling limit theorem.

Definition 3.1 (Uniform Relative Boundedness, [13]) Let $\mathcal{L}$ be a Hilbert space, $L(\Lambda)$, $M(\Lambda)$, $N(\lambda)$ and $O(\lambda)$ ($\Lambda > 0$, $\lambda \in \mathbb{R} \setminus \{0\}$) linear operators on $\mathcal{L}$ satisfying

$$\cap_{\Lambda>0} D(L(\Lambda)) \neq \emptyset, \quad \cap_{\lambda\in\mathbb{R}\setminus\{0\}} D(N(\lambda)) \neq \emptyset.$$

(1) We say that $M(\Lambda)$ is uniformly $L(\Lambda)$-bounded near ∞ if there exist constants $\Lambda_0 > 0$ and $a, b \geq 0$ such that for any $\Lambda \geq \Lambda_0$, $D(M(\Lambda)) \supset D(L(\Lambda))$ and

$$\|M(\Lambda)\psi\| \leq a\|L(\Lambda)\psi\| + b\|\psi\|, \quad \psi \in D(L(\Lambda)).$$

(2) We say that uniformly $M(\Lambda)$ is $L(\Lambda)$-infinitesimally small near ∞ if for any $\varepsilon > 0$, there exist constants $\Lambda_0(\varepsilon) > 0$ and $b(\varepsilon) \geq 0$ such that for any $\Lambda \geq \Lambda_0(\varepsilon)$, $D(M(\Lambda)) \supset D(L(\Lambda))$ and

$$\|M(\Lambda)\psi\| \leq \varepsilon\|L(\Lambda)\psi\| + b(\varepsilon)\|\psi\|, \quad \psi \in D(L(\Lambda)).$$

(3) We say that $O(\lambda)$ is uniformly $N(\lambda)$-bounded near 0 if there exist constants $\lambda_0 > 0$ and $a, b \geq 0$ such that for any $\lambda \in \mathbb{R} \setminus \{0\}$ with $|\lambda| \leq \lambda_0$, $D(O(\lambda)) \supset D(N(\lambda))$ and

$$\|O(\lambda)\psi\| \leq a\|N(\lambda)\psi\| + b\|\psi\|, \quad \psi \in D(N(\lambda)).$$

(4) We say that $O(\lambda)$ is uniformly $N(\lambda)$-infinitesimally small near 0 if for

any $\varepsilon > 0$, there exist constants $\lambda_0(\varepsilon) > 0$ and $b(\varepsilon) \geq 0$ such that for any $\lambda \in \mathbb{R} \setminus \{0\}$ with $|\lambda| \leq \lambda_0$, $D(O(\lambda)) \supset D(N(\lambda))$ and

$$\|O(\lambda)\psi\| \leq \varepsilon\|N(\lambda)\psi\| + b(\varepsilon)\|\psi\|, \quad \psi \in D(N(\lambda)).$$

Now we prepare a setting to state a scaling limit theorem.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and we set

$$\mathcal{X} := \mathcal{H} \otimes \mathcal{K}.$$

Let A and B be non-negative self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$, respectively, with

$$\ker B \neq \{0\}.$$

We denote the orthogonal projection from $\mathcal{K}$ onto $\ker B$ by P_B. We suppose that a family of symmetric operators $\{C_\Lambda\}_{\Lambda>0}$ on $\mathcal{X}$ satisfies the following conditions:

(i) C_Λ is uniformly $(A \otimes I + \Lambda I \otimes B)$-infinitesimally small near ∞.

(ii) There exists a symmetric operator C such that $D(C) \supset D(A) \otimes \ker B$ and

$$s\text{-}\lim_{\Lambda\to\infty} C_\Lambda(A \otimes I + \Lambda I \otimes B - z)^{-1} = C(A - z)^{-1} \otimes P_B$$

holds for all $z \in \mathbb{C} \setminus [0, \infty)$.

The following lemma is used to prove Proposition 4.4 below. For the proof of the lemma, see [12].

Lemma 3.2 *Suppose that C_Λ satisfies the condition* (ii). *Then, for $z \in \mathbb{C} \setminus [0, \infty)$*

$$s\text{-}\lim_{\Lambda\to\infty} C_\Lambda(A \otimes I + \Lambda I \otimes B - z)^{-1}(I \otimes (I - P_B)) = 0.$$

Under the above setting, the following abstract scaling limit theorem holds.

Theorem 3.3 ([1]) *Let A, B, C_Λ and C as above. Then the following hold.*

(a) *For all* $\Lambda > \Lambda_0$ *with some* Λ_0*, the operator*

$$K_\Lambda := A \otimes I + \Lambda I \otimes B + C_\Lambda$$

is self-adjoint on D_{AB} *and bounded below uniformly in* $\Lambda > \Lambda_0$*. Moreover, it is essentially self-adjoint on any core for* $A \otimes I + I \otimes B$*.*

(b) *The operator*

$$K_\infty := A \otimes I + (I \otimes P_B)C(I \otimes P_B)$$

is self-adjoint on $D(A \otimes I)$ *and bouned below. Moreover, it is essentially self-adjoint on any core for* $A \otimes I$*.*

(c) *For all* $z \in [\cap_{\Lambda > \Lambda_0} \rho(K_\Lambda)] \cap \rho(K_\infty)$*,*

$$s\text{-}\lim_{\Lambda \to \infty} (K_\Lambda - z)^{-1} = (K_\infty - z)^{-1}(I \otimes P_B).$$

To express $(K_\infty - z)^{-1}(I \otimes P_B)$ in more explicit way, we need the notion of partial expectation for operators. For this purpose, we now introduce a class of operators on $\mathcal{X}$.

