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Measures orthogonal to tensor products of function algebras

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Abstract. Some fundamental properties of measures orthogonal to tensor products of function algebras are considered. It is shown that if $\mathcal{M}_1$, $\mathcal{M}_2$ are reducing bands of measures for algebras A_1, A_2, then their projection-preserving product is a reducing band for $A_1 \otimes A_2$. As an application, a decomposition of measures orthogonal to the tensor product of some classes of function algebras is obtained. These classes contain among others, multidimensional ball algebras.

Key words: function algebras, tensor product, annihilating measures, marginal measures, reducing bands of measures.

1. Introduction and preliminaries

Let X be a compact Hausdorff space. By $C(X)$ we will denote the Banach algebra of all complex continuous functions on X equipped with the supnorm, and by $M(X)$ the set of all complex Borel regular measures on X. Considered with the total variation norm $\|\cdot\|$, $M(X)$ is a Banach space. For a measure $\mu \in M(X)$ denote by $|\mu|$ its variation measure. If E is a subset of $M(X)$ then E^s will denote the set of all measures in $M(X)$ singular to each measure in E. A subset $\mathcal{M}$ of $M(X)$ is a *band* (see [9], sec. 2) if $(\mathcal{M}^s)^s = \mathcal{M}$. For $E \subset M(X)$, the set $(E^s)^s$ is the smallest band containing E. We call it the band *generated* by E.

One of the important questions concerning tensor products of function algebras is how the structure of the collection of measures orthogonal to these products depends on the structures of analogous collections for the original algebras. It is well known, (see [13]), that the Cartesian product of Gleason parts of two function algebras is a Gleason part for their tensor product. This fact implies that the projection-preserving product of bands generated by the mentioned Gleason parts is a reducing band (the necessary definitions will be established later).

In Section 2 we show that a similar property holds for all reducing

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bands. In Section 3 we give some applications of these general properties. We get there decompositions of measures orthogonal to tensor products of some classes of function algebras.

Apart from deepening our knowledge on tensor products of function algebras, this decomposition technique plays an important role in the theory of operator representations of function algebras and its applications to the invariant subspace problems for multioperators (see for example [8], [9], [10], [11], [12]). Similar decompositions were known up till now only for some function algebras on Cartesian products of compact subsets of the complex plane. Since the results obtained in the present paper are general, one may expect their future application for operator representations of some wider class of function algebras. This remark applies, in particular, to the possible extension of the results of Eschmeier [5] and Pott [14].

It is easy to see that any band is a norm closed subspace of $M(X)$, which is "closed" also with respect to the absolute continuity relation denoted by $\ll$. This property and the consequent decomposition are formulated in the following proposition:

Proposition 1.1 *A subset $\mathcal{M} \subset M(X)$ is a band if and only if it is a closed subspace of $M(X)$ satisfying the implication*

$$\mu \in M(X),\ \mu \ll |\nu|,\ \nu \in \mathcal{M} \implies \mu \in \mathcal{M}.$$

Any measure $\mu \in M(X)$ has a Lebesgue decomposition of the form

$$\mu = \mu^{\mathcal{M}} + \mu^{\mathcal{M}^s}, \tag{1.1}$$

where $\mu^{\mathcal{M}} \in \mathcal{M}$ and $\mu^{\mathcal{M}^s} \in \mathcal{M}^s$.

(For details and terminology concerning bands we refer the reader to [2], sec. 20., [4], V. 17). As a consequence we get

Proposition 1.2 *If $\mathcal{M} \subset M(X)$ is a band then*

$$(f\mu)^{\mathcal{M}} = f\mu^{\mathcal{M}} \quad \text{for } f \in C(X),$$

where $f\mu$ denotes the measure defined by $\int g\, d(f\mu) = \int gf\, d\mu$ for $g \in C(X)$.

In the case when only one band is considered we will write μ^s instead of $\mu^{\mathcal{M}^s}$ for the singular part of μ.