We say that a densely defined operator S on $\mathcal{X}$ is in $\mathbb{E}(\mathcal{X})$ if and only if there exist dense subspaces $D_{\mathcal{H}}(S)$ and $D_{\mathcal{K}}(S)$ in $\mathcal{H}$ and $\mathcal{K}$, respectively, such that

$$D_{\mathcal{H}}(S) \hat{\otimes} D_{\mathcal{K}}(S) \subset D(S).$$

For $S \in \mathbb{E}(S)$, $f \in \mathcal{H}$, $g \in D_{\mathcal{K}}(S)$ and $v \in D_{\mathcal{H}}(S)$, we define the antilinear functional $L_{f,g}$ on $\mathcal{H}$ by

$$L_{f,g}(u) := \langle u \otimes f, S(v \otimes g) \rangle.$$

$L_{f,g}$ is bounded with

$$\left\| L_{f,g}(u) \right\| \leq \|u\| \|f\| \|S(v \otimes g)\|.$$

Therefore, by the Riesz lemma, there exists a unique vector $E_{f,g}(S)v \in \mathcal{H}$ such that

$$L_{f,g}(u) = \left\langle u, E_{f,g}(S)v \right\rangle$$

and

$$\left\|E_{f,g}(S)v\right\| \leq \|f\|\|S(v\otimes g)\|.$$

The map $E_{f,g}(S): v \mapsto E_{f,g}(S)v \in \mathcal{H}$ is linear. Hence, $E_{f,g}(S)$ is a densely defined operator on $\mathcal{H}$ with $D(E_{f,g}) = D_{\mathcal{H}}(S)$. We also define an operator $E_f(S)$ on $\mathcal{H}$ by

$$E_f(S) := E_{f,f}(S).$$

We call the operator $E_{f,g}(S)$ (resp. $E_f(S)$) the *partial expectation* of S with respect to $\{f,g\}$ (resp. f).

Theorem 3.4 ([1]) *Let C be an element of $\mathbb{E}(\mathcal{X})$. Suppose that $\ker B = \mathcal{L}\{f_0\}$ with $\|f_0\| = 1$ and $D_{\mathcal{K}}(C) \supset \ker B$. Then, for $z \in \mathbb{C}\setminus\mathbb{R}$ or for $z < 0$ with $|z|$ sufficiently large,*

$$s\text{-}\lim_{\Lambda\to\infty}(K_\Lambda - z)^{-1} = (K_{\mathrm{eff}} - z)^{-1} \otimes P_B,$$

where

$$K_{\mathrm{eff}} := A + E_{f_0}(C).$$

4. Proof of the Main Result

In this section, we prove Theorem 2.9. The strategy to prove Theorem 2.9 is to apply Theorems 3.3 and 3.4. Our argument is similar to that of Suzuki [12], [13].

4.1. Dressing transformation

We set

$$S := \tilde{\phi}(iv/\omega).$$

S is essentially self-adjoint on $\mathcal{D}_0$ as stated in Subsection 2.2. We denote the closure of S by the same symbol S. Let U be the one-parameter unitary group generated by S:

$$U(t) := e^{itS}, \quad t \in \mathbb{R}.$$

We introduce a subspace $\mathcal{F}_{\text{fin}}(\omega)$ of $\mathcal{F}_{\text{b}}(L^2(\mathbb{R}^d))$ by

$$\mathcal{F}_{\text{fin}}(\omega) := \mathcal{L}\big\{\Omega, a(f_1)^* \cdots a(f_n)^*\Omega \mid n \geq 1, f_j \in D(\omega), j = 1, 2, \ldots, n\big\}.$$

We define a subspace $\mathcal{D}_\omega$ of $\mathcal{F}$ by

$$\mathcal{D}_\omega := C_0^\infty(\mathbb{R}^N) \,\hat{\otimes}\, \mathcal{F}_{\text{fin}}(\omega).$$

Let $\mathcal{A}$ be an algebra. For any element X in $\mathcal{A}$, we define a map $\text{ad}^n(X)$ from $\mathcal{A}$ into $\mathcal{A}$ inductively by

$$\text{ad}^0(X)Y := Y, \quad \text{ad}^n(X)Y := [X, \text{ad}^{n-1}(X)Y].$$

The following lemma is fundamental in our argument below.

Lemma 4.1 *For any $t \in \mathbb{R}$, the following operator equality hold*:

$$U(t)(I \otimes H_{\text{b}})U(t)^{-1} = I \otimes H_{\text{b}} + tH_{\text{I}} + \frac{t^2}{2}\langle\!\langle v/\omega, v\rangle\!\rangle, \tag{4.1}$$

$$U(t)H_{\text{I}}U(t)^{-1} = H_{\text{I}} + t\langle\!\langle v/\omega, v\rangle\!\rangle. \tag{4.2}$$

We need the following well-known lemma to prove Lemma 4.1. Lemma 4.2 is also used to prove Lemma 4.9 below.

Lemma 4.2 *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Let T and S be symmetric operators on $\mathcal{H}$ and $\mathcal{K}$, respectively. Suppose that S is essentially self-adjoint on a dense subspace $\mathcal{D}$ of $\mathcal{K}$. Suppose that a unitary operator U from $\mathcal{H}$ onto $\mathcal{K}$ satisfies the following conditions*:

(i) *$U^{-1}\mathcal{D} \subset D(T)$,*
(ii) *For $\psi \in \mathcal{D}$, $UTU^{-1}\psi = S\psi$ holds.*

Then, T is essentially self-adjoint on $U^{-1}\mathcal{D}$ and the operator equality

$$U\overline{T}U^{-1} = \overline{S}$$

holds.

Proof of Lemma 4.1. Let $\psi \in \mathcal{D}_0 \cap D(I \otimes H_{\text{b}})$. Then, ψ is an entire vector for S. Hence, we have

$$U(t)^{-1}\psi = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} S^n \psi. \tag{4.3}$$

For $m \in \mathbb{N}$, we define ϕ_m by

$$\phi_m := \sum_{n=0}^{m} \frac{(-it)^n}{n!} S^n \psi.$$

By a direct calculation, we have

$$\begin{aligned} I \otimes H_{\mathrm{b}} \phi_m &= \sum_{n=0}^{m} \frac{(-it)^n}{n!} \sum_{j=1}^{n} S^{j-1} [I \otimes H_{\mathrm{b}}, S] S^{n-j} \psi + \sum_{n=0}^{m} \frac{(-it)^n}{n!} S^n I \otimes H_{\mathrm{b}} \psi \\ &= \sum_{n=0}^{m} \frac{(-it)^n}{n!} \sum_{j=1}^{n} S^{j-1} i \tilde{\phi}(v) S^{n-j} \psi + \sum_{n=0}^{m} \frac{(-it)^n}{n!} S^n I \otimes H_{\mathrm{b}} \psi. \end{aligned} \tag{4.4}$$

Here, we use the identity $[I \otimes H_{\mathrm{b}}, S]\psi = i\tilde{\phi}(v)\psi$. By (2.6) and the ratio test for series, the first term of (4.4) converges as $m \to \infty$. The second term of (4.4) also converges as $m \to \infty$ because $I \otimes H_{\mathrm{b}}\psi \in \mathcal{D}_0$. Thus, $I \otimes H_{\mathrm{b}}\phi_m$ converges as $m \to \infty$. By the closedness of $I \otimes H_{\mathrm{b}}$, we have $U(t)^{-1}\psi \in D(I \otimes H_{\mathrm{b}})$ and

$$I \otimes H_{\mathrm{b}} U(t)^{-1} \psi = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} I \otimes H_{\mathrm{b}} S^n \psi. \tag{4.5}$$

By (4.3) and (4.5), for $\phi \in \mathcal{D}_0$, we have

$$\begin{aligned} &\langle \phi, U(t) I \otimes H_{\mathrm{b}} U(t)^{-1} \psi \rangle \\ &\quad = \sum_{N=0}^{\infty} \frac{(it)^N}{N!} \sum_{k=0}^{N} \binom{N}{k} (-1)^k \langle S^{N-k}\phi, I \otimes H_{\mathrm{b}} S^k \psi \rangle \\ &\quad = \sum_{N=0}^{\infty} \frac{(it)^N}{N!} \langle \phi, \mathrm{ad}^N(S) I \otimes H_{\mathrm{b}} \psi \rangle. \end{aligned}$$

One can easily calculate that

$$\begin{aligned}
\mathrm{ad}^0(S)I\otimes H_{\mathrm{b}}\psi &= I\otimes H_{\mathrm{b}}\psi,\\
\mathrm{ad}^1(S)I\otimes H_{\mathrm{b}}\psi &= -i\tilde{\phi}(v)\psi,\\
\mathrm{ad}^2(S)I\otimes H_{\mathrm{b}}\psi &= -\langle\!\langle v/\omega, v\rangle\!\rangle,\\
\mathrm{ad}^n(S)I\otimes H_{\mathrm{b}}\psi &= 0, \quad \text{for} \quad n\geq 3.
\end{aligned}$$

Therefore, we obtain

$$\langle \phi, U(t)I\otimes H_{\mathrm{b}}U(t)^{-1}\psi\rangle = \left\langle \phi, \left(I\otimes H_{\mathrm{b}} + t\tilde{\phi}(v) + \frac{t^2}{2}\langle\!\langle v/\omega, v\rangle\!\rangle\right)\psi\right\rangle.$$

Since $\mathcal{D}_0$ is dense in $\mathcal{F}$, we have

$$U(t)I\otimes H_{\mathrm{b}}U(t)^{-1}\psi = \left(I\otimes H_{\mathrm{b}} + t\tilde{\phi}(v) + \frac{t^2}{2}\langle\!\langle v/\omega, v\rangle\!\rangle\right)\psi. \tag{4.6}$$

From Lemma 4.2 and (4.6), we obtain the operator equality (4.1). The proof of the operator equality (4.2) is same as that of (4.1). □

We set

$$\delta A(t) := U(t)A\otimes IU(t)^{-1} - A\otimes I.$$

From the definition of $\delta A(t)$ and (4.1), we have

$$\begin{aligned}
&U(t)[A\otimes I + \alpha I\otimes H_{\mathrm{b}} + \beta H_{\mathrm{I}}]U(t)^{-1}\psi\\
&\quad = \left[A\otimes I + \alpha I\otimes H_{\mathrm{b}} + (\alpha t+\beta)H_{\mathrm{I}} + \delta A(t) - \left(\frac{\alpha t^2}{2} + \beta t\right)\langle\!\langle v/\omega, v\rangle\!\rangle\right]\psi.
\end{aligned} \tag{4.7}$$

Substituting Λ^2, Λ and $-1/\Lambda$ into α, β and t, respectively, in (4.7), we obtain

$$U(1/\Lambda)^{-1}H(\Lambda)U(1/\Lambda)\psi = [A\otimes I + \Lambda^2 I\otimes H_{\mathrm{b}} + C_\Lambda]\psi, \tag{4.8}$$

where

$$C_\Lambda = \delta A(-1/\Lambda) + \frac{1}{2}\langle\!\langle v/\omega, v\rangle\!\rangle.$$

We need the following two propositions to apply Theorems 3.3 and 3.4 to our case.