In what follows, assume that A is an arbitrary function algebra on X, i.e. a uniformly closed subalgebra of $C(X)$ containing constants and separating the points of X. We say that a measure μ is *orthogonal to* A (or *annihilates* A) and write $\mu \perp A$ if $\int u\, d\mu = 0$ for $u \in A$. The set of all such measures is denoted by $A^{\perp}$. A band $\mathcal{M} \in M(X)$ is said to be *reducing with respect to* A if we have the implication

$$\mu \perp A \implies \mu^{\mathcal{M}} \perp A \tag{1.2}$$

When A is fixed we simply say that $\mathcal{M}$ is *reducing.*

2. Main results

Let X_1, X_2 be two compact Hausdorff spaces, and A_1, A_2 function algebras on X_1, X_2 respectively. Denote by $A_1 \otimes A_2$ the closure in $C(X_1 \times X_2)$ of the algebraic tensor product of A_1 and A_2. Then $A_1 \otimes A_2$ (with the supnorm) is a function algebra on $X_1 \times X_2$ (for further details see [13]). For a measure $\mu \in M(X_1 \times X_2)$ we define its natural projections $\pi_1\mu$, $\pi_2\mu$ (marginal measures) on the sets X_1, X_2 as follows:

$$(\pi_1\mu)(E_1) \stackrel{\text{df}}{=} \mu(E_1 \times X_2),$$
$$(\pi_2\mu)(E_2) \stackrel{\text{df}}{=} \mu(X_1 \times E_2),$$

where E_1, E_2 are Borel subsets of X_1, X_2 respectively.

Let μ be a finite positive regular measure on $X_1 \times X_2$, and let E_1 be a Borel subset of X_1. Then for any $\varepsilon > 0$ there is a compact set $K \subset E_1 \times X_2$ such that $\mu(E_1 \times X_2) - \mu(K) < \varepsilon$. Let K_1 be the projection of K onto X_1. Since the projection mapping in the Cartesian product is continuous, K_1 is also compact. Then

$$\begin{aligned}(\pi_1\mu)(E_1) - (\pi_1\mu)(K_1) &= \mu(E_1 \times X_2) - \mu(K_1 \times X_2)\\ &< \mu(E_1 \times X_2) - \mu(K) < \varepsilon.\end{aligned}$$

It means that $\pi_1\mu$ is inner regular. For the outer regularity observe that if K_1 is compact and $K_1 \subset (X_1 \setminus E_1)$ then $X_1 \setminus K_1 \supset E_1$ and $X_1 \setminus K_1$ is open.

Recall that a complex measure is regular if and only if its variation measure is regular, and if and only if its real and imaginary part are regular. A real measure is regular if and only if its positive and negative part in Jordan decomposition are regular. It is also true that if μ and ν are finite nonnegative measures, μ is regular and $\mu \geq \nu$ then ν is also regular. Hence, if μ is real regular then also $\pi_i\mu$ $(i = 1, 2)$ is regular since $(\pi_i\mu)_{\pm} \leq \pi_i\mu_{\pm}$.

For a complex regular measure μ the regularity of $\pi_i\mu$ follows from the regularity of its real and imaginary part. So we have

Proposition 2.1 *If $\mu \in M(X_1 \times X_2)$ then $\pi_i\mu \in M(X_i)$ for $i = 1, 2$.*

By $\mathcal{N}_+$ we will denote the set of all real nonnegative measures contained in a given set of complex measures $\mathcal{N}$. We have the following lemmas

Lemma 2.2 *If $\mathcal{M} \subset M(X_i)$, $(i = 1, 2)$ is a band then*

$$M(X_1 \times X_2)_+ = \pi_i^{-1}(\mathcal{M})_+ + \pi_i^{-1}(\mathcal{M}^s)_+, \tag{2.1}$$

and the summands on the right hand side are mutually singular.

Proof. Since the proof is analogous for $i = 1$ and $i = 2$, it is sufficient to consider only the case $i = 1$.

For a Borel set F and a measure μ, denote by μ_F the restriction of μ to the set F. It follows directly from the definition of π_i that for any Borel $E \subset X_1$ and any nonnegative measure $\mu \in M(X_1 \times X_2)$ we have

$$(\pi_1\mu)_E = \pi_1(\mu_{E\times X_2}), \tag{2.2}$$

and, by Proposition 2.1, $\pi_1\mu \in M(X_1)$.

When $\mathcal{M} \subset M(X_1)$ and $\mu \geq 0$ is fixed, taking the decomposition (1.1) for $\pi_1\mu$ we can find a Borel set $E \subset X_1$ depending on μ such that

$$(\pi_1\mu)^{\mathcal{M}} = (\pi_1\mu)_E, \quad (\pi_1\mu)^{\mathcal{M}^s} = (\pi_1\mu)_F, \tag{2.3}$$

where $F := X_1 \backslash E$. Hence $\mu = \mu_{E\times X_2} + \mu_{F\times X_2}$, and by (2.2), (2.3), $\mu_{E\times X_2} \in \pi_1^{-1}(\mathcal{M})_+$, $\mu_{F\times X_2} \in \pi_1^{-1}(\mathcal{M}^s)_+$. Since $\mu \geq 0$ has been arbitrarily chosen, we get (2.1) for $i = 1$.

On the other hand, if positive measures μ, $\nu \in M(X_1 \times X_2)$ are such that $\mu \in \pi_1^{-1}(\mathcal{M})$, $\nu \in \pi_1^{-1}(\mathcal{M}^s)$, then $\pi_1(\mu) \in \mathcal{M}$, $\pi_1(\nu) \in \mathcal{M}^s$. So there are two disjoint Borel subsets E, $F \subset X_1$ such that $\pi_1(\mu)$ is supported on E and $\pi_1(\nu)$ is supported on F. Hence, by the positivity, μ is supported on $E \times X_2$ and ν is supported on $F \times X_2$ which implies that they are mutually singular. □

Lemma 2.3 *If $\mu \in \pi_i^{-1}(\mathcal{M})_+ - \pi_i^{-1}(\mathcal{M})_+$ has its Jordan decomposition $\mu = \mu^+ - \mu^-$, then μ^+, $\mu^- \in \pi_i^{-1}(\mathcal{M})_+$.*

Proof. Let $\mu = \lambda_1 - \lambda_2$, where $\lambda_i \in \pi_i^{-1}(\mathcal{M})_+$ $(i = 1, 2)$. By the minimality of Jordan decomposition we get $\lambda_1 \geq \mu^+$ and $\lambda_2 \geq \mu^-$. Hence $\pi_i\lambda_1 \geq$

$\pi_i\mu^+$, $\pi_i\lambda_2 \geq \pi_i\mu^-$. So, by Propositions 1.1 and 2.1, $\pi_i\mu^+$, $\pi_i\mu^- \in \mathcal{M}$, and consequently μ^+, $\mu^- \in \pi_i^{-1}(\mathcal{M})$ which, by their positivity gives μ^+, $\mu^- \in \pi_i^{-1}(\mathcal{M})_+$. □

For the sake of completeness we state and prove below some known property of real measures. The author was unable to find its explicite form in the literature.

Lemma 2.4 *If μ, ν are real measures having Jordan decompositions*

$$\mu = \mu^+ - \mu^-, \quad \nu = \nu^+ - \nu^-$$

into the combinations of nonnegative measures then

$$\|\mu - \nu\| = \|\mu^+ - \nu^+\| + \|\mu^- - \nu^-\|.$$

Proof. It follows from Hahn decomposition theorem that there are Borel subsets E^+, E^-, F^+, F^- such that E^- is the complement of E^+, F^- is the complement of F^+, and

$$\mu^+ = \mu_{|_{E^+}}, \quad \mu^- = \mu_{|_{E^-}}, \quad \nu^+ = \nu_{|_{F^+}}, \quad \nu^- = \nu_{|_{F^-}}. \tag{2.4}$$