Proposition 4.3 *Suppose that* (A.1)–(A.4) *and* (v.1)–(v.2) *hold. Then,* $\overline{C_\Lambda}$ *is* $H_0(\Lambda)$*-infinitesimally small near* ∞.

Proposition 4.4 *Suppose that* (A.1)–(A.4) *and* (v.1)–(v.2) *hold. Then, for* $z \in \mathbb{C} \setminus [E_0(A), \infty)$,

$$s\text{-}\lim_{\Lambda\to\infty} C_\Lambda\big(A \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} - z\big)^{-1} = C(A-z)^{-1} \otimes P_0,$$

where

$$C := \frac{1}{2}\,\mathrm{ad}^2(S) I \otimes H_{\mathrm{b}}.$$

4.2. Proof of Propositions 4.3 and 4.4

To prove Proposition 4.3, we need some lemmas.

Lemma 4.5 $U(t)(I \otimes H_{\mathrm{b}})U(t)^{-1}$ *is uniformly* $(I \otimes H_{\mathrm{b}})$*-bounded near* 0.

Proof. By Lemma 4.1, for $\psi \in \mathcal{D}_\omega$, we have

$$\begin{aligned}
&\|U(t)(I \otimes H_{\mathrm{b}})U(t)^{-1}\psi\| \\
&\quad \le \|I \otimes H_{\mathrm{b}}\psi\| + |t|\big\|\tilde{\phi}(v)\psi\big\| + \frac{t^2}{2}\|v/\omega\|\|v\|\|\psi\| \\
&\quad \le \|I \otimes H_{\mathrm{b}}\psi\| + \sqrt{2}|t|\big\|v/\sqrt{\omega}\big\|\big\|I \otimes H_{\mathrm{b}}^{1/2}\psi\big\| \\
&\qquad + \frac{|t|}{\sqrt{2}}\|v\|\|\psi\| + \frac{t^2}{2}\|v/\omega\|\|v\|\|\psi\|.
\end{aligned}$$

This proves the lemma. □

The following lemma follows from Lemma 4.5 and Theorem B.2.

Lemma 4.6

(i) *For* $v \in L^{\infty,2}$ *with* $v/\sqrt{\omega} \in L^{\infty,2}$, $U(t)\tilde{\phi}(v)U(t)^{-1}$ *is uniformly* $(I \otimes H_{\mathrm{b}})$*-infinitesimally small near* 0.

(ii) *For* $v \in L^{\infty,2}$ *with* $v/\sqrt{\omega} \in L^{\infty,2}$ *and* $w \in L^{\infty,2}$ *with* $\sqrt{\omega}w, w/\sqrt{\omega} \in L^{\infty,2}$, $U(t)\tilde{\phi}(v)\tilde{\phi}(w)U(t)^{-1}$ *is uniformly* $(I \otimes H_{\mathrm{b}})$*-bounded near* 0.

The following lemma is the most important lemma to prove Propositions 4.3 and 4.4. Before proving the lemma, note the following fact:

$$U(t)\mathcal{D}_\omega \subset D(A \otimes I).$$

This is shown in the same way as in the proof of Lemma 4.1.

Lemma 4.7 *The operator $\delta A(t)$ is uniformly $(A \otimes I + I \otimes H_{\mathrm{b}})$-infinitesimally small near* 0 *and for $\psi \in \mathcal{D}_\omega$*

$$\lim_{t \to 0} \delta A(t)\psi = 0 \tag{4.9}$$

holds.

Proof. Let $\psi \in \mathcal{D}_\omega$. By the Taylor expansion, there exists a number $\xi(t)$ between 0 and t such that

$$\delta A(t)\psi = itU(\xi(t))\,\mathrm{ad}(S)[A \otimes I]U(\xi(t))^{-1}\psi.$$

Note that

$$\mathrm{ad}(S)[A \otimes I] = \tilde{\phi}(i\triangle_x v/\omega) + 2\sum_{j=1}^{N} \tilde{\phi}(i\partial_j v/\omega)(D_j \otimes I)$$

on $\mathcal{D}_\omega$. Hence, we have

$$\begin{aligned} \|\delta A(t)\psi\| \le |t| \big\|\tilde{\phi}(i\triangle_x v/\omega)U(\xi(t))^{-1}\psi\big\| \\ + 2|t|\sum_{j=1}^{N} \big\|\tilde{\phi}(i\partial_j v/\omega)(D_j \otimes I)U(\xi(t))^{-1}\psi\big\|. \end{aligned} \tag{4.10}$$

By the strong commutativity between $\tilde{\phi}(i\triangle v/\omega)$ and $U(\xi(t))^{-1}$, the first term of (4.10) is estimated as follows:

$$\begin{aligned} &|t|\big\|\tilde{\phi}(i\triangle_x v/\omega)U(\xi(t))^{-1}\psi\big\| \\ &\quad = |t|\big\|\tilde{\phi}(i\triangle_x v/\omega)\psi\big\| \\ &\quad \le \sqrt{2}|t|\big\|\triangle_x v/\omega^{3/2}\big\|\big\|I \otimes H_{\mathrm{b}}^{1/2}\psi\big\| + \frac{|t|}{\sqrt{2}}\|\triangle_x v/\omega\|\|\psi\|. \end{aligned}$$

We next estimate the second term of (4.10).