Then

$$\begin{aligned}
&\|\mu - \nu\| \\
&= |\mu - \nu|((E^+ \cap F^+) \cup (E^+ \cap F^-) \cup (E^- \cap F^+) \cup (E^- \cap F^-)) \\
&= |\mu - \nu|(E^+ \cap F^+) + |\mu - \nu|(E^+ \cap F^-) \\
&\quad + |\mu - \nu|(E^- \cap F^+) + |\mu - \nu|(E^- \cap F^-) \\
&= |\mu^+ - \nu^+|(E^+ \cap F^+) + (\mu^+ + \nu^-)(E^+ \cap F^-) \\
&\quad + (\mu^- + \nu^+)(E^- \cap F^+) + |\mu^- - \nu^-|(E^- \cap F^-) \\
&= |\mu^+ - \nu^+|(E^+ \cap F^+) + \mu^+(F^-) + \nu^-(E^+) \\
&\quad + \mu^-(F^+) + \nu^+(E^-) + |\mu^- - \nu^-|(E^- \cap F^-),
\end{aligned}$$

because, by (2.4), $\mu^\pm$ is supported on $E^\pm$, and $\nu^\pm$ is supported on $F^\pm$.

But, also by (2.4),

$$\mu^\pm(F^\mp) = |\mu^\pm - \nu^\pm|(F^\mp), \quad \nu^\pm(E^\mp) = |\mu^\pm - \nu^\pm|(E^\mp),$$

and hence

$$
\begin{aligned}
&|\mu^{\pm} - \nu^{\pm}|(E^{\pm} \cap F^{\pm}) + \mu^{\pm}(F^{\mp}) + \nu^{\pm}(E^{\mp}) \\
&= |\mu^{\pm} - \nu^{\pm}|(E^{\pm} \cap F^{\pm}) + |\mu^{\pm} - \nu^{\pm}|(E^{\mp}) + |\mu^{\pm} - \nu^{\pm}|(F^{\mp}) \\
&\geq |\mu^{\pm} - \nu^{\pm}|(E^{\pm} \cap F^{\pm}) + |\mu^{\pm} - \nu^{\pm}|(E^{\mp} \cup F^{\mp}) = \|\mu^{\pm} - \nu^{\pm}\|.
\end{aligned}
$$

Consequently, $\|\mu - \nu\| \geq \|\mu^+ - \nu^+\| + \|\mu^- - \nu^-\|$. The opposite inequality follows immediately from the triangle inequality. □

For $\mathcal{M} \subset M(X_i)$ define its *band preimage* as follows:

$$\widetilde{\pi_i^{-1}}(\mathcal{M}) \overset{\mathrm{df}}{=} \mathrm{Lin}(\pi_i^{-1}(\mathcal{M})_+),$$

where $\mathrm{Lin}(\mathcal{N})$ denotes the set of all finite linear combinations of elements in $\mathcal{N}$. By the definition, and the fact that $\pi_i^{-1}(\mathcal{M})_+$ is a cone we have

$$\widetilde{\pi_i^{-1}}(\mathcal{M}) = \pi_i^{-1}(\mathcal{M})_+ - \pi_i^{-1}(\mathcal{M})_+ + i\pi_i^{-1}(\mathcal{M})_+ - i\pi_i^{-1}(\mathcal{M})_+. \quad (2.5)$$

Theorem 2.5 *If $\mathcal{M} \subset M(X_i)$, $(i = 1, 2)$ is a band then $\widetilde{\pi_i^{-1}}(\mathcal{M})$ is also a band. Moreover*

$$\widetilde{\pi_i^{-1}}(\mathcal{M})^s = \widetilde{\pi_i^{-1}}(\mathcal{M}^s). \quad (2.6)$$

If $\mathcal{M}$ is reducing for A_i then $\widetilde{\pi_i^{-1}}(\mathcal{M})$ is reducing for $A_1 \otimes A_2$.

Proof. Since $\mathcal{M}$ is a band, $\pi_i^{-1}(\mathcal{M})$ is closed in $M(X_1 \times X_2)$ by the continuity of π_i. On the other hand, the set $M(X_1 \times X_2)_+$ is evidently closed. Hence $\pi_i^{-1}(\mathcal{M})_+ = \pi_i^{-1}(\mathcal{M}) \cap M(X_1 \times X_2)_+$ is also closed.