By a similar argument as above, there exists a number $\eta(t)$ betweeen 0 and $\xi(t)$ such that for all $\psi \in \mathcal{D}_\omega$

$$
\begin{aligned}
&U(\xi(t))\tilde{\phi}(i\partial_j v/\omega)(D_j \otimes I)U(\xi(t))^{-1}\psi \\
&\quad = \tilde{\phi}(i\partial_j v/\omega)(D_j \otimes I)\psi \\
&\qquad + i\xi(t)U(\eta(t))\,\mathrm{ad}^1(S)\tilde{\phi}(i\partial_j v/\omega)(D_j \otimes I)U(\eta(t))^{-1}\psi.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\big\|\tilde{\phi}(i\partial_j v/\omega)(D_j \otimes I)U(\xi(t))^{-1}\psi\big\| \\
&\quad \le \big\|\tilde{\phi}(i\partial_j v/\omega)(D_j \otimes I)\psi\big\| \\
&\qquad + |\xi(t)|\big\|\,\mathrm{ad}^1(S)\tilde{\phi}(i\partial_j v/\omega)(D_j \otimes I)U(\eta(t))^{-1}\psi\big\|. \qquad (4.11)
\end{aligned}
$$

The first term of (4.11) is estimated as follows:

$$
\begin{aligned}
&\big\|\tilde{\phi}(i\partial_j v/\omega)(D_j \otimes I)\psi\big\| \\
&\quad \le \sqrt{2}\big\|\partial_j v/\omega^{3/2}\big\|\big\|(I \otimes H_{\mathrm{b}}^{1/2})(D_j \otimes I)\psi\big\| + \frac{1}{\sqrt{2}}\|\partial_j v/\omega\|\|(D_j \otimes I)\psi\| \\
&\quad \le \left(\big\|\partial_j v/\omega^{3/2}\big\| + \frac{\varepsilon}{\sqrt{2}}\|\partial_j v/\omega\|\right)\big\|(\tilde{A} \otimes I + I \otimes H_{\mathrm{b}})\psi\big\| \\
&\qquad + \frac{1}{4\sqrt{2}\varepsilon}\|\partial_j v/\omega\|\|\psi\|.
\end{aligned}
$$

Here we used the facts that

$$
\big\|(I \otimes H_{\mathrm{b}}^{1/2})((D_j \otimes I)\psi\big\| \le \frac{1}{\sqrt{2}}\big\|(\tilde{A} \otimes I + I \otimes H_{\mathrm{b}})\psi\big\|
$$

and

$$
\|(D_j \otimes I)\psi\| \le \varepsilon\big\|(\tilde{A} \otimes I + I \otimes H_{\mathrm{b}})\psi\big\| + \frac{1}{4\varepsilon}\|\psi\|.
$$

It is easy to see that

$$
\begin{aligned}
&\mathrm{ad}^1(S)\tilde{\phi}(i\partial_j v/\omega)(D_j \otimes I)\\
&\qquad = i\mathrm{Im}\langle\!\langle iv/\omega, i\partial_j v/\omega\rangle\!\rangle(D_j \otimes I) - \tilde{\phi}(i\partial_j v/\omega)\tilde{\phi}(i\partial_j v/\omega)
\end{aligned}
$$

on $\mathcal{D}_\omega$. Therefore, the second term of (4.11) is estimated as follows:

$$
\begin{aligned}
&\left\| \mathrm{ad}^1(S)\tilde{\phi}(i\partial_j v/\omega)(D_j \otimes I)U(\eta(t))^{-1}\psi\right\|\\
&\qquad \leq \|v/\omega\|\|i\partial_j v/\omega\|\left\|(D_j \otimes I)U(\eta(t))^{-1}\psi\right\|\\
&\qquad\quad + \left\|\tilde{\phi}(i\partial_j v/\omega)\tilde{\phi}(i\partial_j v/\omega)U(\eta(t))^{-1}\psi\right\|.
\end{aligned}
$$

In the same way as in the proof of Lemma 4.1, $U(\eta(t))(D_j \otimes I)U(\eta(t))^{-1}$ is $(A \otimes I + I \otimes H_{\mathrm{b}})$-infinitesimally small near 0. The operator $U(\eta(t))\tilde{\phi}(i\partial_j v/\omega)\tilde{\phi}(i\partial_j v/\omega)U(\eta(t))^{-1}$ is $(I \otimes H_{\mathrm{b}})$-bounded uniformly near 0 by Lemma 4.6. This proves the lemma. □

Proposition 4.3 is a direct consequence of Lemma 4.7.

We next prove Proposition 4.4. To prove Proposition 4.4, we need the following lemma.

Lemma 4.8 *For $\psi \in \mathcal{D}_\omega$,*

$$\lim_{\Lambda\to\infty} C_\Lambda \psi = C\psi. \tag{4.12}$$

Proof. Let $\psi \in \mathcal{D}_\omega$. By (4.9), we have

$$\lim_{\Lambda\to\infty} \delta A(-1/\Lambda)\psi = 0.$$

This proves the lemma. □

We are now ready to prove Proposition 4.4.

Proof of Proposition 4.4. For $z \in \mathbb{C} \setminus [E_0(A), \infty)$, note that

$$(A \otimes I - z)^{-1} I \otimes P_0 = (A \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} - z)^{-1} I \otimes P_0. \tag{4.13}$$

From (4.13), we have

$$
\begin{aligned}
&\overline{C_\Lambda}(A \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} - z)^{-1}\\
&\qquad = \overline{C_\Lambda}(A \otimes I - z)^{-1} I \otimes P_0 + \overline{C_\Lambda}(A \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} - z)^{-1} I \otimes (I - P_0).
\end{aligned}
$$

By Lemma 3.2, it follows that

$$s\text{-}\lim_{\Lambda\to\infty} \overline{C_\Lambda}(A\otimes I+\Lambda^2 I\otimes H_{\mathrm{b}}-z)^{-1} I\otimes(I-P_0)=0.$$

Thus, it is sufficient to prove

$$s\text{-}\lim_{\Lambda\to\infty} \overline{C_\Lambda}(A\otimes I-z)^{-1} I\otimes P_0=C(A-z)^{-1}\otimes P_0. \tag{4.14}$$

By Proposition 4.3, $\overline{C_\Lambda}(A\otimes I-z)^{-1}(I\otimes P_0)$ is uniformly bounded with respect to Λ. Since $\mathcal{D}$ is a core for A, $(A-z)\mathcal{D}$ is a dense subspace of $L^2(\mathbb{R}^N)$. Thus, to prove (4.14), we only to prove

$$\lim_{\Lambda\to\infty} \overline{C_\Lambda}(A\otimes I-z)^{-1} I\otimes P_0\Psi=C(A-z)^{-1}\otimes P_0\Psi, \tag{4.15}$$

for all $\Psi\in(A-z)\mathcal{D}\,\hat{\otimes}\,\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$.

Let $\psi\in\mathcal{D}$ and $\phi\in\mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$. By Lemma 4.8, we have

$$\begin{aligned}&\big(\overline{C_\Lambda}(A\otimes I-z)^{-1}(I\otimes P_0)-C(A-z)^{-1}\otimes P_0\big)((A-z)\psi\otimes\phi)\\&\quad=(\overline{C_\Lambda}-C)(\psi\otimes P_0\phi)\quad\to 0\quad\text{as}\quad \Lambda\to\infty.\end{aligned}$$

Thus, (4.15) is proved. Therefore, (4.14) follows. This completes the proof of the proposition. □

4.3. Proof of Theorem 2.9

By Proposition 4.3 and the Kato-Rellich theorem, the operator $A\otimes I+\Lambda^2 I\otimes H_{\mathrm{b}}+\overline{C_\Lambda}$ is self-adjoint for sufficiently large Λ. Therefore, from (4.8) and Lemma 4.2, we obtain the following theorem.

Theorem 4.9 *Suppose* (A.1)–(A.4) *and* (v.1)–(v.2) *hold. Then,* $A\otimes I+\Lambda^2 I\otimes H_{\mathrm{b}}+\overline{C_\Lambda}$ *is self-adjoint on* $D(A\otimes I)\cap D(I\otimes H_{\mathrm{b}})$ *and the following operator equality holds*:

$$U(1/\Lambda)H(\Lambda)U(1/\Lambda)^{-1}=A\otimes I+\Lambda^2 I\otimes H_{\mathrm{b}}+\overline{C_\Lambda}. \tag{4.16}$$

We are now in position to prove Theorem 2.9.

Proof of Theorem 2.9. By Proposition 4.3 and 4.4, we can apply Theorem 3.3 to our case. To apply Theorem 3.4, we need to calculate the partial expectation of the operator C with respect to the Fock vacuum Ω.

On $\mathcal{D}_0 \cap D(I \otimes H_{\mathrm{b}})$, we have

$$C = -\frac{1}{2}\langle\!\langle v/\omega, v\rangle\!\rangle.$$

Therefore, for any $u_1, u_2 \in L^2(\mathbb{R}^N)$, we obtain

$$\begin{aligned}\langle u_1 \otimes \Omega, C(u_2 \otimes \Omega)\rangle &= -\frac{1}{2}\langle u_1 \otimes \Omega, \langle\!\langle v/\omega, v\rangle\!\rangle(u_2 \otimes \Omega)\rangle \\ &= -\frac{1}{2}\langle u_1, \langle\!\langle v/\omega, v\rangle\!\rangle_0 u_2\rangle.\end{aligned}$$

Hence, the partial expectation of the operator C with respect to the Fock vacuum Ω is equal to $-1/2\langle\!\langle v/\omega, v\rangle\!\rangle_0$. Therefore, we obtain

$$s\text{-}\lim_{\Lambda\to\infty} \left(A \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} + \overline{C_\Lambda} - z\right)^{-1} = (A + V_{\mathrm{eff}} - z)^{-1} \otimes P_0. \quad (4.17)$$

On the other hand, by (4.16), we have

$$U(1/\Lambda)(H(\Lambda) - z)^{-1}U(1/\Lambda)^{-1} = \left(A \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} - \overline{C_\Lambda} - z\right)^{-1}.$$

Therefore, we obtain

$$s\text{-}\lim_{\Lambda\to\infty} (H(\Lambda) - z)^{-1} = s\text{-}\lim_{\Lambda\to\infty} \left(A \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} + \overline{C_\Lambda} - z\right)^{-1}. \quad (4.18)$$

Since

$$s\text{-}\lim_{t\to 0} U(t) = I,$$

(4.17) and (4.18) imply (2.9). This completes the proof of Theorem 2.9. □

5. Example

As an application of our theory developed in the present paper, we consider a scaling limit for the Nelson model, which describes an interaction between quantum particles and a quantum scalar field. In this section, we write an element $x \in \mathbb{R}^{dn}$ by $x = (x_1, \ldots, x_n), x_j \in \mathbb{R}^d$.