For a complex measure μ we can apply Jordan decomposition of its real and imaginary part, i.e.

$$\mu = \nu^+ - \nu^- + i\eta^+ - i\eta^-, \quad (2.7)$$

where ν^+, ν^-, η^+, η^- are nonnegative. Moreover, if $\mu \in \widetilde{\pi_i^{-1}}(\mathcal{M})$, then by Lemma 2.3 and (2.5), $\nu^+, \nu^-, \eta^+, \eta^- \in \pi_i^{-1}(\mathcal{M})_+$. If $\{\mu_n\}$ is a sequence in $\widetilde{\pi_i^{-1}}(\mathcal{M})$ and $\mu_n \to \mu$ then, by Lemma 2.4, all the parts of μ_n in the decomposition (2.7) tends to the appropriate parts of μ. Consequently, since $\pi_i^{-1}(\mathcal{M})_+$ is closed, $\widetilde{\pi_i^{-1}}(\mathcal{M})$ is also closed. By Lemma 2.2, (2.5) and (2.7), we get

$$
\begin{aligned}
&M(X_1 \times X_2) = \widetilde{\pi_i^{-1}}(\mathcal{M}) + \widetilde{\pi_i^{-1}}(\mathcal{M}^s), \\
&\mu \in \widetilde{\pi_i^{-1}}(\mathcal{M}),\ \nu \in \widetilde{\pi_i^{-1}}(\mathcal{M}^s) \implies \mu \perp \nu,
\end{aligned} \quad (2.8)
$$

which shows (2.6). Applying (2.6) we have

$$\widetilde{\pi_i^{-1}}(\mathcal{M})^{ss} = \widetilde{\pi_i^{-1}}(\mathcal{M}^s)^s = \widetilde{\pi_i^{-1}}(\mathcal{M}^{ss}) = \widetilde{\pi_i^{-1}}(\mathcal{M}),$$

which means that $\widetilde{\pi_i^{-1}}(\mathcal{M})$ is a band.

It remains to show that if the band $\mathcal{M}$ is reducing, then for $i = 1, 2$ the bands $\widetilde{\pi_i^{-1}}(\mathcal{M})$ are also reducing. The property (2.8) implies the equality

$$(\pi_i\mu)^{\mathcal{M}} = \pi_i(\mu^{\widetilde{\pi_i^{-1}}(\mathcal{M})}) \quad \text{for } \mu \in M(X_1 \times X_2). \tag{2.9}$$

From now on we put $i = 1$ for the simplicity of notation and assume that $\mathcal{M} \subset M(X_1)$ is reducing for A_1. Let μ be a complex measure orthogonal to $A_1 \otimes A_2$. Take now the Lebesgue decomposition of μ with respect to the band $\widetilde{\pi_1^{-1}}(\mathcal{M})$ and denote by μ^a the absolutely continuous part of μ. We should show that μ^a is orthogonal to $A_1 \otimes A_2$. For such a purpose it is enough to show

$$\int fg\, d\mu^a = 0 \quad \text{for } f \in A_1,\ g \in A_2. \tag{2.10}$$

Since μ is orthogonal to $A_1 \otimes A_2$, we have

$$0 = \int fg\, d\mu = \int f\, d(g\mu) = \int f\, d(\pi_1(g\mu)).$$

The last equality follows from the fact that the function f depends only on the first variable. So we have shown that the measure $\pi_1(g\mu)$ is orthogonal to A_1. Since $\mathcal{M}$ is reducing for A_1, we have $\int f\, d(\pi_1(g\mu))^{\mathcal{M}} = 0$. But, by (2.9), we have

$$\int f\, d(\pi_1((g\mu)^a)) = \int f\, d(\pi_1(g\mu))^{\mathcal{M}} = 0. \tag{2.11}$$

On the other hand

$$\int fg\, d\mu^a = \int f\, d(g\mu^a) = \int f\, d(g\mu)^a = \int f\, d(\pi_1((g\mu)^a)), \tag{2.12}$$

where the last equality follows from the fact that f depends only on the first variable. By (2.11), (2.12) we get $\int fg\, d\mu^a = 0$, which shows (2.10) and completes the proof. □