We consider a quantum system consisting of n quantum particles with mass $m > 0$ moving in the Euclidean space $\mathbb{R}^d$ under the influence of a potential V. A Hamiltonian of such a system is given by

$$A := -\sum_{j=1}^{n} \frac{1}{2m} \triangle_{x_j} + V,$$

where $\triangle_{x_j}$ is the Laplacian with respect to the variable x_j. In what follows, we assume that the operator A satisfies the assumption (A.1)–(A.4).

We define a function ω on $\mathbb{R}^d$ by

$$\omega(k) := \sqrt{\mu^2 + |k|^2},$$

where $\mu \geq 0$ is the mass of a boson. The function ω denotes the kinetic energy of a boson with momentum k.

Let $g_j \in L^2(\mathbb{R}^d)$ with $\omega g_j \in L^2(\mathbb{R}^d)$ and $g_j/\omega \in L^2(\mathbb{R}^d)$ $(j = 1, \ldots, n)$. We define a function v_{Nelson} on $\mathbb{R}^{dn} \times \mathbb{R}^d$ by

$$v_{\mathrm{Nelson}}(x, k) := \sum_{j=1}^{n} e^{-ix_j \cdot k} g_j(k),$$

where $x_j \cdot k$ denote the Euclidean inner product of x_j and k. It is easily verified that v_{Nelson} satisfies the assumption (v.1)–(v.2).

We define an operator H_{Nelson} on a Hilbert space $\mathcal{F}_{\mathrm{Nelson}} := L^2(\mathbb{R}^{dn}) \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d))$ by

$$H_{\mathrm{Nelson}} := A \otimes I + I \otimes H_{\mathrm{b}} + \tilde{\phi}(v_{\mathrm{Nelson}}).$$

For $\Lambda > 0$, we define the scaled Hamiltonian by

$$H_{\mathrm{Nelson}}(\Lambda) := A \otimes I + \Lambda^2 I \otimes H_{\mathrm{b}} + \Lambda \tilde{\phi}(v_{\mathrm{Nelson}}).$$

In this case, we have

$$\begin{aligned} &\langle v_{\mathrm{Nelson}}(x, k)/\omega(k), v_{\mathrm{Nelson}}(x, k) \rangle_{L^2(\mathbb{R}^d_k)} \\ &= \sum_{i,j=1}^{n} \int_{\mathbb{R}^d} \frac{e^{i(x_i - x_j) \cdot k}}{\omega(k)} \overline{g_i(k)} g_j(k) dk. \end{aligned} \tag{5.1}$$

We set the right hand side of (5.1) by $E(x)$. The multiplication operator by the function E on $L^2(\mathbb{R}^{dn})$ is denoted by the same symbol E. Therefore, we obtain

$$\langle\!\langle v/\omega, v\rangle\!\rangle_0 = E.$$

By Theorem 2.9, we obtain the following theorem.

Theorem 5.1 *Let A, v_{Nelson} and E as above. Then, for $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large,*

$$s\text{-}\lim_{\Lambda\to\infty}(H_{\text{Nelson}}(\Lambda) - z)^{-1} = (A + E - z)^{-1} \otimes P_0$$

holds.

A. Weyl relations for $\tilde{\phi}(v)$

In this appendix, we derive the Weyl relations for $\tilde{\phi}(v)$ (Theorem A.4). As a corollary of the Weyl relations, we obtain a necessary and sufficient condition that $\tilde{\phi}(v)$ and $\tilde{\phi}(w)$ strongly commute.

Lemma A.1 *Let $\mathcal{A}$ be an algebra and X, Y, Z be elments of $\mathcal{A}$. Suppose that Z commutes with X and Y, and that X and Y satisfy the following commutation relation*:

$$[X, Y] = Z. \tag{A.1}$$

Then, the following equation holds:

$$\sum_{k=0}^{n} {}_nC_k X^{n-k} Y^k = \sum_{k,r\geq 0, k+2r=n} \frac{n!}{k!r!}\left(\frac{Z}{2}\right)^r (X+Y)^k. \tag{A.2}$$

Proof. One can easily verify the lemma by induction. So we omit the proof. □

The following lemma is important to prove the Weyl relations for $\tilde{\phi}(v)$. It is proved in the same way as in the proof of Theorem 2.3.