We introduce the notion of *projection-preserving product* of bands $\mathcal{M}_i \subset$

$M(X_i)$, $(i = 1, 2)$ as follows:

$$\mathcal{M}_1 \otimes_{\mathrm{pr}} \mathcal{M}_2 \overset{\mathrm{df}}{=} \widetilde{\pi_1^{-1}}(\mathcal{M}_1) \cap \widetilde{\pi_2^{-1}}(\mathcal{M}_2).$$

It follows almost directly from the definition that the intersection of reducing bands is a reducing band. So, by Theorem 2.5, we have the following

Theorem 2.6 *Let for $i = 1, 2$ the set $\mathcal{M}_i \subset M(X_i)$ be a reducing band for A_i. Then $\mathcal{M}_1 \otimes_{\mathrm{pr}} \mathcal{M}_2$ is a reducing band for $A_1 \otimes A_2$.*

Moreover, we have the following decomposition into mutually singular reducing bands of measures:

$$M(X_1 \times X_2) = \mathcal{M}_1 \otimes_{\mathrm{pr}} \mathcal{M}_2 + \mathcal{M}_1 \otimes_{\mathrm{pr}} \mathcal{M}_2^s + \mathcal{M}_1^s \otimes_{\mathrm{pr}} \mathcal{M}_2 + \mathcal{M}_1^s \otimes_{\mathrm{pr}} \mathcal{M}_2^s.$$

3. Some applications

Denote by Q_i the set of all non-peak points of A_i, and assume that $\mathcal{M}_i$ is generated by all measures representing for points in Q_i. It follows from Lemma II.7.4 [6] that for each $\mu \in \mathcal{M}_1^s$ there is a Borel set $E \subset X_1$ such that μ is supported on E, and E is a nullset for $\mathcal{M}_1$, i.e. each measure in $\mathcal{M}_1$ vanishes on E. The similar property is valid for $\mathcal{M}_2^s$.

Hence, by (2.2), each measure in $\pi_1^{-1}((\mathcal{M}_1)^s)_+$ is supported on a set of the form $E \times X_2$, where E is a nullset for $\mathcal{M}_1$. Consequently, each measure in $\widetilde{\pi_1^{-1}}((\mathcal{M}_1)^s)$ is supported on a set of such a form. Similarly, each measure in $\widetilde{\pi_2^{-1}}((\mathcal{M}_2)^s)$ is supported on a set of the form $X_1 \times E$, where E is a nullset for $\mathcal{M}_2$, and each measure in $\mathcal{M}_1^s \otimes_{\mathrm{pr}} \mathcal{M}_2^s$ is supported on a set of the form $E_1 \times E_2$, where E_i, $(i = 1, 2)$ is a nullset for $\mathcal{M}_i$.

If we assume moreover that there are no measures in $A_i^{\perp}$ which are singular to $\mathcal{M}_i$, then by [6], p. 59, and the regularity property of $M(X_i)$, we can deduce that any measure in $M(X_i)$ which is singular to all measures in $\mathcal{M}_i$ is supported on a set which is a countable union of peak interpolation sets (see [6]). And also the opposite is true: each measure supported on such a set is singular to $\mathcal{M}_i$. If E_i, $(i = 1, 2)$, is a peak interpolation set for A_i then $E_1 \times E_2$ is a peak interpolation set for $A_1 \otimes A_2$, and so, it is a nullset for measures orthogonal to $A_1 \otimes A_2$. Hence all measures orthogonal to $A_1 \otimes A_2$ are singular to $(A_1^{\perp})^s \otimes_{\mathrm{pr}} (A_2^{\perp})^s$. So, using Theorem 2.6, we obtain a result which generalizes partially Bekken's decomposition of measures orthogonal to the algebra $R(K_1 \times K_2)$ where K_i is a compact subset of the complex plane (see [1], Thm. 1). Our result does not give so detailed description of the first part of the decomposition, but instead, it is valid for much larger class of function algebras and has much shorter and elementary proof.

Theorem 3.1 *Let Q_i be the sets of all non-peak points for function algebras $A_i \subset C(X_i)$, $i = 1, 2$, and $\mathcal{M}_i$ the bands of measures generated by all measures representing points in Q_i. Assume moreover that there are no measures orthogonal to A_i which are singular to $\mathcal{M}_i$. Then for each measure μ orthogonal to $A_1 \otimes A_2$ we have the following decomposition into mutually singular measures which are orthogonal to $A_1 \otimes A_2$:*

$$\mu = \mu_0 + \mu_1 + \mu_2,$$

where
(1) *μ_0 belongs to the band $\mathcal{M}_1 \otimes_{\mathrm{pr}} \mathcal{M}_2$,*
(2) *μ_1 is supported on a set of the form $E \times X_2$, where E is a nullset for $A_1^{\perp}$, $\pi_2(\mu_1)$ belongs to band $\mathcal{M}_2$,*
(3) *μ_2 is supported on a set of the form $X_1 \times E$, where E is a nullset for $A_2^{\perp}$, $\pi_1(\mu_2)$ belongs to band $\mathcal{M}_1$.*

For an integer k denote by $A(\mathbb{B}_k)$ the algebra of all functions analytic in the open unit ball $\mathbb{B}_k \subset \mathbb{C}^k$ and continuous on $\overline{\mathbb{B}_k}$. The set of all non-peak points for such an algebra is the entire unit ball $\mathbb{B}_k$. Denote by $\mathcal{M}_0$ the band of measures generated by all measures representing for points in $\mathbb{B}_k$. By Henkin theorem (see [7] or Thm. 9.3.1, [15]) together with Cole-Range theorem (see [3] or Thm. 9.6.1, [15]) we get

Proposition 3.2 *There are no measures orthogonal to $A(\mathbb{B}_k)$ and singular to all measures representing for points in $\mathbb{B}_k$ i.e. we have the following decomposition into mutually singular bands*

$$M(\mathbb{B}_k) = \mathcal{M}_0 + (A(\mathbb{B}_k)^{\perp})^s.$$

By [6] VI.1.2, the band $\mathcal{M}_0$ is generated by all measures representing the evaluation at the origin for the algebra $A(\mathbb{B}_k)$. Hence, in a standard way, like in [8], we can deduce that for any measure $\mu \in \mathcal{M}_0$ one finds a measure ν representing the evaluation at the origin such that μ is absolutely continuous with respect to ν. For integers k, l denote by $A(\mathbb{B}_k \times \mathbb{B}_l)$ the algebra of all functions analytic in $\mathbb{B}_k \times \mathbb{B}_l$ and continuous on $\overline{\mathbb{B}_k \times \mathbb{B}_l}$, by π_k denote the projection from the measure space $M(\overline{\mathbb{B}_k \times \mathbb{B}_l})$ onto $M(\overline{\mathbb{B}_k})$

and by π_l onto $M(\overline{\mathbb{B}_l})$ (like in the beginning of section 2). So, as a corollary from Theorem 3.1 we have the following

Theorem 3.3 *For each measure μ orthogonal to $A(\mathbb{B}_k \times \mathbb{B}_l)$ we have the decomposition into three mutually singular measures which are orthogonal to $A(\mathbb{B}_k \times \mathbb{B}_l)$:*

$$\mu = \mu_0 + \mu_1 + \mu_2,$$

where

(1) *$\pi_k(\mu_0)$ (resp. $\pi_l(\mu_0)$) is absolutely continuous with respect to some measure representing the evaluation at the origin for the algebra $A(\mathbb{B}_k)$ (resp. $A(\mathbb{B}_l)$),*

(2) *μ_1 (resp. μ_2) is supported on a set of the form $E \times X_2$ (resp. $X_1 \times E$), where E is a nullset for $A(\mathbb{B}_k)^\perp$ (resp. $A(\mathbb{B}_l)^\perp$), the measure $\pi_l(\mu_1)$ (resp. $\pi_k(\mu_2)$) is absolutely continuous with respect to some measure representing the evaluation at the origin for the algebra $A(\mathbb{B}_l)$ (resp. $A(\mathbb{B}_k)$).*

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