Lemma A.2 *For any $z \in \mathbb{C}, \psi \in \mathcal{D}_0$ and $v, w \in L^{\infty,2}$, $e^{z\tilde{\phi}(v)}\psi$ is an entire vector for $\tilde{\phi}(w)$. Moreover, for any $\zeta \in \mathbb{C}$,*

$$e^{z\tilde{\phi}(v)}e^{\zeta\tilde{\phi}(w)}\psi = \sum_{n,m=0}^{\infty} \frac{z^n\zeta^m}{n!m!}\tilde{\phi}(v)^n\tilde{\phi}(w)^m\psi. \tag{A.3}$$

Here, the right hand side of (A.3) *is absolutely convergent.*

Proposition A.3

(1) *Let* $\psi \in \mathcal{D}_0$ *and* $z \in \mathbb{C}$. *Then,*

$$e^{z\tilde{\phi}(v+w)}\psi = e^{-iz^2\mathrm{Im}\langle\!\langle v,w\rangle\!\rangle/2}e^{z\tilde{\phi}(v)}e^{z\tilde{\phi}(w)}\psi.$$

(2) *The following operator equalities hold*:

$$e^{i\tilde{\phi}(v+w)} = e^{i\mathrm{Im}\langle\!\langle v,w\rangle\!\rangle/2}e^{i\tilde{\phi}(v)}e^{i\tilde{\phi}(w)}, \tag{A.4}$$

$$e^{i\tilde{\phi}(v)}e^{i\tilde{\phi}(w)} = e^{-i\mathrm{Im}\langle\!\langle v,w\rangle\!\rangle}e^{i\tilde{\phi}(w)}e^{i\tilde{\phi}(v)}. \tag{A.5}$$

Proof.

(1) Direct calculations by using Lemmas A.1 and A.2.
(2) (2) follows from (1). □

Theorem A.4 (Weyl relations for $\tilde{\phi}(v)$) *For all* $s, t \in \mathbb{R}$,

$$e^{is\tilde{\phi}(v)}e^{it\tilde{\phi}(w)} = e^{ist\mathrm{Im}\langle\!\langle v,w\rangle\!\rangle}e^{it\tilde{\phi}(w)}e^{is\tilde{\phi}(v)}. \tag{A.6}$$

Proof. Replacing v with sv and w with tw respectively in (A.5), we obtain (A.6). □

We would like to seek a necessary and sufficient condition that $\tilde{\phi}(v)$ and $\tilde{\phi}(w)$ strongly commute. Remember that two self-adjoint operators strongly commute if and only if their spectral measures commute. This condition is rephrased by the condition that thier strongly one-parameter unitary groups commute. Therefore, we obtain the following corollary of Theorem A.4.

Corollary A.5 *Let* $v, w \in L^{\infty,2}$. *Then* $\tilde{\phi}(v)$ *and* $\tilde{\phi}(w)$ *strongly commute if and only if* $\langle v(x,k), w(x,k)\rangle_{L^2(\mathbb{R}^d_k)} \in \mathbb{R}$ *for almost every* $x \in \mathbb{R}^N$.

B. Relative Boundedness of $\tilde{\phi}(v)\tilde{\phi}(w)$

In this appendix, we establish a relative boundedness of $\tilde{\phi}(v)\tilde{\phi}(w)$ $(v, w \in L^{\infty,2})$ with respect to $I \otimes H_{\mathrm{b}}$. To do so, we need a lemma.

Lemma B.1

(i) *For* $v \in L^{\infty,2}$ *with* $v/\sqrt{\omega} \in L^{\infty,2}$, $D(I \otimes H_{\mathrm{b}}) \subset D(I \otimes H_{\mathrm{b}}^{1/2}\tilde{a}(v))$ *and for* $\psi \in D(I \otimes H_{\mathrm{b}})$,

$$\left\|I \otimes H_{\mathrm{b}}^{1/2}\tilde{a}(v)\psi\right\| \le \left\|v/\sqrt{\omega}\right\| \|I \otimes H_{\mathrm{b}}\psi\|.$$

(ii) *For* $v \in L^{\infty,2}$ *with* $\sqrt{\omega}v, v/\sqrt{\omega} \in L^{\infty,2}$, $D(I \otimes H_{\mathrm{b}}) \subset D(I \otimes H_{\mathrm{b}}^{1/2}\tilde{a}^*(v))$ *and for* $\psi \in D(I \otimes H_{\mathrm{b}})$,

$$\begin{aligned}
&\left\|I \otimes H_{\mathrm{b}}^{1/2}\tilde{a}^*(v)\psi\right\|^2 \\
&\quad \le (1+\varepsilon)\big(\|v/\sqrt{\omega}\|^2 \|I \otimes H_{\mathrm{b}}\psi\|^2 + \|v\|^2 \left\|I \otimes H_{\mathrm{b}}^{1/2}\psi\right\|^2\big) \\
&\qquad + \left(1 + \frac{1}{\varepsilon}\right)\big(\|v\|^2 \left\|I \otimes H_{\mathrm{b}}^{1/2}\psi\right\|^2 + \|\sqrt{\omega}v\|^2 \|\psi\|^2\big),
\end{aligned}$$

where ε *is an arbitrary positive constant.*

Proof. Direct calculations. □

The desired relative estimate of $\tilde{\phi}(v)\tilde{\phi}(w)$ immediately follows from Lemma B.1

Theorem B.2 *For* $v \in L^{\infty,2}$ *with* $v/\sqrt{\omega} \in L^{\infty,2}$ *and* $w \in L^{\infty,2}$ *with* $\sqrt{\omega}w, w/\sqrt{\omega} \in L^{\infty,2}$, $D(\tilde{\phi}(v)\tilde{\phi}(w)) \supset D(I \otimes H_{\mathrm{b}})$ *and there exist constants* $a(v,w), b(v,w) \ge 0$ *such that for* $\psi \in D(I \otimes H_{\mathrm{b}})$

$$\left\|\tilde{\phi}(v)\tilde{\phi}(w)\psi\right\| \le a(v,w)\|I \otimes H_{\mathrm{b}}\psi\| + b(v,w)\|\psi\|$$

holds.

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